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Theory of Hecke algebras to association schemes

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Abstract. In the book entitled "Methods of Representation Theory" by Curtis and Reiner they discuss character tables of Hecke algebras. This paper aims to generalize their argument on Hecke algebras to the adjacency algebra of association schemes.

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§1. Introduction

In the paper [3], the first author focused on characters of the factor scheme by a normal closed subset, so that all the irreducible characters of the factor scheme can be embedded into that of the original association scheme.

But this is not true for the factor scheme by a non-normal closed subset. In this paper, we consider characters of the factor scheme by a non-normal closed subset. The argument is very similar as the argument on Hecke algebras for finite permutation groups. Our argument is going almost parallel to [2, pp. 279 – 291].

Let K be an algebraically closed field of characteristic zero. Let G be an association scheme and H a closed subset of G. We define an idempotent e of the adjacency algebra KG. Then a K-algebra $eKGe$ is isomorphic to the adjacency algebra of the factor scheme $G/\!\!/H$. So we can consider $K(G/\!\!/H)$ is a subset of KG. Using this fact, we consider that the relation between irreducible characters of $K(G/\!\!/H)$ and KG. Namely, if χ is an irreducible character of KG, then the restriction of χ to $K(G/\!\!/H)$ is an irreducible character of $K(G/\!\!/H)$ if it is not zero. Conversely, every irreducible character of $K(G/\!\!/H)$ is obtained in this way.

§2. Notation and terminologies

Most of our notation and terminology stem from [6]. As a standard text to know concepts of association schemes we refer to [1] and [4]. Let (X, G) be an association scheme. We often say that G is an association scheme to simplify our notations. A non-empty subset H of G is called *closed* if $HH \subseteq H$, where the product is the *complex product*. We denote by σ_g the adjacency matrix of $g \in G$. By the definition of an association scheme, $\sigma_f\sigma_g = \sum_{h\in G} a_{fgh}\sigma_h$ for some non-negative integer a_{fgh}. We put $n_g = a_{gg^*1}$, where $g^* = \{(y,x) \mid (x,y) \in g\}$ and $1 = \{(x,x) \mid x \in X\}$. For a subset S of G, we put $\sigma_S = \sum_{g\in S}\sigma_g$ and $n_S = \sum_{g\in S} n_g$. The *adjacency algebra* KG of G over a field K is a matrix algebra generated by $\{\sigma_g \mid g \in G\}$. An algebra homomorphism from KG to the full matrix algebra $M_n(K)$ is called a *representation* of G over K, and the trace of it is called a *character* of G over K. We denote by $\mathrm{Irr}(KG)$ the set of irreducible characters of G over K.

We denote by I_n the identity matrix of degree n, and by J_n the $n \times n$ all-one matrix.

§3. Hecke algebras to association schemes

Throughout of this paper, we use the following notation. Let K be an algebraically closed field of characteristic zero. Let (X, G) be an association scheme, and H a closed subset of G. Then the adjacency algebra KG is semisimple by [6, Theorem 4.1.3]. We put $e = {n_H}^{-1}\sigma_H$. Then e is an idempotent of KG [3, Proposition 3.3]. Put $\mathcal{H} = eKGe$, then $\mathcal{H}$ is a K-algebra with the identity e.

Firstly, we prove that $\mathcal{H}$ is isomorphic to the adjacency algebra of the factor scheme $G/\!\!/H$. Then we consider the relation between irreducible characters of G and $G/\!\!/H$.

Lemma 3.1. *Let H be a closed subset of G. Then $\sigma_g\sigma_H = a_{gHg}\sigma_{gH}$*[1] *and $\sigma_H\sigma_g = a_{Hgg}\sigma_{Hg}$ for any $g \in G$.*

Proof. We have $\sigma_g\sigma_H = \sum_{h\in H}\sigma_g\sigma_h = \sum_{h\in H}\sum_{f\in G} a_{ghf}\sigma_f = \sum_{f\in G} a_{gHf}\sigma_f$. If $f \notin gH$, then $a_{gHf} = 0$. If $f \in gH$, then $a_{gHf} = a_{gHg}$ by [5, Lemma 4.3 (i)]. So we have $\sigma_g\sigma_H = a_{gHg}\sigma_{gH}$. Similarly $\sigma_H\sigma_g = a_{Hgg}\sigma_{Hg}$ holds. □

Lemma 3.2. *Let H be a closed subset of G. Then $\sigma_H\sigma_g\sigma_H$ is a scalar multiple of σ_{HgH}, and we may assume that $\sigma_{HgH} = \sigma_{g^H} \otimes J_{n_H}$ without loss of generality.*

[1] see [6] for the definition a_{DEg} where D, $E \subseteq G$ and $g \in G$.

