# Square Laplacian perturbed by inverse fourth-power potential II. Holomorphic family of type (A) (complex case) 

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(Received September 9, 2010; Revised November 19, 2010)


#### Abstract

It is proved that $\left\{\Delta^{2}+\kappa|x|^{-4} ; \kappa \in \Sigma^{\mathrm{c}}\right\}$ in $L^{2}\left(\mathbb{R}^{N}\right)$ forms a holomorphic family of type (A), where $\Sigma$ is a closed and convex subset of $\mathbb{C}$. In particular, the $m$-accretivity of $\Delta^{2}+\kappa|x|^{-4}$ in $L^{2}\left(\mathbb{R}^{N}\right)$ is established as an application of the perturbation theorem for linear $m$-accretive operators. The key lies in two inequalities derived by positive semi-definiteness of Gram matrix.


AMS 2010 Mathematics Subject Classification. Primary 47B44, Secondary 35G05.

Key words and phrases. Square Laplacian, inverse fourth-power potential, holomorphic family of type (A), $m$-accretive operators.

## §1. Introduction

Let $A:=\Delta^{2}$ with $D(A):=H^{4}\left(\mathbb{R}^{N}\right)$ and $B:=|x|^{-4}$ with $D(B):=D\left(|x|^{-4}\right)=$ $\left\{u \in L^{2}\left(\mathbb{R}^{N}\right) ;|x|^{-4} u \in L^{2}\left(\mathbb{R}^{N}\right)\right\}(N \in \mathbb{N})$, where $\Delta:=\sum_{j=1}^{N}\left(\partial^{2} / \partial x_{j}^{2}\right)$ is a usual Laplacian in $\mathbb{R}^{N}$. This paper is concerned with parameter dependence of the operator sum $A+\kappa B(\kappa \in \mathbb{C})$ in the complex Hilbert space $L^{2}\left(\mathbb{R}^{N}\right)$ :

$$
(A+\kappa B) u:=\Delta^{2} u+\frac{\kappa}{|x|^{4}} u, \quad u \in D(A) \cap D(B)=H^{4}\left(\mathbb{R}^{N}\right) \cap D\left(|x|^{-4}\right) .
$$

In the previous paper [9] Okazawa, Tamura and Yokota have discussed the selfadjointness of $A+\kappa B$ when " $\kappa \in \mathbb{R}$ " in the (complex) Hilbert space $L^{2}\left(\mathbb{R}^{N}\right)$ $(N \in \mathbb{N})$. Namely, it is proved in [9] that $A+\kappa B$ is nonnegative selfadjoint on $D(A) \cap D(B)$ for $\kappa>\kappa_{0}$, where

$$
\kappa_{0}=\kappa_{0}(N):= \begin{cases}k_{1} & N \leq 8 \\ k_{2} & N \geq 9\end{cases}
$$

and $k_{1}, k_{2}$ will be given in Theorem 1.1. In addition we can assert that $A+\kappa_{0} B$ is nonnegative and essentially selfadjoint in $L^{2}\left(\mathbb{R}^{N}\right)$. As a continuation of [9] this paper concerns the $m$-accretivity and the resolvent set of $A+\kappa B$ when " $\kappa \in \mathbb{C}$ ". First we want to find $\Sigma \subset \mathbb{C}$ such that $\left\{A+\kappa B ; \kappa \in \Sigma^{c}\right\}$ is a holomorphic family of type (A) in the sense of Kato [5, Chapter VII]. Next we consider the $m$-accretivity of $A+\kappa B$ for $\kappa$ in the subset $\Sigma^{c}$.

Now we review the notion of holomorphic family in a simple case (the definition of $m$-accretivity will be given in Section 2).

Definition 1. Let $X$ be a reflexive complex Banach space. Let $\Omega$ be a domain in $\mathbb{C}$ and $\{T(\kappa) ; \kappa \in \Omega\}$ a family of linear operators in $X$. Then $\{T(\kappa) ; \kappa \in \Omega\}$ is said to be a holomorphic family of type (A) in $X$ if
(i) $T(\kappa)$ is closed in $X$ and $D(T(\kappa))=D$ independent of $\kappa$;
(ii) $\kappa \mapsto T(\kappa) u$ is holomorphic in $\Omega$ for every $u \in D$.

Kato [6] proved that $\left\{-\Delta+\kappa|x|^{-2} ; \kappa \in \Omega_{1}\right\}$ forms a holomorphic family of type (A) in $L^{2}\left(\mathbb{R}^{N}\right)$, where $\beta:=1-(N-2)^{2} / 4=-N(N-4) / 4$ and

$$
\Omega_{1}:=\left\{\xi+i \eta \in \mathbb{C} ; \eta^{2}>4(\beta-\xi)\right\}=\left\{\xi+i \eta \in \mathbb{C} ; \xi>\gamma(\eta):=\beta-\eta^{2} / 4\right\} .
$$

Borisov-Okazawa [1] proved that $\left\{d / d x+\kappa x^{-1} ; \kappa \in \Omega_{2}\right\}$ forms a holomorphic family of type (A) in $L^{p}(0, \infty)(1<p<\infty)$, where

$$
\Omega_{2}:=\left\{\kappa \in \mathbb{C} ; \operatorname{Re} \kappa>-p^{\prime-1}\right\}, \quad p^{-1}+p^{\prime-1}=1 .
$$

Concerning fourth order elliptic operators, there seems to be no preceding work on holomorphic family of type (A). So we try to clarify the regions where $A+\kappa B$ forms a holomorphic family of type (A) and where $A+\kappa B$ is $m$-accretive.

Our result is stated as follows.
Theorem 1.1. Set $A:=\Delta^{2}, B:=|x|^{-4}$. Let $k_{1}=k_{1}(N)(N \in \mathbb{N})$ be the constant defined as

$$
\begin{equation*}
k_{1}:=112-3(N-2)^{2} . \tag{1.1}
\end{equation*}
$$

Let $\Sigma$ be the closed convex subset of $\mathbb{C}$ defined as
$\Sigma:=\left\{\xi+i \eta \in \mathbb{C} ; \xi \leq k_{1}, \eta^{2} \leq 64\left[\sqrt{k_{1}-\xi}+\left(10+N-\frac{N^{2}}{4}\right)\right]\left(\sqrt{k_{1}-\xi}+8\right)^{2}\right\}$.
Then the following (i)-(iii) hold.
(i) $B$ is $(A+\kappa B)$-bounded for $\kappa \in \Sigma^{\mathrm{c}}$, with

$$
\|B u\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1}\|(A+\kappa B) u\|, \quad u \in D(A) \cap D(B),
$$

and hence $\left\{A+\kappa B ; \kappa \in \Sigma^{c}\right\}$ forms a holomorphic family of type (A) in $L^{2}\left(\mathbb{R}^{N}\right)$. In particular, if $N \geq 9$ then $B$ is $A$-bounded, with

$$
\|B u\| \leq\left|k_{2}\right|^{-1}\|A u\|, \quad u \in D(A) \subset D(B)
$$

where $k_{2}=k_{2}(N)(N \geq 9)$ is the negative constant defined as

$$
\begin{equation*}
k_{2}:=k_{1}-\left[\left(\frac{N-2}{2}\right)^{2}-11\right]^{2}=-\frac{N}{16}(N-8)\left(N^{2}-16\right) \tag{1.2}
\end{equation*}
$$

In addition, $\Sigma$ can be expressed in terms of $k_{2}$ :

$$
\Sigma=\left\{\xi+i \eta \in \mathbb{C} ; \xi \leq k_{2}, \eta^{2} \leq \frac{64\left(k_{2}-\xi\right)\left(\sqrt{k_{1}-\xi}+8\right)^{2}}{\sqrt{k_{1}-\xi}+\left(N^{2} / 4-N-10\right)}\right\}
$$

(ii) $A+\kappa B$ is m-accretive on $D(A) \cap D(B)$ for $\kappa \in \Sigma^{c}$ with $\operatorname{Re} \kappa \geq-\alpha_{0}$ and $A+\kappa B$ is essentially m-accretive in $L^{2}\left(\mathbb{R}^{N}\right)$ for $\kappa \in \partial \Sigma$ with $\operatorname{Re} \kappa \geq-\alpha_{0}$, where $\alpha_{0}$ is defined as

$$
\alpha_{0}=\alpha_{0}(N):= \begin{cases}0, & N \leq 4  \tag{1.3}\\ {\left[\frac{N(N-4)}{4}\right]^{2},} & N \geq 5\end{cases}
$$

