

A test for subvector of mean vector with two-step monotone missing data

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Abstract. In this paper, we consider the one-sample problem of testing for the subvector of a mean vector with two-step monotone missing data. In the case that the data set consists of complete data with $p(= p_1 + p_2 + p_3)$ dimensions and incomplete data with $(p_1 + p_2)$ dimensions, we derive the likelihood ratio criterion for testing the $(p_2 + p_3)$ mean vector under the given mean vector of p_1 dimensions. Furthermore, we propose an approximation for the upper percentile of the likelihood ratio test (LRT) statistic. We investigate the accuracy and asymptotic behavior of this approximation using Monte Carlo simulation. An example is presented in order to illustrate the method.

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§1. Introduction

When analyzing data, it is important to consider missing observations. The existence of missing data is a common problem that is present in almost all statistical data analyses. However, the majority of statistical methods require a comparatively strict assumption concerning the cause of missing data, and are prone to substantial bias. Methods for dealing with missing data by removing incomplete cases or imputing missing values are more vulnerable to the propagation of bias throughout. Statistical analysis involving monotone missing data has been discussed by many authors, because in this case the mathematical complexity is reduced. For example, Anderson (1957) demonstrated an approach for deriving the maximum likelihood estimators (MLEs) of the mean vector and covariance matrix using the likelihood equations for

monotone missing data. Kanda and Fujikoshi (1998) described the properties of MLEs based on two-step and three-step monotone missing samples and a general k -step. Among the many papers that propose methods for testing mean vectors with monotone missing data, we mention those by Krishnamoorthy and Pannala (1999); Yu, Krishnamoorthy, and Pannala (2006); and Chang and Richards (2009). In particular, for testing the mean vector with two-step monotone missing data, Seko, Yamazaki, and Seo (2012), and Seko, Kawasaki, and Seo (2011) have provided a simple approach to deriving the approximate upper percentiles of the Hotelling's T^2 type statistic and LRT statistic for one-sample and two-sample problems. Moreover, various statistical methods have been developed to analyze data with non-monotone missing values by Srivastava (1985), Srivastava and Carter (1986), and Shutoh, Kusumi, Morinaga, Yamada, and Seo (2010), and others. In the case of general k -step monotone missing data, many difficult problems remain unsolved. For simplicity, we assume that $k = 2$. Let the data set $\{x_{i,j}\}$ be of the form

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,p_1} & x_{1,p_1+1} & \cdots & x_{1,p_1+p_2} & x_{1,p_1+p_2+1} & \cdots & x_{1,p} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n_1,1} & \cdots & x_{n_1,p_1} & x_{n_1,p_1+1} & \cdots & x_{n_1,p_1+p_2} & x_{n_1,p_1+p_2+1} & \cdots & x_{n_1,p} \\ x_{n_1+1,1} & \cdots & x_{n_1+1,p_1} & x_{n_1+1,p_1+1} & \cdots & x_{n_1+1,p_1+p_2} & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{n,1} & \cdots & x_{n,p_1} & x_{n,p_1+1} & \cdots & x_{n,p_1+p_2} & * & \cdots & * \end{pmatrix},$$

where $n_2 = n - n_1$ and $n_1 > p$. Here, “*” indicates missing data. That is, we have complete data for n_1 mutually independent observations with p dimensions, and incomplete data for n_2 mutually independent observations with $(p_1 + p_2)$ dimensions. Such a data set is described as two-step monotone missing data.

In this paper, based on two-step monotone missing data, we consider the one-sample problem of testing for the subvector of a mean vector. We derive the MLEs of the mean vector and the covariance matrix and the MLE of the covariance matrix under the null hypothesis. Using these MLEs, we propose the likelihood ratio test statistic and its approximate upper percentile.

In Section 2, we review the test for a subvector based on non-missing data when the first p_1 dimensions of the mean vector $\boldsymbol{\mu}$ is given. In Section 3, we derive the MLEs and the MLEs under the null hypothesis, with two-step monotone missing data. In Section 4, we propose the LRT statistic and its approximate upper percentiles. The accuracy of the approximate upper percentiles of the test statistic is investigated using Monte Carlo simulation in Section 5. In Section 6, we present a numerical example to illustrate our method using the approximate upper percentiles of the test statistic. Finally, Section 7 concludes this paper.

§2. Test for a subvector

We review the case of non-missing data. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be distributed as $N_p(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_p)'$ and Σ are unknown. Let $\boldsymbol{\mu} = (\boldsymbol{\mu}'_1, \boldsymbol{\mu}'_{(23)})'$, where $\boldsymbol{\mu}_1 = (\mu_1, \mu_2, \dots, \mu_{p_1})'$ and $\boldsymbol{\mu}_{(23)} = (\mu_{p_1+1}, \mu_{p_1+2}, \dots, \mu_p)'$, $p_1 < p < n$. Then, the sample mean vector and unbiased covariance matrix are defined as

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i = \begin{pmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_{(23)} \end{pmatrix}, S = \frac{1}{n-1} \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})' = \begin{pmatrix} S_{11} & S_{1(23)} \\ S_{(23)1} & S_{(23)(23)} \end{pmatrix},$$

respectively, where $\bar{\mathbf{x}}_1$ is a p_1 -vector and S_{11} is a $p_1 \times p_1$ matrix. Consider the following hypothesis test problem for the case of two-step monotone missing data in the one-sample problem:

(1.1)

$$H_0 : \boldsymbol{\mu}_{(23)} = \boldsymbol{\mu}_{(23)0} \text{ given } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{10} \text{ vs. } H_1 : \boldsymbol{\mu}_{(23)} \neq \boldsymbol{\mu}_{(23)0} \text{ given } \boldsymbol{\mu}_1 = \boldsymbol{\mu}_{10},$$

where $\boldsymbol{\mu}_{(23)0}$ and $\boldsymbol{\mu}_{10}$ are known. A criterion that is equivalent to the likelihood ratio can be written as

$$U = \frac{T_p^2 - T_{p_1}^2}{n-1 + T_{p_1}^2},$$

where $T_p^2 = n(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' S^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$ and $T_{p_1}^2 = n(\bar{\mathbf{x}}_1 - \boldsymbol{\mu}_{10})' S_{11}^{-1} (\bar{\mathbf{x}}_1 - \boldsymbol{\mu}_{10})$. We note that $U = \lambda^{-2/n} - 1$, where λ is the likelihood ratio criterion. Under H_0 , it follows that $(n-p)U/(p-p_1)$ is distributed as an F distribution with $p-p_1$ and $n-p$ degrees of freedom. This result follows from the one in Siotani, Hayakawa, and Fujikoshi (1985, p. 215). The criterion is called Rao's U statistic (See, Rao (1949) and Giri (1964)). The situation in which a subvector of $\boldsymbol{\mu}$ can be known is not rare. In some situations, partial information concerning the population means may be available to the experimenter. Furthermore, this hypothesis in the two-sample problem is equivalent to a test for additional information. That is, a problem that is closely related to the testing of the mean vectors $\boldsymbol{\mu}^{(1)} = \boldsymbol{\mu}^{(2)}$ is determining whether $\mathbf{x}_{(23)} = (\mathbf{x}'_2, \mathbf{x}'_3)'$ has additional information in the presence of \mathbf{x}_1 , where $\mathbf{x} = (\mathbf{x}'_1, \mathbf{x}'_{(23)})'$ arises from one of two groups $\Pi^{(1)} : N_p(\boldsymbol{\mu}^{(1)}, \Sigma)$ and $\Pi^{(2)} : N_p(\boldsymbol{\mu}^{(2)}, \Sigma)$. Eaton and Kariya (1975) derived tests for the independence of two normally distributed subvectors in the case that an additional random sample is available. Provost (1990) obtained explicit expressions in the case that the MLEs of all of the parameters of the multinormal random vector are given, and the likelihood ratio statistic for testing the independence between subvectors has been obtained. In the next section, we derive the MLEs and MLEs under H_0 , with two-step monotone missing data, to obtain the LRT statistic.

§3. MLEs with two-step monotone missing data

In this section, we obtain the MLEs using the decomposition of the density into conditional densities, which is called the conditional method (Kanda and Fujikoshi, 1998). Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_1}$ be distributed as $N_p(\boldsymbol{\mu}, \Sigma)$, and let $\mathbf{x}_{n_1+1}, \mathbf{x}_{n_1+2}, \dots, \mathbf{x}_n$ be distributed as $N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)})$, where each $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \dots, x_{j,p})'$, $j = 1, 2, \dots, n_1$ is $p \times 1$, each $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \dots, x_{j,p_1+p_2})'$, $j = n_1 + 1, n_1 + 2, \dots, n$ is $(p_1 + p_2) \times 1$, and

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)} \\ \boldsymbol{\mu}_3 \end{pmatrix}, \quad \Sigma = \left(\begin{array}{cc|c} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \hline \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{array} \right) = \begin{pmatrix} \Sigma_{(12)(12)} & \Sigma_{(12)3} \\ \Sigma_{3(12)} & \Sigma_{33} \end{pmatrix}.$$

We partition \mathbf{x}_j into a $p_1 \times 1$ random vector, a $p_2 \times 1$ random vector, and a $p_3 \times 1$ random vector as $\mathbf{x}_j = (\mathbf{x}'_{1j}, \mathbf{x}'_{2j}, \mathbf{x}'_{3j})' = (\mathbf{x}'_{(12)j}, \mathbf{x}'_{3j})'$, where \mathbf{x}_{ij} , $i = 1, 2, 3$, $j = 1, 2, \dots, n_1$ is $p_i \times 1$, and $p = p_1 + p_2 + p_3$. In addition, $\mathbf{x}_{(12)j}$ is partitioned into a $p_1 \times 1$ random vector and a $p_2 \times 1$ random vector as $\mathbf{x}_{(12)j} = (\mathbf{x}'_{1j}, \mathbf{x}'_{2j})'$, where \mathbf{x}_{ij} , $i = 1, 2$, $j = n_1 + 1, n_1 + 2, \dots, n$ is $p_i \times 1$. Then, the joint density function of the observed data set $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_1}, \mathbf{x}_{(12)n_1+1}, \mathbf{x}_{(12)n_1+2}, \dots, \mathbf{x}_{(12)n}$ can be written as

$$\prod_{j=1}^{n_1} f(\mathbf{x}_j; \boldsymbol{\mu}, \Sigma) \times \prod_{j=n_1+1}^n f(\mathbf{x}_{(12)j}; \boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)}),$$

where $f(\mathbf{x}_j; \boldsymbol{\mu}, \Sigma)$ and $f(\mathbf{x}_{(12)j}; \boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)})$ are the density functions of $N_p(\boldsymbol{\mu}, \Sigma)$ and $N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)})$, respectively. That is, the likelihood function is given by

$$\begin{aligned} L(\boldsymbol{\mu}, \Sigma) &= \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x}_j - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu}) \right\} \\ &\times \prod_{j=n_1+1}^n \frac{1}{(2\pi)^{(p_1+p_2)/2} |\Sigma_{(12)(12)}|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x}_{(12)j} - \boldsymbol{\mu}_{(12)})' \Sigma_{(12)(12)}^{-1} (\mathbf{x}_{(12)j} - \boldsymbol{\mu}_{(12)}) \right\}. \end{aligned}$$

The sample mean vectors are defined as

$$\begin{aligned} \bar{\mathbf{x}}_{1T} &= \frac{1}{n} \sum_{j=1}^n \mathbf{x}_{1j}, \quad \bar{\mathbf{x}}_{2T} = \frac{1}{n} \sum_{j=1}^n \mathbf{x}_{2j}, \\ \bar{\mathbf{x}}_F &= (\bar{\mathbf{x}}'_{(12)F}, \bar{\mathbf{x}}'_{3F})' = \left(\frac{1}{n_1} \sum_{j=1}^{n_1} \mathbf{x}'_{(12)j}, \frac{1}{n_1} \sum_{j=1}^{n_1} \mathbf{x}'_{3j} \right)'. \end{aligned}$$