Proof. The first assertion is a direct consequence of [5, Lemma 4.3 (i)]. The second assertion follows from the fact that

$$(\sigma_{HgH})_{x,y} = \begin{cases} 1, & \text{if } (xH, yH) \in g^H, \\ 0, & \text{otherwise,} \end{cases}$$

since $\{xH \mid x \in X\}$ is a partition of X. □

Lemma 3.3. *The left KG-modules KGe and $KG \otimes_{KH} Ke$ are isomorphic.*

Proof. We can define a KG-homomorphism $\Phi : KG \otimes_{KH} Ke \to KGe$ by $\Phi(\sigma_g \otimes e) = \sigma_g e$. This is clearly an epimorphism. By Lemma 3.1, KGe has a basis $\{\sigma_{gH} \mid g \in G\}$. On the other hand, $\sigma_g \otimes e = \sigma_g e \otimes e = n_H^{-1} \sigma_{gH} \otimes e$, so $KG \otimes_{KH} Ke$ is spanned by $\{\sigma_{gH} \otimes e \mid g \in G\}$. Thus we have $\dim_K KG \otimes_{KH} Ke \leq \dim_K KGe$ and Φ is an isomorphism. □

Proposition 3.4. *As K-algebras, $\mathcal{H} \cong K(G/\!\!/H)$.*

Proof. By Lemma 3.2, $\{\sigma_{HgH} \mid g \in G\}$ is a K-basis of $\mathcal{H}$ and we may assume that $\sigma_{HgH} = \sigma_{g^H} \otimes J_{n_H}$. Then it is easy to verify that the map $\sigma_{HgH} \mapsto n_H \sigma_{g^H}$ is an algebra isomorphism from $\mathcal{H}$ to $K(G/\!\!/H)$. □

We define the inner product of characters of KG as follows. For all χ, $\chi' \in \mathrm{Irr}(KG)$, we put $(\chi, \chi') = \delta_{\chi,\chi'}$ and for other characters it is linearly extended.

We shall denote by ${1_H}^G$ the character afforded by the KG-module $KG \otimes_{KH} Ke$.

Proposition 3.5. *For each $\xi \in \mathrm{Irr}(KG)$, we have $(\xi, {1_H}^G) = \xi(e) = \dim_K eM$ where M is a KG-module affording ξ.*

Proof. Let Φ be a matrix representation of KG defined by M. Then $\xi(e) = \operatorname{rank} \Phi(e) = \dim_K eM$, since e is an idempotent. By the semisimplicity of KG, we have $\dim_K eM = \dim_K \mathrm{Hom}_{KG}(KGe, M) = (\xi, {1_H}^G)$. □

Lemma 3.6. *An idempotent $u \in \mathcal{H}$ is primitive if and only if u is primitive in KG.*

Proof. For a semisimple K-algebra A, an idempotent $v \in A$ is primitive if and only if $vAv = Kv$. Since e is the identity of $\mathcal{H}$, $u\mathcal{H}u = ueKGeu = uKGu$, and the result follows. □

Lemma 3.7. *Let $\xi \in \mathrm{Irr}(KG)$. Then the restriction $\xi|_{\mathcal{H}} \neq 0$ if and only if $(\xi, {1_H}^G) \neq 0$.*

Proof. Let M be an irreducible KG-module affording ξ. If $\xi|_{\mathcal{H}} \neq 0$, then $\xi(eae) \neq 0$ for some $a \in KG$. This implies that $\dim_K eM \neq 0$. It follows from Lemma 3.5 that $(\xi, {1_H}^G) \neq 0$. Conversely, if $(\xi, {1_H}^G) \neq 0$, then, by Lemma 3.5 $\xi(e) \neq 0$, implying $\xi|_{\mathcal{H}} \neq 0$. □

The next is the main result in this paper.

Theorem 3.8. *There is a one-to-one correspondence between* $\{\xi \in \mathrm{Irr}(KG) \mid \xi|_{\mathcal{H}} \neq 0\}$ *and* $\mathrm{Irr}(\mathcal{H})$ *by the map* $\xi \mapsto \xi|_{\mathcal{H}}$.