In particular, if $\kappa \in \mathbb{R}$, then $m$-accretivity is replaced with nonnegative selfadjointness.
(iii) Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa<-\alpha_{0}$. Let $c_{\alpha_{0}}(\kappa)$ and $\theta_{\alpha_{0}}$ be defined as

$$
\begin{aligned}
c_{\alpha_{0}}(\kappa) & := \begin{cases}\min \left\{\frac{\left|-\alpha_{0}+i \eta-\kappa\right|}{\operatorname{dist}\left(-\alpha_{0}+i \eta, \Sigma\right)} ; \eta_{0}<\eta<\infty\right\}, & \operatorname{Im} \kappa>0 \\
\min \left\{\frac{\left|-\alpha_{0}+i \eta-\bar{\kappa}\right|}{\operatorname{dist}\left(-\alpha_{0}+i \eta, \Sigma\right)} ; \eta_{0}<\eta<\infty\right\}, & \operatorname{Im} \kappa<0\end{cases} \\
\theta_{\alpha_{0}} & :=\tan ^{-1}\left(\frac{1-c_{\alpha_{0}}(\kappa)}{\sqrt{c_{\alpha_{0}}(\kappa)\left(2-c_{\alpha_{0}}(\kappa)\right)}}\right),
\end{aligned}
$$

where $\eta_{0}:=\max \left\{\eta \geq 0 ;-\alpha_{0}+i \eta \in \Sigma\right\}$. Then $c_{\alpha_{0}}(\kappa) \in(0,1)$ and $\theta_{\alpha_{0}} \in$ ( $0, \pi / 2$ ).
(a) If $\operatorname{Im} \kappa>0$, then the resolvent set $\rho(-(A+\kappa B))$ contains the sector $S_{+}(\kappa)$, where

$$
S_{+}(\kappa):=\left\{\lambda \in \mathbb{C} ;-\theta_{\alpha_{0}}<\arg \lambda<\pi / 2\right\} .
$$

(b) If $\operatorname{Im} \kappa<0$, then the resolvent set $\rho(-(A+\kappa B))$ contains the sector $S_{-}(\kappa)$, where

$$
S_{-}(\kappa):=\left\{\lambda \in \mathbb{C} ;-\pi / 2<\arg \lambda<\theta_{\alpha_{0}}\right\} .
$$

Remark 1.1. When $N \geq 5, \alpha_{0}$ in (1.3) appears in the Rellich inequality (cf. Davies-Hinz [3, Corollary 14], Okazawa [8, Lemma 3.8], [9, Lemma 3.2]).

Remark 1.2. Theorem 1.1 (iii) (and also Theorem 2.1 (iii), Theorem 2.7 (vi)) can be improved. Actually, the referee ${ }^{1}$ informed us that $\theta_{\alpha_{0}}$ in Theorem 1.1 can be replaced with

$$
\tan ^{-1}\left(\frac{\sqrt{1-c_{\alpha_{0}}(\kappa)^{2}}}{c_{\alpha_{0}}(\kappa)}\right)
$$



$$
N=5
$$



Figure 1: The images of $\Sigma$ for $N=4,5,8,9$ and the value of $-\alpha_{0}$
In Section 2 we propose abstract theorems based on Kato [6]. However, the assumption and conclusions are slightly changed. In the proof of Theorem 1.1 we need some generalized forms of the inequalities obtained in [9]. Section 3 starts with their proofs depending on the positive semi-definiteness of Gram matrix. At the end of Section 3 we complete the proof of Theorem 1.1 by applying abstract theorems prepared in Section 2.

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## §2. Abstract theory toward Theorem 1.1

First we review some definitions required to state Theorems 2.1 and 2.7. Let $A$ be a linear operator with domain $D(A)$ and range $R(A)$ in a (complex) Hilbert space $H$. Then $A$ is said to be accretive if $\operatorname{Re}(A u, u) \geq 0$ for every $u \in D(A)$. An accretive operator $A$ is said to be $m$-accretive if $R(A+1)=H$.

Let $A$ be $m$-accretive in $H$. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda>0, R(A+$ $\lambda)=H$ holds with

$$
\left\|(A+\lambda)^{-1}\right\| \leq(\operatorname{Re} \lambda)^{-1}
$$

Therefore we can define the Yosida approximation $\left\{A_{\varepsilon} ; \varepsilon>0\right\}$ of $A$ :

$$
A_{\varepsilon}:=A(1+\varepsilon A)^{-1}, \quad \varepsilon>0
$$

A nonnegative selfadjoint operator is a typical example of $m$-accretive operator, while a symmetric $m$-accretive operator is nonnegative and selfadjoint (see Brézis [2, Proposition VII.6] or Kato [5, Problem V.3.32]).

Next we consider the $m$-accretivity of $A+\kappa B(\kappa \in \mathbb{C})$ where $A$ and $B$ are nonnegative selfadjoint operators in $H$. Since $m$-accretive operators are closed and densely defined, we will first find $\Omega \subset \mathbb{C}$ where $\{A+\kappa B ; \kappa \in \Omega\}$ forms a holomorphic family of type (A). Next we will find a set of $\kappa \in \Omega$ where $A+\kappa B$ is $m$-accretive. We also consider the resolvent set of $A+\kappa B$ for each $\kappa \in \Omega$.

Theorem 2.1. Let $A$ and $B$ be nonnegative selfadjoint operators in $H$. Let $\Sigma \subset \mathbb{C}$, and $\gamma: \mathbb{R} \rightarrow \mathbb{R}$. Assume that $\Sigma$ and $\gamma$ satisfy $(\gamma \mathbf{1})-(\gamma \mathbf{4})$ and $(\gamma \mathbf{5})_{0}$ :
$(\gamma \mathbf{1}) \gamma$ is continuous and concave,
$(\gamma \mathbf{2}) \gamma(\eta)=\gamma(-\eta)$ for $\eta \in \mathbb{R}$,
$(\gamma \mathbf{3}) \Sigma=\{\xi+i \eta \in \mathbb{C} ; \xi \leq \gamma(\eta)\}$,
$(\boldsymbol{\gamma} \mathbf{)})-\left(A u, B_{\varepsilon} u\right) \in \Sigma$ for $u \in D(A)$ with $\left\|B_{\varepsilon} u\right\|=\left\|B(1+\varepsilon B)^{-1} u\right\|=1$ for any $\varepsilon>0$,
$(\gamma 5)_{0} 0 \leq \gamma(0)(\Leftrightarrow 0 \in \Sigma)$.
Then the following (i)-(iii) hold.
(i) $B$ is $(A+\kappa B)$-bounded for $\kappa \in \Sigma^{\mathrm{c}}$, with

$$
\begin{equation*}
\|B u\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1}\|(A+\kappa B) u\|, \quad u \in D(A) \cap D(B) \tag{2.1}
\end{equation*}
$$

and $\left\{A+\kappa B ; \kappa \in \Sigma^{c}\right\}$ forms a holomorphic family of type (A).
(ii) $A+\kappa B$ is m-accretive on $D(A) \cap D(B)$ for $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa \geq 0$ and $A+\kappa B$ is essentially $m$-accretive in $H$ for $\kappa \in \partial \Sigma$ with $\operatorname{Re} \kappa \geq 0$.
(iii) Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa<0$. Let $c_{0}(\kappa)$ and $\theta_{0}$ be defined as

$$
\begin{align*}
c_{0}(\kappa) & := \begin{cases}\min \left\{\frac{|i \eta-\kappa|}{\operatorname{dist}(i \eta, \Sigma)} ; \eta_{0}<\eta<\infty\right\}, & \operatorname{Im} \kappa>0, \\
\min \left\{\frac{|i \eta-\bar{\kappa}|}{\operatorname{dist}(i \eta, \Sigma)} ; \eta_{0}<\eta<\infty\right\}, & \operatorname{Im} \kappa<0,\end{cases}  \tag{2.2}\\
\theta_{0} & :=\tan ^{-1}\left(\frac{1-c_{0}(\kappa)}{\sqrt{c_{0}(\kappa)\left(2-c_{0}(\kappa)\right)}}\right), \tag{2.3}
\end{align*}
$$

where $\eta_{0}:=\max \{\eta \geq 0 ;$ i $\eta \in \Sigma\}$. Then $c_{0}(\kappa) \in(0,1)$ and $\theta_{0} \in(0, \pi / 2)$, and the resolvent set is described by $\theta_{0}$ as follows.
(a) If $\operatorname{Im} \kappa>0$, then the resolvent set $\rho(-(A+\kappa B))$ contains the sector $S_{+}(\kappa)$, where

$$
S_{+}(\kappa):=\left\{\mu \in \mathbb{C} ;-\theta_{0}<\arg \mu<\pi / 2\right\} .
$$

(b) If $\operatorname{Im} \kappa<0$, then the resolvent set $\rho(-(A+\kappa B))$ contains the sector $S_{-}(\kappa)$, where

$$
S_{-}(\kappa):=\left\{\mu \in \mathbb{C} ;-\pi / 2<\arg \mu<\theta_{0}\right\} .
$$

Remark 2.1. Let $A$ and $B$ be as in Theorem 2.1 with $\gamma(0) \geq 0$. Consider the closed interval $(-\infty, \gamma(0)]$ as a subset of $\Sigma \cap \mathbb{R}$ (instead of $\Sigma \subset \mathbb{C}$ itself). Then it is proved in [8, Theorem 1.6] that $B$ is $(A+t B)$-bounded for $t>\gamma(0)$ (that is, $t \in(-\infty, \gamma(0)]^{c}$ ), with

$$
\|B u\| \leq(t-\gamma(0))^{-1}\|(A+t B) u\|, \quad u \in D(A) \cap D(B),
$$

and $A+t B$ is selfadjoint on $D(A) \cap D(B)$ for $t>\gamma(0)$; in particular, if $\gamma(0)>0$, then $A+\gamma(0) B$ is essentially selfadjoint in $H$. These facts are regarded as a restriction of Theorem 2.1 (i) and (ii) to the subset $\Sigma^{\mathrm{c}} \cap \mathbb{R}$.

