# A CELLULAR SIMPLEX WITH PRESCRIBED NUMBERS OF POINTS IN REGIONS DETERMINED BY ITS FACETS 

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#### Abstract

Let $P$ be a finite set of points in the 3 -dimensional Euclidean space $\mathbb{R}^{3}$ in general position. For $x_{0}, x_{1}, x_{2}, x_{3} \in P$, let $H^{+}\left(x_{0} ; x_{1}, x_{2}, x_{3}\right)$ (resp. $\left.H^{-}\left(x_{0}, x_{1}, x_{2}, x_{3}\right)\right)$ denote the open half space containing $x_{0}$ (resp. not containing $x_{0}$ ) and bounded by the plane containing $x_{1}, x_{2}, x_{3}$. Further let $$
\begin{aligned} P\left(x_{0} ; x_{1}, x_{2}, x_{3}\right):=P & \cap H^{+}\left(x_{1} ; x_{0}, x_{2}, x_{3}\right) \\ & \cap H^{+}\left(x_{2} ; x_{0}, x_{1}, x_{3}\right) \\ & \cap H^{+}\left(x_{3} ; x_{0}, x_{1}, x_{3}\right) . \end{aligned}
$$


In this paper, we show the following statement: if $|P| \geq 4$, and if $k_{1}, k_{2}, k_{3}, k_{4}$ are integers with $k_{1}+k_{2}+k_{3}+k_{4}=|P|-4,0 \leq k_{1}, k_{2}, k_{3}, k_{4} \leq \frac{|P|-2}{2}$ and $k_{1}+k_{2} \leq \frac{|P|-2}{2}$, then for any $p_{1}, p_{2} \in P\left(p_{1} \neq p_{2}\right)$, there exist $q_{1}, q_{2} \in P$ such that the convex hull of $\left\{p_{1}, p_{2}, q_{1}, q_{2}\right\}$ is a 3 -simplex (tetrahedron) containing no point of $P$ in its interior and such that

$$
\begin{aligned}
& \left|P\left(p_{1} ; p_{2}, q_{1}, q_{2}\right)\right| \leq k_{1} \leq P \cap H^{-}\left(p_{1} ; p_{2}, q_{1}, q_{2}\right) \mid, \\
& \left|P\left(p_{2} ; p_{1}, q_{1}, q_{2}\right)\right| \leq k_{2} \leq P \cap H^{-}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right) \mid, \\
& \left|P\left(q_{1} ; q_{2}, p_{1}, p_{2}\right)\right| \leq k_{3} \leq P \cap H^{-}\left(q_{1} ; q_{2}, p_{1}, p_{2}\right) \mid, \\
& \left|P\left(q_{2} ; q_{1}, p_{1}, p_{2}\right)\right| \leq k_{4} \leq P \cap H^{-}\left(q_{2} ; q_{1}, p_{1}, p_{2}\right) \mid .
\end{aligned}
$$

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## §1. Introduction.

For a subset $V$ of the $d$-dimensional Euclidean space $\mathbb{R}^{d}$, let conv $(V)$ denote the convex hull of $V$, and let aff $(V)$ denote the affine flat spanned by $V$. For $d+1$ points $x_{0}, x_{2}, \cdots, x_{d}$ not lying in the same (affine) $(d-1)$-flat in $\mathbb{R}^{d}$,
let $H^{+}\left(x_{0} ; x_{1}, \cdots, x_{d}\right)\left(\right.$ resp. $\left.H^{-}\left(x_{0} ; x_{1}, \cdots, x_{d}\right)\right)$ denote the open half-space which is bounded by aff $\left(\left\{x_{1}, \cdots, x_{d}\right\}\right)$ and contains $x$ (resp. does not contain $x)$. Now let $P$ be a fixed set of points in $\mathbb{R}^{d}$. We say that $P$ is in general position if no $d+1$ points of $P$ lie in the same $(d-1)$-flat. For $d+1$ points $x_{0}, x_{1}, \cdots, x_{d}$ not lying in the same ( $d-1$ )-flat, let

$$
P\left(x_{0} ; x_{1}, \cdots, x_{d}\right):=P \cap \bigcap_{1 \leq i \leq d} H^{+}\left(x_{i} ; x_{0}, x_{1}, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{d}\right) .
$$

If a subset $V$ of $\mathbb{R}^{3}$ contains no point of $P$ in its interior, $V$ is said to be vacuum. Further, following Kupitz[2], we call a polyhedron $D$ cellular if $D$ is vacuum and all vertices of $D$ are points of $P$. In this paper, we show the following theorem as a 3 -dimensional version of Lemma 3 in [1]:

Theorem 1. Let $P$ be a finite set of points in $\mathbb{R}^{3}$ in general position. Suppose that $|P| \geq 4$, and let $k_{1}, k_{2}, k_{3}, k_{4}$ be integers such that $k_{1}+k_{2}+k_{3}+k_{4}=$ $|P|-4,0 \leq k_{1}, k_{2}, k_{3}, k_{4} \leq \frac{|P|-2}{2}$ and $k_{1}+k_{2} \leq \frac{|P|-2}{2}$. Further let $p_{1}, p_{2}$ be specified points of $P$ with $p_{1} \neq p_{2}$. Then there exist two points $q_{1}, q_{2}$ of $P$ such that $\operatorname{conv}\left(\left\{p_{1}, p_{2}, q_{1}, q_{2}\right\}\right)$ is a cellular 3-simplex and the following inequalities hold:

$$
\begin{align*}
& \left|P\left(p_{1} ; p_{2}, q_{1}, q_{2}\right)\right| \leq k_{1} \leq\left|P \cap H^{-}\left(p_{1} ; p_{2}, q_{1}, q_{2}\right)\right|,  \tag{1.1}\\
& \left|P\left(p_{2} ; p_{1}, q_{1}, q_{2}\right)\right| \leq k_{2} \leq\left|P \cap H^{-}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right)\right|,  \tag{1.2}\\
& \left|P\left(q_{1} ; q_{2}, p_{1}, p_{2}\right)\right| \leq k_{3} \leq\left|P \cap H^{-}\left(q_{1} ; q_{2}, p_{1}, p_{2}\right)\right|  \tag{1.3}\\
& \left|P\left(q_{2} ; q_{1}, p_{1}, p_{2}\right)\right| \leq k_{4} \leq\left|P \cap H^{-}\left(q_{2} ; q_{1}, p_{1}, p_{2}\right)\right| . \tag{1.4}
\end{align*}
$$

## §2. Proof of Theorem 1.

Let $P, k_{1}, k_{2}, k_{3}, k_{4}, p_{1}, p_{2}$ be as in Theorem 1 . Let $\pi: \mathbb{R}^{3} \rightarrow \mathbb{R}^{2}$ be the orthogonal projection in the direction of $\overrightarrow{p_{1} p_{2}}$. We use the following result in the plane case (a slight modification of Claim 1 in [1]):

Proposition 1. Let $P^{\prime}$ be a finite set of points in $\mathbb{R}^{2}$, and let $r_{0}^{\prime}$ be a specified point of $P^{\prime}$. Suppose that $P^{\prime} \geq 3$ and any line passing through $r_{0}^{\prime}$ contains at most one point of $P^{\prime}$ other than $r_{0}^{\prime}$. Let $k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}$ be integers satisfying $0 \leq k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime} \leq \frac{\left|P^{\prime}\right|-1}{2}$ and $k_{1}^{\prime}+k_{2}^{\prime}+k_{3}^{\prime}=\left|P^{\prime}\right|-3$. Then there exist $x^{\prime} \in \mathbb{R}^{2}-P$ and $r_{1}^{\prime}, r_{2}^{\prime} \in P^{\prime}-\left\{r_{0}^{\prime}\right\}$ such that

$$
P^{\prime}=\left\{r_{0}^{\prime}, r_{1}^{\prime}, r_{2}^{\prime}\right\} \cup P^{\prime}\left(r_{0}^{\prime} ; x^{\prime}, r_{1}^{\prime}\right) \cup P^{\prime}\left(r_{0}^{\prime} ; r_{1}^{\prime}, r_{2}^{\prime}\right) \cup P^{\prime}\left(r_{0}^{\prime} ; r_{2}^{\prime}, x^{\prime}\right)
$$

and

$$
\left|P^{\prime}\left(r_{0}^{\prime} ; x^{\prime}, r_{1}^{\prime}\right)\right|=k_{1}^{\prime},\left|P^{\prime}\left(r_{0}^{\prime} ; r_{1}^{\prime}, r_{2}^{\prime}\right)\right|=k_{2}^{\prime},\left|P^{\prime}\left(r_{0}^{\prime} ; r_{2}^{\prime}, x^{\prime}\right)\right|=k_{3}^{\prime} .
$$