In our situation, we first multiply the observation vectors \mathbf{x}_j by the transformation matrix

$$\Gamma_1 = \left(\begin{array}{cc|c} I_{p_1} & O & O \\ O & I_{p_2} & O \\ \hline -\Sigma_{3(12)}\Sigma_{(12)(12)}^{-1} & I_{p_3} & \end{array} \right)$$

on the left side, so that the transformed observation vectors are

$$\begin{aligned} \mathbf{x}_{(12)j} &\sim N_{p_1+p_2}(\boldsymbol{\mu}_{(12)}, \Sigma_{(12)(12)}), \quad j = 1, 2, \dots, n, \\ \mathbf{x}_{3j} - \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}\mathbf{x}_{(12)j} &\sim N_{p_3}(\boldsymbol{\mu}_3 - \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}\boldsymbol{\mu}_{(12)}, \Sigma_{33 \cdot (12)}), \quad j = 1, 2, \dots, n_1, \end{aligned}$$

where $\Sigma_{33 \cdot (12)} = \Sigma_{33} - \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}\Sigma_{(12)3}$. Next, we multiply the above observation vectors by the transformation matrix

$$\left(\begin{array}{cc|c} I_{p_1} & O & O \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p_2} & O \\ \hline O & & I_{p_3} \end{array} \right)$$

on the left side, so that the transformed observation vectors are

$$\begin{aligned} \mathbf{x}_{1j} &\sim N_{p_1}(\boldsymbol{\eta}_1, \Psi_{11}), \quad j = 1, 2, \dots, n, \\ \mathbf{x}_{2j} - \Psi_{21}\mathbf{x}_{1j} &\sim N_{p_2}(\boldsymbol{\eta}_2, \Psi_{22}), \quad j = 1, 2, \dots, n, \\ \mathbf{x}_{3j} - \Psi_{3(12)}\mathbf{x}_{(12)j} &\sim N_{p_3}(\boldsymbol{\eta}_3, \Psi_{33}), \quad j = 1, 2, \dots, n_1, \end{aligned}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \\ \boldsymbol{\eta}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_2 - \Sigma_{21}\Sigma_{11}^{-1}\boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_3 - \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}\boldsymbol{\mu}_{(12)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_2 - \Psi_{21}\boldsymbol{\mu}_{10} \\ \boldsymbol{\mu}_3 - \Psi_{3(12)}\boldsymbol{\mu}_{(12)} \end{pmatrix},$$

$$\Psi = \left(\begin{array}{cc|c} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} \\ \hline \Psi_{31} & \Psi_{32} & \Psi_{33} \end{array} \right) = \begin{pmatrix} \Psi_{(12)(12)} & \Psi_{(12)3} \\ \Psi_{3(12)} & \Psi_{33} \end{pmatrix},$$

$$\Psi_{(12)(12)} = \begin{pmatrix} \Sigma_{11} & \Sigma_{11}^{-1}\Sigma_{12} \\ \Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22 \cdot 1} \end{pmatrix}, \quad \Psi_{3(12)} = \Psi'_{(12)3} = \Sigma_{3(12)}\Sigma_{(12)(12)}^{-1}, \quad \Psi_{33} = \Sigma_{33 \cdot (12)},$$

and $\Sigma_{22 \cdot 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{21}$. It should be noted that \mathbf{x}_{1j} , $\mathbf{x}_{2j} - \Psi_{21}\mathbf{x}_{1j}$, and $\mathbf{x}_{3j} - \Psi_{3(12)}\mathbf{x}_{(12)j}$ are independent. Because $(\boldsymbol{\eta}, \Psi)$ has a one-to-one correspondence with $(\boldsymbol{\mu}, \Sigma)$, it is sufficient to derive the MLEs of $(\boldsymbol{\eta}, \Psi)$ instead of $(\boldsymbol{\mu}, \Sigma)$. Using the above transformation matrices, we will derive the MLEs of $\boldsymbol{\eta}_2$, $\boldsymbol{\eta}_3$, $\Psi_{(12)(12)}$, $\Psi_{3(12)}$, and Ψ_{33} .

Theorem 1. Suppose that the data set has a two-step monotone missing pattern. Then, the maximum likelihood estimators of $\boldsymbol{\eta}_2, \boldsymbol{\eta}_3, \Psi_{11}, \Psi_{21}, \Psi_{22}, \Psi_{3(12)}$, and Ψ_{33} are given by

$$\begin{aligned}\hat{\boldsymbol{\eta}}_2 &= \bar{\mathbf{x}}_{2T} - \hat{\Psi}_{21} \bar{\mathbf{x}}_{1T}, \quad \hat{\boldsymbol{\eta}}_3 = \bar{\mathbf{x}}_{3F} - \hat{\Psi}_{3(12)} \bar{\mathbf{x}}_{(12)F}, \\ \hat{\Psi}_{11} &= \frac{1}{n} \sum_{j=1}^n (\mathbf{x}_{1j} - \boldsymbol{\mu}_{10})(\mathbf{x}_{1j} - \boldsymbol{\mu}_{10})', \\ \hat{\Psi}_{21} &= \left\{ \sum_{j=1}^n (\mathbf{x}_{2j} - \bar{\mathbf{x}}_{2T})(\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})' \right\} \left\{ \sum_{j=1}^n (\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})(\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})' \right\}^{-1}, \\ \hat{\Psi}_{22} &= \frac{1}{n} \sum_{j=1}^n (\mathbf{x}_{2j} - \hat{\Psi}_{21} \mathbf{x}_{1j} - \hat{\boldsymbol{\eta}}_2)(\mathbf{x}_{2j} - \hat{\Psi}_{21} \mathbf{x}_{1j} - \hat{\boldsymbol{\eta}}_2)', \\ \hat{\Psi}_{3(12)} &= \left\{ \sum_{j=1}^{n_1} (\mathbf{x}_{3j} - \bar{\mathbf{x}}_{3F})(\mathbf{x}_{(12)j} - \bar{\mathbf{x}}_{(12)F})' \right\} \\ &\quad \times \left\{ \sum_{j=1}^{n_1} (\mathbf{x}_{(12)j} - \bar{\mathbf{x}}_{(12)F})(\mathbf{x}_{(12)j} - \bar{\mathbf{x}}_{(12)F})' \right\}^{-1}, \\ \hat{\Psi}_{33} &= \frac{1}{n_1} \sum_{j=1}^{n_1} (\mathbf{x}_{3j} - \hat{\Psi}_{3(12)} \mathbf{x}_{(12)j} - \hat{\boldsymbol{\eta}}_3)(\mathbf{x}_{3j} - \hat{\Psi}_{3(12)} \mathbf{x}_{(12)j} - \hat{\boldsymbol{\eta}}_3)',\end{aligned}$$