As stated above Theorem 2.1 is proved along the idea in the proof of $[8$, Theorem 1.6]. We shall divide the proof into several lemmas.

Lemma 2.2. The assertion (i) of Theorem 2.1 holds.
Proof. Let $\kappa \in \Sigma^{\mathrm{c}}$ and $\varepsilon>0$. To prove (2.1) we shall show that

$$
\begin{equation*}
\left\|B_{\varepsilon} u\right\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1}\left\|\left(A+\kappa B_{\varepsilon}\right) u\right\|, \quad u \in D(A) . \tag{2.4}
\end{equation*}
$$

Here we may assume that $B_{\varepsilon} u=B(1+\varepsilon B)^{-1} u \neq 0$ for $u \in D(A)$. Setting $v:=\left\|B_{\varepsilon} u\right\|^{-1} u$, we have $v \in D(A)$ and $\left\|B_{\varepsilon} v\right\|=1$. It then follows from ( $\gamma 4$ ) that

$$
-\left(A v, B_{\varepsilon} v\right) \in \Sigma
$$

Since $\Sigma$ is closed and convex by ( $\gamma \mathbf{1}$ ), we have

$$
0<\operatorname{dist}(\kappa, \Sigma) \leq\left|\kappa+\left(A v, B_{\varepsilon} v\right)\right|=\frac{\left.\mid\left(A+\kappa B_{\varepsilon}\right) u, B_{\varepsilon} u\right) \mid}{\left\|B_{\varepsilon} u\right\|^{2}}
$$

and hence $\left\|B_{\varepsilon} u\right\|^{2} \leq \operatorname{dist}(\kappa, \Sigma)^{-1}\left|\left(\left(A+\kappa B_{\varepsilon}\right) u, B_{\varepsilon} u\right)\right|$. Applying the CauchySchwarz inequality, we have (2.4). Letting $\varepsilon \downarrow 0$ in (2.4) with $u \in D(A) \cap D(B)$ we obtain (2.1). The closedness of $A+\kappa B$ is a consequence of (2.1). This completes the proof of assertion (i) in Theorem 2.1.

Lemma 2.3. $A+\kappa B$ is $m$-accretive in $H$ for $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa \geq 0$.
Proof. Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa \geq 0$. Then it remains to show that

$$
\begin{equation*}
R(A+\kappa B+1)=H . \tag{2.5}
\end{equation*}
$$

Since $A+\kappa B_{\varepsilon}$ is also $m$-accretive (see Pazy [10, Corollary 3.3.3]), for $f \in H$ and $\varepsilon>0$ there exists a unique solution $u_{\varepsilon} \in D(A)$ of the approximate equation

$$
\begin{equation*}
A u_{\varepsilon}+\kappa B_{\varepsilon} u_{\varepsilon}+u_{\varepsilon}=f \tag{2.6}
\end{equation*}
$$

satisfying $\left\|u_{\varepsilon}\right\| \leq\|f\|$ and hence $\left\|\left(A+\kappa B_{\varepsilon}\right) u_{\varepsilon}\right\|=\left\|f-u_{\varepsilon}\right\| \leq 2\|f\|$. Therefore we see from (2.4) that

$$
\left\|B_{\varepsilon} u_{\varepsilon}\right\| \leq 2 \operatorname{dist}(\kappa, \Sigma)^{-1}\|f\| .
$$

This implies that $\left\|B_{\varepsilon}\left(A+\kappa B_{\varepsilon}+1\right)^{-1}\right\|$ is bounded. Thus we obtain (2.5) (see [7, Proposition 2.2] or [4, Exercise 6.12.7 Chapter 1]).

Lemma 2.4. The closure of $A+\kappa B$ (denoted by $\left.(A+\kappa B)^{\Upsilon}\right)$ is m-accretive in $H$ for $\kappa \in \partial \Sigma$ with $\operatorname{Re} \kappa \geq 0$.

Proof. Let $\kappa \in \partial \Sigma$ with Re $\kappa \geq 0$. First we note that $A+\kappa B$ is closable and its closure is also accretive (cf. [10, Theorem 1.4.5]). Now ( $\gamma \mathbf{1}$ ) means that there exists $\nu \in \mathbb{C}$ satisfying $|\nu|=1$ and

$$
\begin{equation*}
\operatorname{Re}[\nu(\overline{z-\kappa})] \leq 0 \quad \forall z \in \Sigma \tag{2.7}
\end{equation*}
$$

(if $\partial \Sigma$ is smooth at a neighborhood of $\kappa$, then $\nu$ is uniquely defined as a unit outward normal vector of $\partial \Sigma$ at $\kappa$ ). (2.7) implies that the function $\zeta \in \Sigma \mapsto$ $|(\kappa+\nu)-\zeta|$ attains to its minimum at $\zeta=\kappa$ (cf. [2, Theorem V.2]). We can show for every $t>0$ that

$$
\begin{align*}
& \operatorname{Re}(\kappa+t \nu) \geq 0,  \tag{2.8}\\
& \operatorname{dist}(\kappa+t \nu, \Sigma)=t . \tag{2.9}
\end{align*}
$$

In fact, $(\gamma \mathbf{3})$ and $\kappa \in \partial \Sigma$ implies $\kappa-1 \in \Sigma$. Setting $z=\kappa-1$ in (2.7), we have $\operatorname{Re} \nu \geq 0$ and (2.8). (2.9) is a consequence of (2.7) multiplied by $t>0$. (2.8) implies that $A+(\kappa+(\nu / n)) B$ is $m$-accretive for each $n \in \mathbb{N}$ (see Lemma 2.3), that is, for every $f \in H$ there is a unique solution $u_{n} \in D(A) \cap D(B)$ of

$$
\begin{equation*}
A u_{n}+(\kappa+(\nu / n)) B u_{n}+u_{n}=f, \tag{2.10}
\end{equation*}
$$

satisfying

$$
\begin{equation*}
\left\|u_{n}\right\| \leq\|f\| . \tag{2.11}
\end{equation*}
$$

Now we can prove that $\left\|(\nu / n) B u_{n}\right\|=n^{-1}\left\|B u_{n}\right\| \leq 2\|f\|$. In fact, we see from (2.1) that

$$
\begin{aligned}
\left\|B u_{n}\right\| & \leq \operatorname{dist}(\kappa+\nu / n, \Sigma)^{-1}\left\|(A+(\kappa+\nu / n) B) u_{n}\right\|=n\left\|f-u_{n}\right\| \\
& \leq 2 n\|f\| .
\end{aligned}
$$

This yields together with (2.10) that $\left\|(A+\kappa B) u_{n}\right\| \leq 4\|f\|$. To finish the proof we show that $(\nu / n) B u_{n}$ converges to zero weakly in $H$. It follows from (2.11) that for every $v \in D(B)$,

$$
\left|\left((\nu / n) B u_{n}, v\right)\right|=n^{-1}\left|\left(u_{n}, B v\right)\right| \leq n^{-1}\|f\| \cdot\|B v\| \rightarrow 0 \quad(n \rightarrow \infty) .
$$

Since $D(B)$ is dense in $H$ and $n^{-1}\left\|B u_{n}\right\|$ is bounded, we see that $n^{-1} B u_{n} \rightarrow 0$ $(n \rightarrow \infty)$ weakly. (2.11) implies that we can choose a subsequence $\left\{u_{n_{k}}\right\} \subset$ $\left\{u_{n}\right\}$ such that $u:=\mathrm{w}-\lim _{k \rightarrow \infty} u_{n_{k}}$ exists. Then we have

$$
\begin{aligned}
(A+\kappa B) u_{n_{k}} & =f-u_{n_{k}}-\left(\nu / n_{k}\right) B u_{n_{k}} \\
& \rightarrow f-u(k \rightarrow \infty) \text { weakly. }
\end{aligned}
$$

It follows from the (weak) closedness of $(A+\kappa B)^{\sim}$ that $u \in D\left((A+\kappa B)^{\sim}\right)$ and $(A+\kappa B)^{\sim} u=f-u$. This proves the essential $m$-accretivity of $A+\kappa B$ for $\kappa \in \partial \Sigma$ with $\operatorname{Re} \kappa \geq 0$.