Proof. We put $\mathrm{Irr}(KG) = \{\chi_1, \chi_2, \cdots, \chi_\ell\}$, and $d_i = \chi_i(1)$. Then, by the semisimplicity of KG, we have an isomorphism $\Phi : KG \to \bigoplus_{i=1}^{\ell} M_{d_i}(K)$. We consider the decomposition of e in this direct sum, $e = e_1 + \cdots + e_\ell$, where $\Phi(e_i) \in M_{d_i}(K)$. Without loss of generality, we may assume that $\Phi(e_i)$ is the diagonal matrix with the first r_i diagonal entries are 1 and 0 otherwise, where $r_i = \chi_i(e)$. Then we have $\mathcal{H} = eKGe \cong \bigoplus_{i=1}^{\ell} M_{r_i}(K)$. Since $\chi_i|_{\mathcal{H}} \neq 0$ if and only if $\chi_i(e) = r_i \neq 0$, the result follows. □

Corollary 3.9. *Let* $\{\varepsilon_i | i = 1, 2, \ldots, l\}$ *be the set of central primitive idempotents of* KG. *Then* $\{e\varepsilon_i | i = 1, 2, \ldots, l\} - \{0\}$ *is the set of central primitive idempotents of* $\mathcal{H}$.

Proof. This is an immediate consequence of the proof of Theorem 3.8. □

We consider the representation of KG which sends σ_g to itself. We call this the *standard representation* of G. Its character is called the *standard character* of G and denoted by γ_G. Obviously $\gamma_G(\sigma_1) = n_G$ and $\gamma_G(\sigma_g) = 0$ for $1 \neq g \in G$. We consider the irreducible decomposition of γ_G :

$$\gamma_G = \sum_{\xi \in \mathrm{Irr}(KG)} m_\xi \xi,$$

and we call m_ξ the *multiplicity* of ξ. The multiplicity plays an important role in the theory of association schemes.

Theorem 3.10. *The multiplicity of* $\xi|_{\mathcal{H}}$ *is equal to that of* ξ *if* $\xi|_{\mathcal{H}} \neq 0$.

Proof. For each $x \in \mathcal{H}$, $\gamma_G(x)$ is the standard character of $K(G/\!\!/H)$ since $\gamma_G(e) = n_G/n_H$ and $\gamma_G(n_H^{-1}\sigma_{HgH}) = 0$ for each $g \in G - H$. Let $\{\varepsilon_\xi \mid \xi \in \mathrm{Irr}(KG)\}$ be the set of the central primitive idempotents of KG. Note that $\gamma_G(\varepsilon_\xi e) = m_\xi \xi(\varepsilon_\xi e)$ and $\gamma_G(\varepsilon_\xi e) = \gamma_{G/\!\!/H}(\varepsilon_\xi e) = m_{\xi|_{\mathcal{H}}} \xi|_{\mathcal{H}}(\varepsilon_\xi e)$. It follows that $m_\xi = m_{\xi|_{\mathcal{H}}}$. □

Theorem 3.11. *Let* $\xi \in \mathrm{Irr}(KG)$ *with* $\xi|_{\mathcal{H}} \neq 0$. *Then* m_ξ *divides* $(n_G/n_H)\mathrm{lcm}\{n_{g^H} \mid g \in G\}$.

Proof. Let ε be the central primitive idempotent of $\mathcal{H}$ corresponding to $\xi|_{\mathcal{H}}$. Then by [6, Lemma 4.1.4], we have

$$\varepsilon = \frac{m_{\xi_{\mathcal{H}}}}{n_{G/\!\!/H}} \sum_{g^H \in G/\!\!/H} \frac{1}{n_{g^H}} \xi \left(\frac{1}{n_H} \sigma_{Hg^*H} \right) \frac{1}{n_H} \sigma_{HgH}.$$

We set $L = \mathrm{lcm}\{n_{g^H} \mid g \in G\}$ and set

$$w = L \sum_{g^H \in G/\!\!/H} \frac{1}{n_{g^H}} \xi \left(\frac{1}{n_H} \sigma_{Hg^*H} \right) \frac{1}{n_H} \sigma_{HgH}.$$

Since w is a scalar multiple of ε, w is central. Therefore, $\xi(w) = \alpha\xi(\varepsilon)$ where α is an algebraic integer since $\xi(n_H^{-1}\sigma_{HgH}) = \xi(\sigma_{g^H})$ is an algebraic integer for each $g \in G$. On the other hand, since $w = Ln_G n_H^{-1} m_\xi^{-1}\varepsilon$, $\xi(w) = Ln_G n_H^{-1} m_\xi^{-1}\xi(\varepsilon)$. Therefore, $Ln_G n_H^{-1} m_\xi^{-1} = \alpha$ is an algebraic integer and rational, implying that m_ξ divides $Ln_G n_H^{-1}$. □

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