Proposition 1 is essentially the same as Claim 1 in [1], so we omit the proof. Since $k_{1}+k_{2} \leq \frac{|P|-2}{2}$, we can apply Proposition 1 to $\pi(P)=\{\pi(p) \mid p \in P\}$ with $r_{0}^{\prime}=\pi\left(p_{1}\right)=\pi\left(p_{2}\right)$ and $k_{1}^{\prime}=k_{3}, k_{2}^{\prime}=k_{1}+k_{2}, k_{3}^{\prime}=k_{4}$. Let $x^{\prime}, r_{1}^{\prime}, r_{2}^{\prime}$ be as in the conclusion of the Proposition 1. We use the same technique as in the proof of Lemma 3 in [1]. Let $l_{0}$ be a line passing through $r_{0}^{\prime}$ and $x^{\prime}$. Take $s_{1}^{\prime}, s_{2}^{\prime} \in P^{\prime}\left(r_{0}^{\prime} ; r_{1}^{\prime}, r_{2}^{\prime}\right) \cup\left\{r_{1}^{\prime}, r_{2}^{\prime}\right\}$ so that for $i=1,2, s_{i}^{\prime}$ lies in the same side of $l_{0}$ as $r_{i}^{\prime}$, and
the line segment $\overline{s_{1}^{\prime} s_{2}^{\prime}}$ is an edge of $\operatorname{conv}\left(P^{\prime}\left(r_{0}^{\prime} ; r_{1}^{\prime}, r_{2}^{\prime}\right) \cup\left\{r_{1}^{\prime}, r_{2}^{\prime}\right\}\right)$ satisfying $\operatorname{conv}\left(\left\{r_{0}^{\prime}, s_{1}^{\prime}, s_{2}^{\prime}\right\}\right) \cap H^{-}\left(r_{0}^{\prime} ; s_{1}^{\prime}, s_{2}^{\prime}\right)=\emptyset$.

Now we return to $\mathbb{R}^{3}$. Let $x, r_{i}, s_{i}(i=1,2)$ be the points of $P$ such that $x$, $\pi\left(r_{i}\right)=r_{i}^{\prime}, \pi\left(s_{i}\right)=s_{i}^{\prime}$, respectively. Let

$$
\begin{aligned}
& K_{1}:=H^{+}\left(x ; r_{1}, p_{1}, p_{2}\right) \cap H^{+}\left(r_{1} ; x, p_{1}, p_{2}\right), \\
& K_{2}:=H^{+}\left(r_{1} ; r_{2}, p_{1}, p_{2}\right) \cap H^{+}\left(r_{2} ; r_{1}, p_{1}, p_{2}\right), \\
& K_{3}:=H^{+}\left(r_{2} ; x, p_{1}, p_{2}\right) \cap H^{+}\left(x ; r_{2}, p_{1}, p_{2}\right) .
\end{aligned}
$$

Then the conclusion of Proposition 1 implies that $K_{i} \cap K_{j}=\emptyset$ if $i \neq j$, and

$$
\begin{align*}
& \left|P \cap K_{1}\right|=k_{1}^{\prime}=k_{3},  \tag{2.6}\\
& \left|P \cap K_{2}\right|=k_{2}^{\prime}=k_{1}+k_{2},  \tag{2.7}\\
& \left|P \cap K_{3}\right|=k_{3}^{\prime}=k_{4} . \tag{2.8}
\end{align*}
$$

Let $H_{0}:=\pi^{-1}\left(l_{0}\right)$ and let $S=\left(P \cap K_{2}\right) \cup\left\{r_{1}, r_{2}\right\} . \operatorname{By}(2.5)$,

$$
\Delta:=H^{+}\left(r_{1}^{\prime} ; r_{0}^{\prime}, r_{2}^{\prime}\right) \cap H^{+}\left(r_{2}^{\prime} ; r_{0}^{\prime}, r_{1}^{\prime}\right) \cap H^{+}\left(r_{0}^{\prime} ; r_{1}^{\prime}, r_{2}^{\prime}\right)
$$

is vacuum. Since $K_{2} \cap H^{+}\left(p_{1} ; p_{2}, s_{1}, s_{2}\right) \cap H^{+}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right) \subset \pi^{-1}(\Delta)$, this implies that $S \cap H^{+}\left(p_{1} ; p_{2}, s_{1}, s_{2}\right) \cap H^{+}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)=\emptyset$. Thus by (2.7), $\left|S \cap H^{+}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)\right| \leq k_{1}+2$ or $\left|S \cap H^{+}\left(p_{1} ; p_{2}, s_{1}, s_{2}\right)\right| \leq k_{2}+2$ holds. By symmetry, we may assume

$$
\begin{equation*}
\left|S \cap H^{+}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)\right| \leq k_{1}+2 . \tag{2.9}
\end{equation*}
$$