Lemma 2.5. Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa<0$. Let $c_{0}(\kappa)$ be defined in (2.2).
(a) If $\operatorname{Im} \kappa>0$, then $\rho(-(A+\kappa B))$ contains the sector $\{\lambda \in \mathbb{C} ; 0 \leq \arg \lambda<$ $\pi / 2\}$, with

$$
\begin{equation*}
\left\|(A+\kappa B+\lambda)^{-1}\right\| \leq\left[1-c_{0}(\kappa)\right]^{-1}(\operatorname{Re} \lambda)^{-1}, \quad \operatorname{Re} \lambda>0, \operatorname{Im} \lambda \geq 0 \tag{2.12}
\end{equation*}
$$

(b) If $\operatorname{Im} \kappa<0$, then $\rho(-(A+\kappa B))$ contains the sector $\{\lambda \in \mathbb{C} ;-\pi / 2<$ $\arg \lambda \leq 0\}$, with

$$
\begin{equation*}
\left\|(A+\kappa B+\lambda)^{-1}\right\| \leq\left[1-c_{0}(\kappa)\right]^{-1}(\operatorname{Re} \lambda)^{-1}, \quad \operatorname{Re} \lambda>0, \operatorname{Im} \lambda \leq 0 . \tag{2.13}
\end{equation*}
$$

Proof. Let $\kappa \in \Sigma^{c}$ with Re $\kappa<0$. Since $\Sigma$ is symmetric with respect to the real axis by $(\boldsymbol{\gamma} \mathbf{2})$, it suffices to prove the assertion (a).
(a) Let $\operatorname{Im} \kappa>0$. Then we shall show that $\lambda \in \rho(-(A+\kappa B))$ for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda>0$ and $\operatorname{Im} \lambda \geq 0$. This is equivalent to the unique solvability of the equation for each $f \in H$

$$
\begin{equation*}
A u+\kappa B u+\lambda u=f \tag{2.14}
\end{equation*}
$$

Let $\zeta \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \zeta=0$ and $\operatorname{Im} \zeta>0$. Then $A+\zeta B$ is $m$-accretive in $H$ (see Lemma 2.3). Setting $K:=(\zeta-\kappa) B(A+\zeta B+\lambda)^{-1}$, (2.14) can be written as

$$
\begin{equation*}
(1-K)(A+\zeta B+\lambda) u=f \tag{2.15}
\end{equation*}
$$

Thus it remains to show the unique solvability of the equation $(1-K) v=f$, since $A+\zeta B+\lambda$ is invertible. To do so it suffices to show that

$$
\begin{equation*}
\|K\|=|\zeta-\kappa| \cdot\left\|B(A+\zeta B+\lambda)^{-1}\right\|<1 \tag{2.16}
\end{equation*}
$$

Now let $\kappa \in \Sigma^{\mathrm{c}}$ (with $\operatorname{Re} \kappa<0$ and $\operatorname{Im} \kappa>0$ ) satisfy $|\zeta-\kappa|<\operatorname{dist}(\zeta, \Sigma)$ (see Figure 2); in this connection note that if $\operatorname{Im} \zeta<0$ then we have $|\zeta-\kappa|>$ $\operatorname{dist}(\zeta, \Sigma)$.


Figure 2: $|\zeta-\kappa|<\operatorname{dist}(\zeta, \Sigma)$
Then we can solve (2.15). It follows from (2.1) that

$$
\begin{equation*}
\|B u\| \leq \operatorname{dist}(\zeta, \Sigma)^{-1}\|(A+\zeta B) u\| \tag{2.17}
\end{equation*}
$$

On the other hand, we can show that

$$
\begin{equation*}
\|(A+\zeta B) u\| \leq\|v\| \tag{2.18}
\end{equation*}
$$

In fact, making the inner product of $(A+\zeta B+\lambda) u=v$ with $(A+\zeta B) u$ gives

$$
\|(A+\zeta B) u\|^{2}+(\operatorname{Re} \lambda)\left\|A^{1 / 2} u\right\|^{2}+\operatorname{Re}(\lambda \bar{\zeta})\left\|B^{1 / 2} u\right\|^{2}=\operatorname{Re}(v,(A+\zeta B) u) .
$$

Since $\operatorname{Re} \zeta=0$ and $\operatorname{Im} \zeta>0$, we have $\operatorname{Re}(\lambda \bar{\zeta})=(\operatorname{Im} \lambda)(\operatorname{Im} \zeta) \geq 0$. Hence applying the Cauchy-Schwarz inequality gives (2.18). Combining (2.17) with (2.18), we have

$$
\|B u\|=\left\|B(A+\zeta B+\lambda)^{-1} v\right\| \leq \operatorname{dist}(\zeta, \Sigma)^{-1}\|v\| .
$$

Therefore, since $|\zeta-\kappa|<\operatorname{dist}(\zeta, \Sigma)$, we obtain (2.16):

$$
\|K\| \leq|\zeta-\kappa| \operatorname{dist}(\zeta, \Sigma)^{-1}<1
$$

This completes the proof of $\lambda \in \rho(-(A+\kappa B))$ for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda>0$ and $\operatorname{Im} \lambda \geq 0$.

Now we prove the estimate (2.12). Since $\|v\|=\left\|(1-K)^{-1} f\right\| \leq(1-$ $\|K\|)^{-1}\|f\|$, it follows from (2.15) that

$$
\left\|(A+\kappa B+\lambda)^{-1} f\right\|=\left\|(A+\zeta B+\lambda)^{-1} v\right\| \leq \frac{(\operatorname{Re} \lambda)^{-1}\|f\|}{1-|\zeta-\kappa| \operatorname{dist}(\zeta, \Sigma)^{-1}}
$$

Here we note that the function $\varphi(\eta):=|i \eta-\kappa| \operatorname{dist}(i \eta, \Sigma)^{-1}$ is continuous on the open interval $\left(\eta_{0}, \infty\right)$, where $\eta_{0}:=\max \{\eta \geq 0 ; i \eta \in \Sigma\}$. We show that $\inf \left\{\varphi(\eta) ; \eta>\eta_{0}\right\}=\min \left\{\varphi(\eta) ; \eta>\eta_{0}\right\}<1$. Let $P: \mathbb{C} \rightarrow \Sigma$ be the projection. Let $\eta_{1} \in\left(\eta_{0}, \infty\right)$ satisfy that $P \kappa, \kappa$ and $i \eta_{1}$ are on the same line. Then we have $\inf \left\{\varphi(\eta) ; \eta>\eta_{0}\right\} \leq \varphi\left(\eta_{1}\right)<1$. On the other hand, we have for every $\eta>\eta_{0}$

$$
\begin{aligned}
\varphi(\eta) & =\frac{|i \eta-\kappa|}{\left|i \eta-i \eta_{0}\right|} \frac{\left|i \eta-i \eta_{0}\right|}{\operatorname{dist}(i \eta, \Sigma)} \\
& \geq \frac{|i \eta-\kappa|}{\left|i \eta-i \eta_{0}\right|},
\end{aligned}
$$

which implies

$$
\liminf _{\eta \rightarrow \infty} \varphi(\eta) \geq 1
$$

Thus we can find $\eta_{2} \geq \eta_{1}$ such that $\inf \left\{\varphi(\eta) ; \eta>\eta_{2}\right\} \geq \varphi\left(\eta_{1}\right)$. Therefore we obtain $\inf \left\{\varphi(\eta) ; \eta>\eta_{0}\right\}=\min \left\{\varphi(\eta) ; \eta>\eta_{0}\right\}$. Setting $c_{0}(\kappa):=\min \{\varphi(\eta) ; \eta>$ $\left.\eta_{0}\right\}$, we obtain (2.12).

Lemma 2.6. Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa<0$. Let $\theta_{0}$ be defined in (2.3). Then
(a) If $\operatorname{Im} \kappa>0$, then $\rho(-(A+\kappa B))$ contains $S_{+}(\kappa)=\left\{\lambda \in \mathbb{C} ;-\theta_{0}<\arg \lambda<\right.$ $\pi / 2\}$.
(b) If $\operatorname{Im} \kappa<0$, then $\rho(-(A+\kappa B))$ contains $S_{-}(\kappa)=\{\lambda \in \mathbb{C} ;-\pi / 2<$ $\left.\arg \lambda<\theta_{0}\right\}$.