respectively.

Proof. The likelihood function for the parameters $\boldsymbol{\eta}$ and Ψ can be written as

$$\begin{aligned}L(\boldsymbol{\eta}, \Psi) &= \prod_{j=1}^n \frac{1}{(2\pi)^{p_1/2} |\Psi_{11}|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x}_{1j} - \boldsymbol{\eta}_1)' \Psi_{11}^{-1} (\mathbf{x}_{1j} - \boldsymbol{\eta}_1) \right\} \\ &\quad \times \prod_{j=1}^n \frac{1}{(2\pi)^{p_2/2} |\Psi_{22}|^{1/2}} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\mathbf{x}_{2j} - \Psi_{21} \mathbf{x}_{1j} - \boldsymbol{\eta}_2)' \Psi_{22}^{-1} (\mathbf{x}_{2j} - \Psi_{21} \mathbf{x}_{1j} - \boldsymbol{\eta}_2) \right\} \\ &\quad \times \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p_3/2} |\Psi_{33}|^{1/2}} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\mathbf{x}_{3j} - \Psi_{3(12)} \mathbf{x}_{(12)j} - \boldsymbol{\eta}_3)' \Psi_{33}^{-1} (\mathbf{x}_{3j} - \Psi_{3(12)} \mathbf{x}_{(12)j} - \boldsymbol{\eta}_3) \right\}.\end{aligned}$$

Then, the partial derivative of $\log L(\boldsymbol{\eta}, \Psi)$ with respect to Ψ_{11} (see Seber (1984,

p.530)) is

$$\frac{\partial \log L(\boldsymbol{\eta}, \Psi)}{\partial \Psi_{11}} = -\frac{n}{2} \Psi_{11}^{-1} + \frac{1}{2} \sum_{j=1}^n \Psi_{11}^{-1} (\mathbf{x}_{1j} - \boldsymbol{\eta}_1)(\mathbf{x}_{1j} - \boldsymbol{\eta}_1)' \Psi_{11}^{-1}.$$

Thus, by solving $\partial \log L(\boldsymbol{\eta}, \Psi) / \partial \Psi_{11} = 0$ we obtain

$$\hat{\Psi}_{11} = \frac{1}{n} \sum_{j=1}^n (\mathbf{x}_{1j} - \boldsymbol{\mu}_{10})(\mathbf{x}_{1j} - \boldsymbol{\mu}_{10})'.$$

Similarly, the partial derivative of $\log L(\boldsymbol{\eta}, \Psi)$ with respect to Ψ_{21} is

$$\begin{aligned} \frac{\partial \log L(\boldsymbol{\eta}, \Psi)}{\partial \Psi_{21}} &= \sum_{j=1}^n \{ \Psi_{22}^{-1} (\mathbf{x}_{2j} - \bar{\mathbf{x}}_{2T})(\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})' \\ &\quad - \Psi_{22}^{-1} \Psi_{21} (\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})(\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})' \}. \end{aligned}$$

Thus, by solving $\partial \log L(\boldsymbol{\eta}, \Psi) / \partial \Psi_{21} = 0$ we obtain

$$\hat{\Psi}_{21} = \left\{ \sum_{j=1}^n (\mathbf{x}_{2j} - \bar{\mathbf{x}}_{2T})(\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})' \right\} \left\{ \sum_{j=1}^n (\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})(\mathbf{x}_{1j} - \bar{\mathbf{x}}_{1T})' \right\}^{-1}.$$

In the same manner as for Ψ_{11} and Ψ_{21} , we solve the equations resulting from setting the partial derivative of $\log L(\boldsymbol{\eta}, \Psi)$ with respect to each of $\boldsymbol{\eta}_2$, $\boldsymbol{\eta}_3$, Ψ_{22} , $\Psi_{3(12)}$, and Ψ_{33} to zero, and obtain the MLEs. \square