For a plane $H$ and a point $x \notin H$, let $H^{+}(x)\left(\right.$ resp. $\left.\bar{H}^{+}(x)\right)$ denote the open (resp. closed) half-space which is bounded by $H$ and contains $x$, and let $H^{-}(x)$ (resp. $\left.\bar{H}^{-}(x)\right)$ denote the open (resp. closed) half-space which is bounded by $H$ and does not contain $x$. Let $H_{1}$ be a plane containing $p_{1}$ such that

$$
\begin{gather*}
\left|S \cap \bar{H}_{1}^{+}\left(p_{2}\right)\right|=k_{1}+2,  \tag{2.10}\\
S \cap \bar{H}_{1}^{+}\left(p_{2}\right) \cap H_{0}^{+}\left(r_{i}\right) \neq \emptyset \text { for } i=1,2 . \tag{2.11}
\end{gather*}
$$

Note that by (2.9), there exists a plane satisfying (2.10) and (2.11). We choose $H_{1}$ so that the angle between $H_{0} \cap H_{1} \cap K_{2}$ and $\overrightarrow{p_{1} p_{2}}$ is as small as possible. Take $q_{1}, q_{2}$ so that

$$
\begin{align*}
& q_{1} \in S \cap \bar{H}_{1}^{+}\left(p_{2}\right) \cap H_{0}^{+}\left(r_{1}\right),  \tag{2.12}\\
& q_{2} \in S \cap \bar{H}_{1}^{+}\left(p_{2}\right) \cap H_{0}^{+}\left(r_{2}\right), \tag{2.13}
\end{align*}
$$

and

$$
\begin{gather*}
\triangle p_{2} q_{1} q_{2} \text { is a facet of } \operatorname{conv}\left(\left(S \cup\left\{p_{2}\right\}\right) \cap \bar{H}_{1}^{+}\left(p_{2}\right)\right) \text { satisfying }  \tag{2.14}\\
\operatorname{conv}\left(\left\{p_{1}, p_{2}, q_{1}, q_{2}\right\}\right) \cap H^{-}\left(p_{1} ; p_{2}, q_{1}, q_{2}\right)=\emptyset
\end{gather*}
$$

By (2.14), $\operatorname{conv}\left(\left\{p_{1}, p_{2}, q_{1}, q_{2}\right\}\right)$ is vacuum. We now proceed to verify the inequalities in the conclusion of Theorem 1. By (2.12) and (2.13),

$$
\begin{aligned}
& P\left(q_{1} ; q_{2}, p_{1}, p_{2}\right) \subseteq P \cap K_{1} \subseteq P \cap H^{-}\left(q_{1} ; q_{2}, p_{1}, p_{2}\right) \text { and } \\
& P\left(q_{2} ; q_{1}, p_{1}, p_{2}\right) \subseteq P \cap K_{3} \subseteq P \cap H^{-}\left(q_{2} ; q_{1}, p_{1}, p_{2}\right)
\end{aligned}
$$

hold, and hence $(2.6),(2.8)$ imply (1.3), (1.4), respectively. Similarly by (2.14),

$$
P\left(p_{1} ; p_{2}, q_{1}, q_{2}\right) \subseteq S \cap \bar{H}_{1}^{+}\left(p_{2}\right)-\left\{q_{1}, q_{2}\right\} \subseteq P \cap H^{-}\left(p_{1} ; p_{2}, q_{1}, q_{2}\right)
$$

holds, and hence (2.10) implies (1.1). Further, it also follows from the choice of $q_{1}, q_{2}$ that

$$
P\left(p_{2} ; p_{1}, q_{1}, q_{2}\right) \subseteq S \cap H_{1}^{-}\left(p_{2}\right)
$$

Since

$$
\left|S \cap H_{1}^{-}\left(p_{2}\right)\right|=\left(k_{1}+k_{2}+2\right)-\left(k_{1}+2\right)=k_{2}
$$

by (2.7) and (2.10), this immediately implies the first inequality in (1.2).
We are now left with the verification of the second inequality in (1.2). Suppose

$$
\left|P \cap H^{-}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right)\right|<k_{2}
$$

Then clearly

$$
\begin{equation*}
\left|S \cap H^{-}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right)\right|<k_{2} \tag{2.15}
\end{equation*}
$$