Proof. We prove only (a) as in the proof of Lemma 2.5.
(a) Let $\operatorname{Im} \kappa>0$. Then it remains to prove that the sector $\left\{\lambda \in \mathbb{C} ;-\theta_{0}<\right.$ $\arg \lambda<0\}$ is contained in $\rho(-(A+\kappa B)$ ) (see Lemma 2.5 (a)). Let $\xi>0$. Then $\xi \in \rho(-(A+\kappa B))$, with $\left\|(A+\kappa B+\xi)^{-1}\right\| \leq\left[1-c_{0}(\kappa)\right]^{-1} \xi^{-1}[$ see $(2.12)]$. Now let $f \in H$. Then we want to solve the equation $A u+\kappa B u+\lambda u=f$, with $\operatorname{Re} \lambda>0$. Setting $K:=(\xi-\lambda)(A+\kappa B+\xi)^{-1}$, we have

$$
\begin{equation*}
(1-K)(A+\kappa B+\xi) u=f \tag{2.19}
\end{equation*}
$$

Noting that if $\operatorname{Im} \lambda>-(\operatorname{Re} \lambda) \tan \theta_{0}$, then there exists some $\xi>0$ such that $|\xi-\lambda|<\left[1-c_{0}(\kappa)\right] \xi$ (see Figure 2) and hence $\|K\| \leq|\xi-\lambda|\left[1-c_{0}(\kappa)\right]^{-1} \xi^{-1}<1$.


Figure 2: $\tan \theta_{0}=\left(1-c_{0}(\kappa)\right) / \sqrt{c_{0}(\kappa)\left(2-c_{0}(\kappa)\right)}$
Therefore $u:=(A+\kappa B+\xi)^{-1}(1-K)^{-1} f$ is a unique solution of (2.19), with

$$
\begin{aligned}
\|u\|=\left\|(A+\kappa B+\xi)^{-1} v\right\| & \leq\left[1-c_{0}(\kappa)\right]^{-1} \xi^{-1}\|v\| \\
& \leq \frac{\|f\|}{\left[1-c_{0}(\kappa)\right] \xi-|\xi-\lambda|}
\end{aligned}
$$

where we have used the inequality

$$
\|v\| \leq\left[1-|\xi-\lambda|\left[1-c_{0}(\kappa)\right]^{-1} \xi^{-1}\right]^{-1}\|f\|
$$

derived from (2.19). Therefore we can conclude that $\lambda \in \rho(-(A+\kappa B))$ for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda>0$ and $\operatorname{Im} \lambda>-(\operatorname{Re} \lambda) \tan \theta_{0}$.

Next, we state two particular cases of Theorem 2.1 in which $B^{1 / 2}$ is $A^{1 / 2}$ bounded or $B$ is $A$-bounded (under the condition $\gamma(0)<0$ ).

Theorem 2.7. Let $A, B, \Sigma$ and $\gamma$ be the same as those in Theorem 2.1 with $(\gamma \mathbf{1})-(\gamma \mathbf{4})$. Assume that there exists $\alpha_{0}>0$ such that

$$
\begin{equation*}
\alpha_{0}\left(B_{\varepsilon} u, u\right) \leq(A u, u), \quad u \in D(A) \tag{2.20}
\end{equation*}
$$

If $(\boldsymbol{\gamma} \mathbf{5})_{0}$ is replaced with
$(\gamma 5)_{\alpha_{0}}-\alpha_{0} \leq \gamma(0)$,
then, in addition to (i) of Theorem 2.1, the following (iv)-(vi) hold.
(iv) If $\gamma(0)<0\left(\Leftrightarrow 0 \in \Sigma^{\mathrm{c}}\right)$, then $B$ is $A$-bounded with

$$
\begin{equation*}
\|B u\| \leq|\gamma(0)|^{-1}\|A u\|, \quad u \in D(A) \subset D(B) \tag{2.21}
\end{equation*}
$$

(v) $A+\kappa B$ is m-accretive on $D(A) \cap D(B)$ for $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa \geq-\alpha_{0}$ and $A+\kappa B$ is essentially m-accretive in $H$ for $\kappa \in \partial \Sigma$ with $\operatorname{Re} \kappa \geq-\alpha_{0}$.
(vi) Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \kappa<-\alpha_{0}$. Let $c_{\alpha_{0}}(\kappa)$ and $\theta_{\alpha_{0}}$ be defined as

$$
\begin{aligned}
& c_{\alpha_{0}}(\kappa):= \begin{cases}\min \left\{\frac{\left|-\alpha_{0}+i \eta-\kappa\right|}{\operatorname{dist}\left(-\alpha_{0}+i \eta, \Sigma\right)} ; \eta_{0}<\eta<\infty\right\}, & \operatorname{Im} \kappa>0, \\
\min \left\{\frac{\left|-\alpha_{0}+i \eta-\bar{\kappa}\right|}{\operatorname{dist}\left(-\alpha_{0}+i \eta, \Sigma\right)} ; \eta_{0}<\eta<\infty\right\}, & \operatorname{Im} \kappa<0,\end{cases} \\
& \theta_{\alpha_{0}}:=\tan ^{-1}\left(\frac{1-c_{\alpha_{0}}(\kappa)}{\sqrt{c_{\alpha_{0}}(\kappa)\left(2-c_{\alpha_{0}}(\kappa)\right)}}\right)
\end{aligned}
$$

where $\eta_{0}:=\max \left\{\eta \geq 0 ;-\alpha_{0}+i \eta \in \Sigma\right\}$. Then $c_{\alpha_{0}}(\kappa) \in(0,1)$ and $\theta_{\alpha_{0}} \in$ $(0, \pi / 2)$.
(a) If $\operatorname{Im} \kappa>0$, then the resolvent set $\rho(-(A+\kappa B))$ contains the sector $S_{+}(\kappa)$, where

$$
S_{+}(\kappa):=\left\{\lambda \in \mathbb{C} ;-\theta_{\alpha_{0}}<\arg \lambda<\pi / 2\right\}
$$

(b) If $\operatorname{Im} \kappa<0$, then the resolvent set $\rho(-(A+\kappa B))$ contains the sector $S_{-}(\kappa)$, where

$$
S_{-}(\kappa):=\left\{\lambda \in \mathbb{C} ;-\pi / 2<\arg \lambda<\theta_{\alpha_{0}}\right\} .
$$

Remark 2.2. Let $A$ and $B$ be as in Theorem 2.7, satisfying (2.20), with $-\alpha_{0} \leq \gamma(0)<0$. Then it is proved in [8, Theorem 1.7] that $B$ is $A$-bounded:

$$
\|B u\| \leq|\gamma(0)|^{-1}\|A u\|, \quad u \in D(A) \subset D(B)
$$

and $A+t B$ is selfadjoint on $D(A)$ for $t>\gamma(0)$; in particular, $A+\gamma(0) B$ is essentially selfadjoint in $H$. These facts are regarded as a restriction of Theorem 2.7 (iv) and (v) to the subset $\Sigma^{\mathrm{c}} \cap \mathbb{R}$.

Proof. (iv) Let $\gamma(0)<0$. To prove (2.21) it suffices to show that

$$
\begin{equation*}
\left\|B_{\varepsilon} u\right\| \leq \operatorname{dist}(0, \Sigma)^{-1}\|A u\|=|\gamma(0)|^{-1}\|A u\|, \quad \varepsilon>0, u \in D(A) \tag{2.22}
\end{equation*}
$$

As in the proof of Lemma 2.2, we see from $(\boldsymbol{\gamma} \mathbf{4})$ that

$$
-\operatorname{Re}\left(A v, B_{\varepsilon} v\right) \leq \gamma(0)<0
$$

where $v:=\left\|B_{\varepsilon} u\right\|^{-1} u$. So we obtain $\operatorname{Re}\left(A u, B_{\varepsilon} u\right) \geq|\gamma(0)| \cdot\left\|B_{\varepsilon} u\right\|^{2}$ and hence (2.22).
(v) Let $\kappa \in \Sigma^{\mathrm{c}}$ with $\alpha_{0}+\operatorname{Re} \kappa \geq 0$. Then the accretivity of $A+\kappa B_{\varepsilon}$ (and $A+\kappa B$ ) is a consequence of (2.20):

$$
\operatorname{Re}\left(\left(A+\kappa B_{\varepsilon}\right) u, u\right) \geq\left(\alpha_{0}+\operatorname{Re} \kappa\right)\left(B_{\varepsilon} u, u\right) \geq 0 .
$$

Now we can consider the unique solvability of the equation for each $f \in H$ and $\lambda>0$

$$
A u_{\varepsilon}+\kappa B_{\varepsilon} u_{\varepsilon}+\lambda u_{\varepsilon}=f
$$