Then, the MLEs of $\boldsymbol{\mu}_{(23)}$ and Σ are expressed as

$$\begin{aligned} \hat{\boldsymbol{\mu}}_{(23)} &= \begin{pmatrix} \bar{\mathbf{x}}_{2T} - \hat{\Sigma}_{21} \hat{\Sigma}_{11}^{-1} (\bar{\mathbf{x}}_{1T} - \boldsymbol{\mu}_{10}) \\ \bar{\mathbf{x}}_{3T} - \hat{\Sigma}_{3(12)} \hat{\Sigma}_{(12)(12)}^{-1} \begin{pmatrix} \bar{\mathbf{x}}_{1F} - \boldsymbol{\mu}_{10} \\ \bar{\mathbf{x}}_{2F} - \hat{\boldsymbol{\mu}}_2 \end{pmatrix} \end{pmatrix}, \\ \hat{\Psi} &= \left(\begin{array}{cc|c} \hat{\Sigma}_{11} & \hat{\Sigma}_{12} & \hat{\Sigma}_{13} \\ \hat{\Sigma}_{21} & \hat{\Sigma}_{22} & \hat{\Sigma}_{23} \\ \hline \hat{\Sigma}_{31} & \hat{\Sigma}_{32} & \hat{\Sigma}_{33} \end{array} \right) = \begin{pmatrix} \hat{\Sigma}_{(12)(12)} & \hat{\Sigma}_{(12)3} \\ \hat{\Sigma}_{3(12)} & \hat{\Sigma}_{33} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \hat{\Sigma}_{(12)(12)} &= \begin{pmatrix} \hat{\Psi}_{11} & \hat{\Psi}_{11} \hat{\Psi}_{12} \\ \hat{\Psi}_{21} \hat{\Psi}_{11} & \hat{\Psi}_{22} + \hat{\Psi}_{21} \hat{\Psi}_{11} \hat{\Psi}_{12} \end{pmatrix}, \quad \hat{\Sigma}_{3(12)} = \hat{\Sigma}'_{(12)3} = \hat{\Psi}_{3(12)} \hat{\Sigma}_{(12)(12)}, \\ \hat{\Sigma}_{33} &= \hat{\Psi}_{33} + \hat{\Psi}_{3(12)} \hat{\Psi}_{(12)(12)} \hat{\Psi}_{(12)3}. \end{aligned}$$

Next, we represent the MLEs under H_0 in order to obtain the LRT statistic. The null hypothesis in (1.1) can be written as $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 (= (\boldsymbol{\mu}'_{10}, \boldsymbol{\mu}'_{20}, \boldsymbol{\mu}'_{30})' = (\boldsymbol{\mu}'_{(12)0}, \boldsymbol{\mu}'_{30})')$. Let $\mathbf{x}_j = (\mathbf{x}'_{(12)j}, \mathbf{x}'_{3j})'$ be distributed as $N_p(\boldsymbol{\mu}_0, \Sigma)$, $j = 1, 2, \dots, n_1$, and $\mathbf{x}_{(12)j}$ be distributed as $N_{p_1+p_2}(\boldsymbol{\mu}_{(12)0}, \Sigma_{(12)(12)})$, for $j = n_1 + 1, n_1 + 2, \dots, n$. Then, the likelihood function is given by

$$\begin{aligned} L(\boldsymbol{\mu}_0, \Sigma) &= \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x}_j - \boldsymbol{\mu}_0)' \Sigma^{-1} (\mathbf{x}_j - \boldsymbol{\mu}_0) \right\} \\ &\quad \times \prod_{j=n_1+1}^n \frac{1}{(2\pi)^{(p_1+p_2)/2} |\Sigma_{(12)(12)}|^{1/2}} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\mathbf{x}_{(12)j} - \boldsymbol{\mu}_{(12)0})' \Sigma_{(12)(12)}^{-1} (\mathbf{x}_{(12)j} - \boldsymbol{\mu}_{(12)0}) \right\}. \end{aligned}$$

By multiplying the observation vectors by Γ_1 on the left side, we obtain

$$\begin{aligned} \mathbf{x}_{(12)j} &\sim N_{p_1+p_2}(\boldsymbol{\xi}_{(12)}, \Phi_{(12)(12)}), \quad j = 1, 2, \dots, n, \\ \mathbf{x}_{3j} - \Phi_{3(12)} \mathbf{x}_{(12)j} &\sim N_{p_3}(\boldsymbol{\xi}_3, \Phi_{33}), \quad j = 1, 2, \dots, n_1, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\xi} &= \begin{pmatrix} \boldsymbol{\xi}_{(12)} \\ \boldsymbol{\xi}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)0} \\ \boldsymbol{\mu}_{30} - \Sigma_{3(12)} \Sigma_{(12)(12)}^{-1} \boldsymbol{\mu}_{(12)0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_{(12)0} \\ \boldsymbol{\mu}_{30} - \Phi_{3(12)} \boldsymbol{\mu}_{(12)0} \end{pmatrix}, \\ \Phi &= \begin{pmatrix} \Phi_{(12)(12)} & \Phi_{(12)3} \\ \Phi_{3(12)} & \Phi_{33} \end{pmatrix} = \begin{pmatrix} \Sigma_{(12)(12)} & \Sigma_{(12)(12)}^{-1} \Sigma_{(12)3} \\ \Sigma_{3(12)} \Sigma_{(12)(12)}^{-1} & \Sigma_{33 \cdot (12)} \end{pmatrix}. \end{aligned}$$

These have a one-to-one correspondence with $\boldsymbol{\mu}_0$ and Σ . For the parameters $\boldsymbol{\xi}$ and Φ , the likelihood function can be written as

$$\begin{aligned} L(\boldsymbol{\xi}, \Phi) &= \prod_{j=1}^n \frac{1}{(2\pi)^{(p_1+p_2)/2} |\Phi_{(12)(12)}|^{1/2}} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\mathbf{x}_{(12)j} - \boldsymbol{\xi}_{(12)})' \Phi_{(12)(12)}^{-1} (\mathbf{x}_{(12)j} - \boldsymbol{\xi}_{(12)}) \right\} \\ &\quad \times \prod_{j=1}^{n_1} \frac{1}{(2\pi)^{p_3/2} |\Phi_{33}|^{1/2}} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\mathbf{x}_{3j} - \Phi_{3(12)} \mathbf{x}_{(12)j} - \boldsymbol{\xi}_3)' \Phi_{33}^{-1} (\mathbf{x}_{3j} - \Phi_{3(12)} \mathbf{x}_{(12)j} - \boldsymbol{\xi}_3) \right\}. \end{aligned}$$

Similarly, as Theorem 1, we have the following Corollary. Note that $\Phi_{3(12)}$ and Φ_{33} correspond to the $\Psi_{3(12)}$ and Ψ_{33} of Theorem 1, respectively.