On the other hand, by (2.7) and (2.9),

$$
\begin{equation*}
\left|S \cap H^{-}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)\right| \geq\left(k_{1}+k_{2}+2\right)-\left(k_{1}+2\right)=k_{2} \tag{2.16}
\end{equation*}
$$

holds. Let $y, z$ be the intersection points of the line passing through $s_{1}, s_{2}$ and $\operatorname{aff}\left(\left\{p_{1}, p_{2}, r_{1}\right\}\right)$, aff $\left(\left\{p_{1}, p_{2}, r_{2}\right\}\right)$, respectively. Then (2.15) and (2.16) imply that $S \cap H^{-}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right) \nsubseteq S \cap H^{-}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right)$, which implies that at least one of $y, z$ belongs to $H^{+}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right)$. We may assume

$$
\begin{equation*}
y \in H^{+}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right) \tag{2.17}
\end{equation*}
$$

without loss of generality. We now show the existence of a plane containing $p_{0}$ which gives rise to a contradiction to the choice of $H_{1}$. Toward this end, we divide the situation into two cases according to the location of $q_{2}$.

Case $1 q_{2}=s_{2}$ or $q_{2} \in H^{+}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)$
In this case,

$$
S \cap H^{-}\left(p_{2} ; p_{1}, y, q_{2}\right) \supseteq S \cap H^{-}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)
$$

holds and hence by (2.16),

$$
\begin{equation*}
\left|S \cap H^{-}\left(p_{2} ; p_{1}, y, q_{2}\right)\right| \geq\left|S \cap H^{-}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)\right| \geq k_{2} . \tag{2.18}
\end{equation*}
$$

Let $l_{1}$ be the line passing through $p_{1}, q_{2}$, and let $H$ be a (movable) plane containing $l_{1}$. If we gradually rotate $H$ with $l_{1}$ as the axis, the value of $\mid S \cap$ $H^{-}\left(p_{2}\right) \mid$ changes by one at each moment when $H$ hits a point of $P$. Therefore by (2.15) and (2.18), there exists $H_{2} \in l_{1} \cup H^{+}\left(p_{2}, p_{1}, q_{1}, q_{2}\right) \cup H^{+}\left(p_{2}, p_{1}, y, q_{2}\right)$ such that $l_{1} \in H_{2}$ and $\left|S \cap H_{2}^{-}\left(p_{2}\right)\right|=k_{2}$, or equivalently, $\left|S \cap \bar{H}_{2}^{+}\left(p_{2}\right)\right|=k_{1}+2$. Now to get a contradiction, we let $K_{2}^{\prime}:=H^{+}\left(q_{1} ; q_{2}, p_{1}, p_{2}\right) \cap H^{+}\left(q_{2} ; q_{1}, p_{1}, p_{2}\right)$ (note that by (2.12) and (2.13), $H_{0}$ intersects with $K_{2}^{\prime}$ ). Then by (2.17), it is easy to see that

$$
\begin{aligned}
H_{0} \cap K_{2}^{\prime} \cap H_{2}^{+}\left(p_{2}\right) & \subset H_{0} \cap K_{2}^{\prime} \cap H^{+}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right) \\
& \subseteq H_{0} \cap K_{2}^{\prime} \cap H_{1}^{+}\left(p_{2}\right),
\end{aligned}
$$

which yields a contradiction to the minimality of the angle between $H_{0} \cap H_{1} \cap$ $K_{2}$ and $\overrightarrow{p_{1} p_{2}}$.

Case $2 q_{2} \in H^{-}\left(p_{2} ; p_{1}, s_{1}, s_{2}\right)$
If (2.18) holds, a contradiction can be derived in the same way as in Case 2. Thus we may assume

$$
\begin{equation*}
\left|S \cap H^{-}\left(p_{2} ; p_{1}, y, q_{2}\right)\right|<k_{2} . \tag{2.19}
\end{equation*}
$$

Let $l_{2}$ be the line passing through $p_{1}, y$. Then again as in Case 1 , (2.16) and (2.19) imply that we can find a plane $H_{3} \in l_{2} \cup H^{+}\left(p_{2}, p_{1}, y, q_{2}\right) \cup$ $H^{+}\left(p_{2}, p_{1}, s_{1}, s_{2}\right)$ such that $l_{2} \in H_{3}$ and $\left|S \cap \bar{H}_{3}^{+}\left(p_{2}\right)\right|=k_{1}+2$ by considering the rotation of a plane containing $l_{2}$ with $l_{2}$ as the axis. Thus again it is easy to see that

$$
\begin{aligned}
H_{0} \cap K_{2}^{\prime} \cap H_{3}^{+}\left(p_{2}\right) & \subset H_{0} \cap K_{2}^{\prime} \cap H^{+}\left(p_{2} ; p_{1}, q_{1}, q_{2}\right) \\
& \subseteq H_{0} \cap K_{2}^{\prime} \cap H_{1}^{+}\left(p_{2}\right),
\end{aligned}
$$

which yields a contradiction.

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