In order to prove $R(A+\kappa B+\lambda)=H$ we only have to show that $\left\|u_{\varepsilon}\right\|$ and $\left\|B_{\varepsilon} u_{\varepsilon}\right\|$ are bounded as $\varepsilon$ tends to zero. The $m$-accretivity of $A+\kappa B_{\varepsilon}$ yields that $\left\|u_{\varepsilon}\right\| \leq \lambda^{-1}\|f\|$ and hence $\left\|A u_{\varepsilon}+\kappa B_{\varepsilon} u_{\varepsilon}\right\| \leq 2\|f\|$. In the same way as in the proof of Lemma 2.5 we can show that there exists $c>0$ such that $\left\|A u_{\varepsilon}\right\|+\left\|B_{\varepsilon} u_{\varepsilon}\right\| \leq c\|f\|$. This concludes that $R(A+\kappa B+\lambda)=H$. The proof of the essential $m$-accretivity of $A+\kappa B$ for $\kappa \in \partial \Sigma$ with $\operatorname{Re} \kappa \geq-\alpha_{0}$ is similar to that of Lemma 2.4.
(vi) Let $\kappa \in \Sigma^{c}$ with $\operatorname{Re} \kappa<-\alpha_{0}$ and $\operatorname{Im} \kappa>0$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda>0$. To show that $\lambda \in \rho(-(A+\kappa B))$ let $f \in H$. Then we want to solve the equation

$$
\begin{equation*}
A u+\kappa B u+\lambda u=f \tag{2.23}
\end{equation*}
$$

Set $v:=(A+\zeta B+\lambda) u$ for $\zeta \in \Sigma^{\mathrm{c}}$ with $\operatorname{Re} \zeta=-\alpha_{0}$. Since $A+\zeta B$ is $m$-accretive in $H$ [see ( $\mathbf{v}$ )], we can write (2.23) as

$$
v-(\zeta-\kappa) B(A+\zeta B+\lambda)^{-1} v=f
$$

Proceeding as in the proof of Lemma 2.5, we can show that $|\zeta-\kappa| \cdot \| B(A+$ $\zeta B+\lambda)^{-1} \|<1$ if $|\zeta-\kappa|<\operatorname{dist}(\zeta, \Sigma)$. Replacing $c_{0}(\kappa)$ with $c_{\alpha_{0}}(\kappa)$, the similar argument to Lemma 2.5 and Lemma 2.6 yields the assertion (a). Considering $\bar{\kappa}$ instead of $\kappa$ when $\operatorname{Im} \kappa<0$, we can also obtain the assertion (b).

Remark 2.3. Let $\left\{\kappa_{n}=\xi_{n}+i \eta\right\} \subset \Sigma^{\mathrm{c}}$ be a sequence satisfying $\xi_{n} \uparrow-\alpha_{0}$ $(n \rightarrow \infty)$ in assertion ( $\mathbf{v i}$ ). Then $c_{\alpha_{0}}\left(\kappa_{n}\right) \rightarrow 0$ and hence the resolvent sets $\rho\left(-\left(A+\kappa_{n} B\right)\right)$ extend from the sectors to the right half-plane as $n \rightarrow \infty$, which suggests the $m$-accretivity of the limiting operator $A+\left(-\alpha_{0}+i \eta\right) B$. This is nothing but the conclusion of ( $\mathbf{v}$ ).

## §3. Proof of Theorem 1.1

In this section we prepare some inequalities to apply Theorems 2.1 and 2.7 to $A:=\Delta^{2}$ and $B:=|x|^{-4}$. In [9, Lemmas 3.1 and 3.3] we have proved the following

Lemma 3.0. Let $v \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$. Then
(i) $\operatorname{Re}((x \cdot \nabla) v, v)=-\frac{N}{2}\|v\|^{2}$,
(ii) $\|(x \cdot \nabla) v\|^{2}-\left(N^{2} / 4\right)\|v\|^{2} \geq 0$,
(iii) $\left\||x|^{2} \Delta v\right\|^{2}\|v\|^{2}+2 N\||x| \nabla v\|^{2}\|v\|^{2}-\||x| \nabla v\|^{4}-4\|(x \cdot \nabla) v\|^{2}\|v\|^{2} \geq 0$.

The following lemma is a strict version of Lemma 3.0 (ii).
Lemma 3.1. Let $v \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$. Then

$$
\begin{equation*}
|\operatorname{Im}(v,(x \cdot \nabla) v)|^{2} \leq\|v\|^{2}\left(\|(x \cdot \nabla) v\|^{2}-\frac{N^{2}}{4}\|v\|^{2}\right) . \tag{3.1}
\end{equation*}
$$

Proof. Let $v \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$. From the Schwarz inequality we have

$$
|\operatorname{Im}(v,(x \cdot \nabla) v)|^{2}+|\operatorname{Re}(v,(x \cdot \nabla) v)|^{2}=|(v,(x \cdot \nabla) v)|^{2} \leq\|v\|^{2}\|x \cdot \nabla v\|^{2} .
$$

Combining this with Lemma 3.0 (i), we obtain (3.1).
The following lemma together with Lemma 3.1 give a strict version of Lemma 3.0 (iii).
Lemma 3.2. Let $v \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$. Then

$$
\begin{align*}
& {\left[\|v\|^{2} \operatorname{Im}\left((x \cdot \nabla) v,|x|^{2} \Delta v\right)-\||x| \nabla v\|^{2} \operatorname{Im}(v,(x \cdot \nabla) v)\right]^{2} }  \tag{3.2}\\
\leq & \left\{\|v\|^{2}\left[\|(x \cdot \nabla) v\|^{2}-\frac{N^{2}}{4}\|v\|^{2}\right]-|\operatorname{Im}(v,(x \cdot \nabla) v)|^{2}\right\} \\
\times & {\left[\left\||x|^{2} \Delta v\right\|^{2}\|v\|^{2}+2 N\||x| \nabla v\|^{2}\|v\|^{2}-\||x| \nabla v\|^{4}-4\|(x \cdot \nabla) v\|^{2}\|v\|^{2}\right] . }
\end{align*}
$$

Proof. For each $v \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ set $v_{1}:=|x|^{2} \Delta v, v_{2}:=(x \cdot \nabla) v, v_{3}:=v$. Let $G:=\left(\left(v_{j}, v_{k}\right)\right)_{j k}$. Let $a, b, c \geq 0$ and $\alpha, \beta, \gamma \in \mathbb{C}$ be defined as

$$
\left(\begin{array}{ccc}
c & \bar{\alpha} & \beta \\
\alpha & b & \bar{\gamma} \\
\bar{\beta} & \gamma & a
\end{array}\right):=\left(\begin{array}{ccc}
\left\||x|^{2} \Delta v\right\|^{2} & \left(|x|^{2} \Delta v,(x \cdot \nabla) v\right) & \left(|x|^{2} \Delta v, v\right) \\
\left((x \cdot \nabla) v,|x|^{2} \Delta v\right) & \|(x \cdot \nabla) v\|^{2} & ((x \cdot \nabla) v, v) \\
\left(v,|x|^{2} \Delta v\right) & (v,(x \cdot \nabla) v) & \|v\|^{2}
\end{array}\right) .
$$

Since $G$ is positive semi-definite, we have $\operatorname{det} G \geq 0$;

$$
a|\alpha|^{2}+b|\beta|^{2}+c|\gamma|^{2} \leq a b c+2 \operatorname{Re}(\alpha \beta \gamma) .
$$

Setting $\alpha=\alpha_{1}+i \alpha_{2}, \beta=\beta_{1}+i \beta_{2}, \gamma=\gamma_{1}+i \gamma_{2}$ with $\alpha_{j}, \beta_{j}, \gamma_{j} \in \mathbb{R}(j=1,2)$, we have

$$
\begin{align*}
& a \alpha_{2}^{2}+b \beta_{2}^{2}+c \gamma_{2}^{2}+2\left(\alpha_{1} \beta_{2} \gamma_{2}+\alpha_{2} \beta_{1} \gamma_{2}+\alpha_{2} \beta_{2} \gamma_{1}\right)  \tag{3.3}\\
\leq & a b c+2 \alpha_{1} \beta_{1} \gamma_{1}-\left(a \alpha_{1}^{2}+b \beta_{1}^{2}+c \gamma_{1}^{2}\right)
\end{align*}
$$

Now it is easy to see that

$$
\begin{align*}
\alpha_{1} & =\operatorname{Re} \alpha=\frac{N}{2} \widetilde{b}-2 b  \tag{3.4}\\
\beta_{1} & =\operatorname{Re} \beta=N a-\widetilde{b}  \tag{3.5}\\
\gamma_{1} & =\operatorname{Re} \gamma=-\frac{N}{2} a \tag{3.6}
\end{align*}
$$

where $\widetilde{b}:=\||x| \nabla v\|^{2}$ (see $[\mathbf{9}$, Section 3]). It follows (3.4)-(3.6) that the righthand side of (3.3) equals

$$
\left(b-\left(N^{2} / 4\right) a\right)\left(a c+2 N a \widetilde{b}-\widetilde{b}^{2}-4 a b\right)
$$

Multiplying (3.3) by $a$ and using the equality $\beta_{2}=2 \gamma_{2}$, we have

$$
\begin{align*}
& a^{2} \alpha_{2}^{2}+2 a\left(\beta_{1}+2 \gamma_{1}\right) \alpha_{2} \gamma_{2}+a\left(4 \alpha_{1}+4 b+c\right) \gamma_{2}^{2}  \tag{3.7}\\
\leq & a\left(b-\left(N^{2} / 4\right) a\right)\left(a c+2 N a \widetilde{b}-\widetilde{b}^{2}-4 a b\right)
\end{align*}
$$

We see from (3.4)-(3.6) that the left-hand side of (3.7) equals

$$
\left(a \alpha_{2}-\widetilde{b} \gamma_{2}\right)^{2}+\left(a c+2 N a \widetilde{b}-\widetilde{b}^{2}-4 a b\right) \gamma_{2}^{2}
$$

which implies that

$$
\begin{equation*}
\left(a \alpha_{2}-\widetilde{b} \gamma_{2}\right)^{2} \leq\left(a b-\left(N^{2} / 4\right) a^{2}-\gamma_{2}^{2}\right)\left(a c+2 N a \widetilde{b}-\widetilde{b}^{2}-4 a b\right) \tag{3.8}
\end{equation*}
$$

(3.8) is nothing but (3.2).