Corollary 1. *Suppose that the data have a two-step monotone missing pattern. The maximum likelihood estimators of $\xi_3, \Phi_{(12)(12)}, \Phi_{3(12)}$ and Φ_{33} under H_0 are given by*

$$\begin{aligned}\tilde{\xi}_3 &= \mu_{30} - \tilde{\Phi}_{3(12)} \mu_{(12)0}, \quad \tilde{\Phi}_{(12)(12)} = \frac{1}{n} \sum_{j=1}^n (\mathbf{x}_{(12)j} - \mu_{(12)0})(\mathbf{x}_{(12)j} - \mu_{(12)0})', \\ \tilde{\Phi}_{3(12)} &= \left\{ \sum_{j=1}^{n_1} (\mathbf{x}_{3j} - \mu_{30})(\mathbf{x}_{(12)j} - \mu_{(12)0})' \right\} \\ &\quad \times \left\{ \sum_{j=1}^{n_1} (\mathbf{x}_{(12)j} - \mu_{(12)0})(\mathbf{x}_{(12)j} - \mu_{(12)0})' \right\}^{-1}, \\ \tilde{\Phi}_{33} &= \frac{1}{n_1} \sum_{j=1}^{n_1} (\mathbf{x}_{3j} - \tilde{\Phi}_{3(12)} \mathbf{x}_{(12)j} - \tilde{\xi}_3)(\mathbf{x}_{3j} - \tilde{\Phi}_{3(12)} \mathbf{x}_{(12)j} - \tilde{\xi}_3)',\end{aligned}$$

respectively.

§4. Likelihood ratio test

In this section, we derive the LRT statistic for testing the subvector of a mean vector with two-step monotone missing data. In the hypothesis in (1.1), the parameter space Ω and the subspace ω when H_0 holds, respectively, are as follows:

$$\begin{aligned}\Omega &= \{(\mu, \Sigma) : -\infty < \mu_i < \infty, i = p_1 + 1, p_1 + 2, \dots, p, \mu_1 = \mu_{10}, \\ &\quad \Sigma > 0 \text{ and } \Sigma_{(23)(23)} > 0\}, \\ \omega &= \{(\mu, \Sigma) : \mu = \mu_0, \Sigma > 0 \text{ and } \Sigma_{(23)(23)} > 0\},\end{aligned}$$

where $\Sigma > 0$ and $\Sigma_{(23)(23)} > 0$ indicate that Σ and $\Sigma_{(23)(23)}$ are positive definite matrices. We note that $L(\tilde{\eta}, \tilde{\Psi}) = L(\tilde{\xi}, \tilde{\Phi})$, where $\tilde{\eta}$ and $\tilde{\Psi}$ are the MLEs of η and Ψ under H_0 . We have that $|\Phi_{(12)(12)}| = |\Psi_{11}| \cdot |\Psi_{22}|$ by Siotani, Hayakawa, and Fujikoshi (1985, p.591). Therefore, using the MLEs in Section 2, the likelihood ratio criterion is given by

$$\lambda_M = \frac{\max_{\omega} L(\mu, \Sigma)}{\max_{\Omega} L(\mu, \Sigma)} = \lambda_{M(12)}^{\frac{n}{2}} \cdot \lambda_{M3}^{\frac{n_1}{2}},$$

where $\lambda_{M(12)} = |\hat{\Psi}_{11}| \cdot |\hat{\Psi}_{22}| / |\tilde{\Phi}_{(12)(12)}|$ and $\lambda_{M3} = |\hat{\Psi}_{33}| / |\tilde{\Phi}_{33}|$ are independent.

Next, we consider the null distribution of $-2 \log \lambda_M$. The characteristic function of $-2 \log \lambda_M$ can be written as

$$\mathbb{E}[e^{it(-2 \log \lambda_M)}] = \mathbb{E}[\lambda_M^{-2it}] = \mathbb{E}[\lambda_{M(12)}^{-itn} \cdot \lambda_{M3}^{-itn_1}].$$

We set

$$\begin{aligned} \mathbf{z}_{(23)F} &= (\mathbf{z}'_{2F}, \mathbf{z}'_{3F})' = \sqrt{n_1} \begin{pmatrix} \bar{\mathbf{x}}_{2F} - \boldsymbol{\mu}_2 \\ \bar{\mathbf{x}}_{3F} - \boldsymbol{\mu}_3 \end{pmatrix}, \quad V_F = \sqrt{n_1 - 1}(S_F - I_p), \\ \mathbf{z}_{2L} &= \sqrt{n_2}(\bar{\mathbf{x}}_{2L} - \boldsymbol{\mu}_2), \quad V_L = \sqrt{n_2 - 1}(S_L - I_{p(12)}), \end{aligned}$$

where $p_{(12)} = p_1 + p_2$, $\mathbf{x}_{(23)j} = (\mathbf{x}'_{2j}, \mathbf{x}'_{3j})'$, and the following hold:

$$\begin{aligned} \bar{\mathbf{x}}_{(23)F} &= \frac{1}{n_1} \sum_{j=1}^{n_1} \mathbf{x}_{(23)j}, \quad \bar{\mathbf{x}}_L = (\bar{\mathbf{x}}'_{1L}, \bar{\mathbf{x}}'_{2L})' = \left(\frac{1}{n_2} \sum_{j=n_1+1}^n \mathbf{x}'_{1j}, \frac{1}{n_2} \sum_{j=n_1+1}^n \mathbf{x}'_{2j} \right)', \\ S_F &= \frac{1}{n_1 - 1} \sum_{j=1}^{n_1} (\mathbf{x}_j - \bar{\mathbf{x}}_F)(\mathbf{x}_j - \bar{\mathbf{x}}_F)', \\ S_L &= \frac{1}{n_2 - 1} \sum_{j=n_1+1}^n (\mathbf{x}_{(12)j} - \bar{\mathbf{x}}_L)(\mathbf{x}_{(12)j} - \bar{\mathbf{x}}_L)'. \end{aligned}$$

Then, we can obtain the expansions of $\mathbb{E}[\lambda_{M(12)}^{-itn}]$ and $\mathbb{E}[\lambda_{M3}^{-itn_1}]$ as follows:

$$\begin{aligned} \mathbb{E}[\lambda_{M(12)}^{-itn}] &= \mathbb{E}[e^{-itn \log \lambda_{M(12)}}] = \mathbb{E}[e^{it(\mathbf{z}'_{2T} \mathbf{z}_{2T} + O_p(n^{-\frac{1}{2}}))}] = (1 - 2it)^{-\frac{p_2}{2}} + O(n^{-1}), \\ \mathbb{E}[\lambda_{M3}^{-itn_1}] &= \mathbb{E}[e^{-itn_1 \log \lambda_{M3}}] = \mathbb{E}[e^{it(\mathbf{z}'_{3F} \mathbf{z}_{3F} + O_p(n_1^{-\frac{1}{2}}))}] = (1 - 2it)^{-\frac{p_3}{2}} + O(n_1^{-1}), \end{aligned}$$

where $\mathbf{z}_{2T} = \sqrt{n}(\bar{\mathbf{x}}_{2T} - \boldsymbol{\mu}_2)$. Thus, we have that

$$\mathbb{E}[e^{it(-2 \log \lambda_M)}] = (1 - 2it)^{-\frac{p_2 + p_3}{2}} + O(n^{-1}).$$

From the above that under the null hypothesis with $\Sigma = I_p$, the LRT statistic $-2 \log \lambda_M$ is asymptotically distributed as χ^2 with $p_2 + p_3$ degrees of freedom, when $n_1, n_2 \rightarrow \infty$ with $n_i/n \rightarrow \delta \in (0, 1]$, $i = 1, 2$. Even for the case of general Σ it should be possible to prove that this holds in a similar manner. However, this becomes very complicated, and is left as a problem for a future study.

However, the upper percentile of the χ^2 distribution is not a good approximation to that of the LRT statistic when the sample size is not large. We will consider an approximate upper percentile of the LRT statistic, because the exact one is not easy to obtain. In this paper, we present a simple approximation using the $n_1 \times p$ and $n \times p$ complete data sets (see, e.g., Seko et al.

(2012)). As in Section 1, we make use of a property that is present in the case of complete data. That is, the exact upper 100α percentile of λ is given by

$$q_n(\alpha) = \left\{ 1 + \frac{(p_2 + p_3)F_{p_2+p_3, n-p}(\alpha)}{n-p} \right\}^{-\frac{n}{2}},$$

and $F_{a,b}(\alpha)$ is the upper 100α percentile of the F distribution with a and b degrees of freedom. Thus, we can formulate an approximate upper 100α percentile of the LRT statistic $-2\log \lambda_M$ as

$$q_M^*(\alpha) = -2\log \left\{ \frac{p_3}{p} q_{n_1}(\alpha) + \frac{p_1 + p_2}{p} q_n(\alpha) \right\},$$

where

$$q_{n_1}(\alpha) = \left\{ 1 + \frac{(p_2 + p_3)F_{p_2+p_3, n_1-p}(\alpha)}{n_1-p} \right\}^{-\frac{n_1}{2}}.$$

Therefore, we reject H_0 if $-2\log \lambda_M > q_M^*(\alpha)$. In the next section, the accuracy and asymptotic behavior of the approximation are investigated using Monte Carlo simulation.

§5. Simulation studies

In this section, we compute the upper 100α percentiles of the LRT statistic $q_{\text{sim}}(\alpha)$ using Monte Carlo simulation for $\alpha = 0.05$ and 0.01 . We generate artificial two-step missing data from $N_p(\mathbf{0}, I_p)$ for various conditions of p_1, p_2, p_3, n_1 , and n_2 . We simulate the upper percentiles of the LRT statistic, $q_M^*(\alpha)$, and the type I error rates under the simulated LRT statistic when the null hypothesis is rejected using $q_M^*(\alpha)$ and $\chi_{p_2+p_3}^2$, where

$$P_{q^*} = \Pr\{-2\log \lambda_M > q_M^*(\alpha)\}, \quad P_c = \Pr\{-2\log \lambda_M > \chi_{p_2+p_3}^2(\alpha)\},$$

and $\chi_f^2(\alpha)$ is the upper 100α percentile of the χ^2 distribution with f degrees of freedom. In Tables 1-4, we present the simulation results for the following

four cases:

- Case I : $(p_1, p_2, p_3) = (2, 2, 4), (2, 3, 3), (2, 4, 2),$
 $(n_1, n_2) = (n_1, 2n_1), (n_1, n_1), (n_1, n_1/2), n_1 = 20, 40, 80, 160;$
- Case II : $(p_1, p_2, p_3) = (2, 2, 4), (2, 3, 3), (2, 4, 2),$
 $(n_1, n_2), n_1 = 20, 40, 80, 160, n_2 = 10, 20, 40;$
- Case III : $(p_1, p_2, p_3) = (2, 2, 2), (4, 2, 2), (8, 2, 2),$
 $(n_1, n_2) = (n_1, 2n_1), (n_1, n_1), (n_1, n_1/2), n_1 = 20, 40, 80, 160;$
- Case IV : $(p_1, p_2, p_3) = (2, 2, 4), (4, 3, 3), (6, 2, 2),$
 $(n_1, n_2), n_1 = 20, 40, 80, 160, n_2 = 10, 20, 40.$

We note that the cases for $p = 8$ and $p_1 = 2$ are given in Tables 1 and 2. That is, the values of p and p_1 are fixed. Furthermore, Tables 3 and 4 present the case where $p_2 = p_3$, and p_2 and p_3 are fixed.

From Tables 1 and 2, we can see that the proposed approximation $q_M^*(\alpha)$ provides a good result in the case that the sample sizes n_1 and n_2 are large or the sample size n_1 is large and n_2 is fixed. Our results also indicate that the type I error rate is close to α when the sample size n_1 is large. From Tables 3 and 4, we can see that the approximation $q_M^*(\alpha)$ is good in the case that $p_2 = p_3 = 2$ and the sample size n_1 is large. It can be seen from Tables 3 and 4 that the value of $q_M^*(\alpha)$ is close to that of the LRT when p_1 is small. However, we note that the proposed approximation performs better than the χ^2 approximation for all cases.