Lemma 3.3. Let $k_{1}$ be the constants defined in (1.1):

$$
k_{1}=112-3(N-2)^{2}
$$

For $u \in H^{4}\left(\mathbb{R}^{N}\right)$ and $\varepsilon>0$ put

$$
\mathrm{IP}:=\left(\Delta^{2} u,\left(|x|^{4}+\varepsilon\right)^{-1} u\right)
$$

and $a:=\left\|\left(|x|^{4}+\varepsilon\right)^{-1} u\right\|^{2}$. Then $k_{1} a+\operatorname{ReIP} \geq 0$ and

$$
\begin{align*}
|\operatorname{Im~IP}|^{2} & \leq 64 \sqrt{a}\left(\sqrt{k_{1} a+\operatorname{ReIP}}-\left(\left(N^{2} / 4\right)-N-10\right) \sqrt{a}\right)  \tag{3.9}\\
& \times\left(\sqrt{k_{1} a+\operatorname{ReIP}}+8 \sqrt{a}\right)^{2}
\end{align*}
$$

If $N \geq 9$, then $k_{2} a+\operatorname{Re} I P \geq 0$ and

$$
\begin{equation*}
|\operatorname{Im} \operatorname{IP}|^{2} \leq \frac{64 \sqrt{a}\left(k_{2} a+\operatorname{ReIP}\right)\left(\sqrt{k_{1} a+\operatorname{ReIP}}+8 \sqrt{a}\right)^{2}}{\sqrt{k_{1} a+\operatorname{ReIP}}+\left(\left(N^{2} / 4\right)-N-10\right) \sqrt{a}} \tag{3.10}
\end{equation*}
$$

where $k_{2}=k_{1}-\left[(N-2)^{2} / 4-11\right]^{2}=-(N / 16)(N-8)\left(N^{2}-16\right)<0(N \geq 9)$.

Proof. Let $u \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ and $\varepsilon>0$. Put $v:=\left(|x|^{4}+\varepsilon\right)^{-1} u$. By using the same notations as in the proof of Lemma 3.2 (3.8) is written as

$$
\begin{equation*}
L:=\frac{\left(a \alpha_{2}-\widetilde{b} \gamma_{2}\right)^{2}}{a b-\left(N^{2} / 4\right) a^{2}-\gamma_{2}^{2}} \leq a c+2 N a \widetilde{b}-\widetilde{b}^{2}-4 a b=: R . \tag{3.11}
\end{equation*}
$$

Here we note (see [9, Proof of Lemma 3.4]) that

$$
\mathrm{IP}=\left\||x|^{2} \Delta v\right\|^{2}+8\left((x \cdot \nabla) v,|x|^{2} \Delta v\right)+4(N+2)\left(v,|x|^{2} \Delta v\right)+\varepsilon\|\Delta v\|^{2} .
$$

It follows that

$$
\begin{align*}
c & =\left\||x|^{2} \Delta v\right\|^{2} \leq \operatorname{Re} \operatorname{IP}+16 b+8 \widetilde{b}-4 N(N+2) a,  \tag{3.12}\\
\alpha_{2} & =\operatorname{Im}\left((x \cdot \nabla) v,|x|^{2} \Delta v\right)=\frac{1}{8} \operatorname{Im} \operatorname{IP}+(N+2) \gamma_{2} . \tag{3.13}
\end{align*}
$$

In fact, (3.13) holds as a consequence $\beta_{2}=2 \gamma_{2}$. Applying (3.13) to $L$ yields

$$
L=\frac{\left(\frac{a}{8} \operatorname{Im} \operatorname{IP}+((N+2) a-\widetilde{b}) \gamma_{2}\right)^{2}}{a\left(b-\left(N^{2} / 4\right) a\right)-\gamma_{2}^{2}}=\frac{\left(c_{1} \gamma_{2}+c_{2}\right)^{2}}{c_{0}-\gamma_{2}^{2}}
$$

where

$$
\begin{align*}
c_{0} & :=a\left(b-\left(N^{2} / 4\right) a\right) \geq \gamma_{2}^{2},  \tag{3.14}\\
c_{1} & :=(N+2) a-\widetilde{b},  \tag{3.15}\\
c_{2} & :=\frac{a}{8} \operatorname{Im} \text { IP; } \tag{3.16}
\end{align*}
$$

note that the inequality in (3.14) is nothing but (3.1). Since the quadratic equation $L\left(c_{0}-t^{2}\right)=\left(c_{1} t+c_{2}\right)^{2}$ has a real root $t=\gamma_{2}$, the discriminant is nonnegative:

$$
\begin{equation*}
L\left(c_{0} L+c_{0} c_{1}^{2}-c_{2}^{2}\right) \geq 0 . \tag{3.17}
\end{equation*}
$$

It is clear that $L \geq 0$. If $L>0$, then (3.17) yields

$$
\begin{equation*}
L \geq\left(c_{2}^{2} / c_{0}\right)-c_{1}^{2} . \tag{3.18}
\end{equation*}
$$

If $L=0$, then $\gamma_{2}=-c_{2} / c_{1}$ and hence (3.14) yields that $0 \geq\left(c_{2}^{2} / c_{0}\right)-c_{1}^{2}$. This means that (3.18) holds for $L \geq 0$. Hence it follows from (3.14)-(3.16) and (3.18) that

$$
\begin{equation*}
L \geq \frac{a|\operatorname{Im~IP}|^{2}}{64\left(b-\left(N^{2} / 4\right) a\right)}-(\widetilde{b}-(N+2) a)^{2} . \tag{3.19}
\end{equation*}
$$

On the other hand, since $b \leq \widetilde{b},(3.11)$ and (3.12) yields

$$
\begin{align*}
R & \leq a \operatorname{ReIP}+12 a b+2(N+4) a \widetilde{b}-\widetilde{b}^{2}-4 N(N+2) a^{2}  \tag{3.20}\\
& \leq a\left(k_{1} a+\operatorname{ReIP}\right)-(\widetilde{b}-(N+10) a)^{2}
\end{align*}
$$

where $k_{1}:=(N+10)^{2}-4 N(N+2)=112-3(N-2)^{2}$. Since $L \leq R$, it follows from (3.19) and (3.20) that
(3.21) $\frac{a|\operatorname{Im} \operatorname{IP}|^{2}}{64\left(b-N^{2} a / 4\right)}-(\widetilde{b}-(N+2) a)^{2} \leq a\left(k_{1} a+\operatorname{ReIP}\right)-(\widetilde{b}-(N+10) a)^{2}$.

Therefore we obtain

$$
\begin{equation*}
\frac{|\operatorname{Im} \mathrm{IP}|^{2}}{64\left(b-\left(N^{2} / 4\right) a\right)}-16(\widetilde{b}-(N+6) a) \leq k_{1} a+\operatorname{Re} \mathrm{IP}=: K \tag{3.22}
\end{equation*}
$$

Now we see from (3.20) that

$$
(\widetilde{b}-(N+10) a)^{2} \leq R+(\widetilde{b}-(N+10) a)^{2} \leq a K
$$

and hence

$$
\begin{equation*}
b \leq \widetilde{b} \leq \sqrt{a K}+(N+10) a \tag{3.23}
\end{equation*}
$$

Applying (3.23) to (3.22), we obtain
$\frac{|\operatorname{Im~IP}|^{2}}{64 \sqrt{a}\left[\sqrt{K}-\left(\left(N^{2} / 4\right)-N-10\right) \sqrt{a}\right]} \leq K+16(\sqrt{a K}+4 a)=(\sqrt{K}+8 \sqrt{a})^{2}$.
This proves (3.9) for $u \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$. Next note that $N^{2} / 4-N-10 \geq 0$ for $N \geq 9$. To obtain (3.10), we have only to use the equality

$$
\sqrt{K}-\left(\left(N^{2} / 4\right)-N-10\right) \sqrt{a}=\frac{k_{2} a+\operatorname{ReIP}}{\sqrt{K}+\left(\left(N^{2} / 4\right)-N-10\right) \sqrt{a}}
$$

where $k_{2}=-N(N-8)\left(N^{2}-16\right) / 16$. Since $C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ is dense in $H^{4}\left(\mathbb{R}^{N}\right)$, we obtain (3.9) for every $u \in H^{4}\left(\mathbb{R}^{N}\right)$.

Proof of Theorem 1.1. Let $H:=L^{2}\left(\mathbb{R}^{N}\right), A:=\Delta^{2}$ with $D(A):=H^{4}\left(\mathbb{R}^{N}\right)$ and $B:=|x|^{-4}$ with $D(B):=\left\{u \in H ;|x|^{-4} u \in H\right\}$. For $u \in D(A)$ and $\varepsilon>0$ take $v:=B_{\varepsilon} u=\left(|x|^{4}+\varepsilon\right)^{-1} u$ with $\sqrt{a}:=\|v\|=1$. Set $\xi, \eta \in \mathbb{R}$ as

$$
\xi+i \eta:=-\mathrm{IP}=-\left(A u, B_{\varepsilon} u\right)
$$

We shall prove that there exist $\gamma$ independent of $\varepsilon>0$ satisfying $(\gamma \mathbf{1}),(\gamma \mathbf{2})$, $(\gamma \mathbf{5})_{0}$ in Theorem 2.1 (or $(\gamma \mathbf{5})_{\alpha_{0}}$ in Theorem 2.7) and $\Sigma$ defined in $(\gamma \mathbf{3})$ such
that $-\mathrm{IP} \in \Sigma$ for every $u \in D(A)$ and $\varepsilon>0$, i.e., ( $\gamma \mathbf{4}$ ) holds. First it follows from Lemma 3.3 with $\operatorname{Re} \operatorname{IP}=-\xi, \operatorname{Im} \operatorname{IP}=-\eta, a=1$ that

$$
\left\{\begin{array}{l}
k_{1} a+\operatorname{ReIP}=k_{1}-\xi \geq 0,  \tag{3.24}\\
|\eta|^{2} \leq \varphi_{N}(\xi),
\end{array}\right.
$$

where $\varphi_{N}:\left(-\infty, k_{1}\right] \rightarrow \mathbb{R}$ is given as follows (see (3.9)):

$$
\begin{equation*}
\varphi_{N}(t):=64\left[\sqrt{k_{1}-t}+\left(10+N-\left(N^{2} / 4\right)\right)\right]\left(\sqrt{k_{1}-t}+8\right)^{2} . \tag{3.25}
\end{equation*}
$$

We can easily see that $\varphi_{N}$ is monotone decreasing and $\lim _{t \rightarrow-\infty} \varphi_{N}(t)=\infty$. According to the sign of $\varphi_{N}\left(k_{1}\right)$ we consider two cases $N \leq 8$ and $N \geq 9$.

In the case $N \leq 8$ it holds from $10+N-\left(N^{2} / 4\right)>0$ that $\varphi_{N}\left(k_{1}\right)=$ $\min \left\{\varphi_{N}(t) ; t \leq k_{1}\right\}>0$. If $|\eta|^{2} \leq \varphi_{N}\left(k_{1}\right)$, then $|\eta|^{2} \leq \varphi_{N}(\xi)$ holds. If $|\eta|^{2} \geq \varphi_{N}\left(k_{1}\right)$, then $|\eta|^{2} \leq \varphi_{N}(\xi)$ is equivalent to $\xi \leq \varphi_{N}^{-1}\left(|\eta|^{2}\right)$. Thus we have

$$
\left\{\begin{array}{l}
\xi \leq k_{1} \text { when }|\eta|^{2} \leq \varphi_{N}\left(k_{1}\right)  \tag{3.26}\\
\xi \leq \varphi_{N}^{-1}\left(|\eta|^{2}\right) \quad \text { when }|\eta|^{2} \geq \varphi_{N}\left(k_{1}\right) .
\end{array}\right.
$$

Set

$$
\gamma(t):=\left\{\begin{array}{l}
k_{1} \text { when }|t|^{2} \leq \varphi_{N}\left(k_{1}\right) \\
\varphi_{N}^{-1}\left(|t|^{2}\right) \quad \text { when }|t|^{2} \geq \varphi_{N}\left(k_{1}\right) .
\end{array}\right.
$$

$(\gamma \mathbf{2})$ is clearly satisfied. Let $\Sigma$ be defined in ( $\gamma \mathbf{3}$ ). We show that $\gamma$ is concave. (3.24) implies that

$$
\begin{equation*}
\Sigma=\left\{\xi+i \eta \in \mathbb{C} ; \xi \leq k_{1},|\eta| \leq \sqrt{\varphi_{N}(\xi)}\right\} . \tag{3.27}
\end{equation*}
$$

Since $\sqrt{\varphi_{N}}$ is concave, (3.27) shows that $\Sigma$ is convex. Hence $\gamma$ is concave and $(\gamma \mathbf{1})$ is satisfied. (3.24) and (3.27) imply that $(\gamma \mathbf{4})$ is satisfied. Noting $\gamma(0)=k_{1}>0$, we see that $(\gamma \mathbf{5})_{0}$ is satisfied. When $N \leq 4$, we apply Theorem 2.1 with $A, B, \gamma$ and $\Sigma$ to obtain the assertion of Theorem 1.1 in the case $N \leq 4$. When $N \geq 5$, we have the Rellich inequality

$$
\frac{N(N-4)}{4}\left\|\left(|x|^{2}+\varepsilon\right)^{-1} u\right\| \leq\|\Delta u\|, \quad u \in H^{2}\left(\mathbb{R}^{N}\right)
$$

which implies (2.20) with $\alpha_{0}:=[N(N-4) / 4]^{2}$. Since $\gamma(0)=k_{1}>0>-\alpha_{0}$, $(\gamma \mathbf{5})_{\alpha_{0}}$ is satisfied. Thus we can apply Theorem 2.7 with $A, B, \gamma$ and $\Sigma$ to obtain Theorem $2.7(\mathbf{v})$, ( $\mathbf{v i}$ ). Therefore we obtain the assertion of Theorem 1.1 in the case $5 \leq N \leq 8$.

In the case $N \geq 9$ it follows from Lemma 3.3 with $\operatorname{Re} \operatorname{IP}=-\xi, a=1$ that

$$
\begin{equation*}
\xi \leq k_{2}:=-(N / 16)(N-8)\left(N^{2}-16\right) . \tag{3.28}
\end{equation*}
$$

In particular, (3.10) implies that $\varphi_{N}$ has another expression:

$$
\varphi_{N}(t)=\frac{64\left(k_{2}-t\right)\left(\sqrt{k_{1}-t}+8\right)}{\sqrt{k_{1}-t}+\left(\left(N^{2} / 4\right)-N-10\right)}
$$

Then $\varphi_{N}\left(k_{2}\right)=0$ and $\sqrt{\varphi_{N}}$ is concave on $\left(-\infty, k_{2}\right]$. Set

$$
\gamma(t):=\varphi_{N}^{-1}\left(|t|^{2}\right), \quad t \in \mathbb{R}
$$

It is clear that $(\gamma \mathbf{2})$ is satisfied. Let $\Sigma$ be defined in $(\gamma \mathbf{3})$. Noting $k_{2}<k_{1}$, we see from (3.24) and (3.28) that

$$
\begin{equation*}
\Sigma=\left\{\xi+i \eta \in \mathbb{C} ; \xi \leq k_{2},|\eta| \leq \sqrt{\varphi_{N}(\xi)}\right\} \tag{3.29}
\end{equation*}
$$

Since $\sqrt{\varphi_{N}}$ is concave, we see from (3.29) that $\Sigma$ is convex. Hence $\gamma$ is concave and $(\gamma \mathbf{1})$ is satisfied. (3.24), (3.28) and (3.29) imply that $(\gamma \mathbf{4})$ is satisfied. Applying the Rellich inequality again, we have $(2.20)$ with $\alpha_{0}:=[N(N-4) / 4]^{2}$. Since $\gamma(0)=k_{2}>-\alpha_{0},(\gamma \mathbf{5})_{\alpha_{0}}$ is satisfied. Since $\gamma(0)=k_{2}<0$, we obtain Theorem 2.7 (iv). Therefore we obtain the assertion of Theorem 1.1 in the case $N \geq 9$. This completes the proof of Theorem 1.1.

Acknowledgments. The author feels extremely thankful to the referee for the essential comments on our result. As stated in Remark 1.2 the referee's suggestion notified us that we can improve angles $\theta_{\alpha_{0}}$ or $\theta_{0}$ in Theorem 1.1 (iii), Theorem 2.1 (iii) and Theorem 2.7 (vi). Also a lot of comments are helpful to improve the quality of the paper. The author would like to express his deep gratitude to his PhD advisor Professor N. Okazawa for a lot of valuable guidance to complete this paper. The author also thanks Professor T. Yokota for a lot of helpful advice.

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[^0]:    ${ }^{1}$ The author would like to thank the referee for this comment.