In addition, we used Monte Carlo simulation for some selected parameters to estimate the powers of the LRT based on two-step monotone missing data and the LRT based on partially complete data of $n_1 \times p$. In the case that the type I error is close to α , each part of the data is set to the same degree. We expected the results for the powers of the LRT based on $q_M^*(\alpha)$ to be larger than the corresponding powers of the LRT based on $q_{n_1}(\alpha)$. Because the type I error is not stable, the power of the LRT based on $\chi_{p_2+p_3}^2(\alpha)$ is not comparing. We note that the upper 100α percentile of the $\chi_{p_2+p_3}^2$ is smaller than q_M^* , the power becomes large. This should be investigated in further detail using Monte Carlo simulation. Furthermore, we plan to discuss the power in a theoretical context in future work. In particular, we will consider the non-null distribution under local alternatives.

TABLE 1 : p_1 and p are fixed, and $\alpha = 0.05, 0.01$

		$\alpha=0.05$				$\alpha=0.01$			
n_1	n_2	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 4)$									
20	40	17.69	15.17	.093	.171	23.81	19.99	.028	.062
40	80	14.54	13.90	.061	.091	19.50	18.50	.014	.024
80	160	13.48	13.24	.054	.067	18.06	17.66	.011	.016
160	320	13.01	12.91	.052	.058	17.40	17.24	.011	.012
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20	20	17.83	15.92	.080	.176	24.00	21.05	.022	.064
40	40	14.61	14.15	.058	.093	19.52	18.85	.013	.025
80	80	13.52	13.33	.053	.068	18.05	17.80	.011	.016
160	160	13.02	12.95	.051	.058	17.43	17.29	.011	.013
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20	10	17.93	16.71	.068	.180	23.98	22.21	.016	.066
40	20	14.69	14.39	.055	.095	19.63	19.20	.012	.025
80	40	13.54	13.43	.052	.069	18.10	17.93	.011	.016
160	80	13.06	13.00	.051	.059	17.43	17.35	.010	.013
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$(p_1, p_2, p_3) = (2, 3, 3)$									
20	40	17.09	14.81	.090	.154	23.07	19.58	.026	.054
40	80	14.32	13.72	.060	.086	19.16	18.27	.014	.022
80	160	13.35	13.15	.054	.065	17.82	17.55	.011	.015
160	320	12.96	12.87	.052	.057	17.31	17.18	.010	.012
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20	20	17.37	15.59	.079	.163	23.35	20.68	.021	.058
40	40	14.46	14.00	.058	.089	19.31	18.66	.013	.023
80	80	13.43	13.27	.053	.067	17.95	17.71	.011	.015
160	160	13.00	12.92	.051	.058	17.38	17.25	.010	.012
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20	10	17.62	16.45	.067	.171	23.66	21.90	.016	.061
40	20	14.56	14.29	.055	.092	19.45	19.06	.011	.024
80	40	13.49	13.39	.052	.068	18.00	17.87	.010	.015
160	80	13.01	12.98	.051	.058	17.35	17.33	.010	.012
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$(p_1, p_2, p_3) = (2, 4, 2)$									
20	40	16.34	14.51	.081	.134	22.11	19.25	.022	.044
40	80	14.06	13.55	.059	.080	18.78	18.06	.013	.020
80	160	13.24	13.07	.053	.063	17.67	17.44	.011	.014
160	320	12.89	12.83	.051	.055	17.26	17.13	.011	.012
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20	20	16.74	15.32	.073	.146	22.55	20.36	.019	.049
40	40	14.23	13.87	.056	.084	19.02	18.49	.012	.021
80	80	13.34	13.21	.052	.064	17.85	17.63	.011	.015
160	160	12.93	12.89	.051	.056	17.31	17.21	.010	.012
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20	10	17.24	16.22	.065	.159	23.21	21.62	.015	.056
40	20	14.44	14.19	.054	.089	19.27	18.94	.011	.023
80	40	13.43	13.34	.051	.067	17.92	17.81	.010	.015
160	80	12.98	12.96	.050	.057	17.33	17.30	.010	.012

Note : $\chi_6^2(0.05) = 12.59$, $\chi_6^2(0.01) = 16.81$

TABLE 2 : p_1, p and n_2 are fixed, and $\alpha = 0.05, 0.01$

		$\alpha=0.05$				$\alpha=0.01$			
n_1	n_2	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 4)$									
20	10	17.93	16.71	.068	.180	23.98	22.21	.016	.066
40	10	14.75	14.58	.053	.097	19.72	19.47	.011	.026
80	10	13.58	13.56	.050	.070	18.13	18.10	.010	.016
160	10	13.08	13.07	.050	.059	17.50	17.45	.010	.013
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20	20	17.83	15.92	.080	.176	24.00	21.05	.022	.064
40	20	14.69	14.39	.055	.095	19.63	19.20	.012	.025
80	20	13.56	13.51	.051	.069	18.07	18.04	.010	.016
160	20	13.08	13.06	.050	.059	17.43	17.43	.010	.013
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20	40	17.69	15.17	.093	.171	23.81	19.99	.028	.062
40	40	14.61	14.15	.058	.093	19.52	18.85	.013	.025
80	40	13.54	13.43	.052	.069	18.10	17.93	.011	.016
160	40	13.06	13.03	.051	.059	17.42	17.40	.010	.013
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$(p_1, p_2, p_3) = (2, 3, 3)$									
20	10	17.62	16.45	.067	.171	23.66	21.90	.016	.061
40	10	14.68	14.51	.053	.094	19.60	19.38	.011	.025
80	10	13.56	13.54	.050	.069	18.11	18.08	.010	.016
160	10	13.07	13.07	.050	.059	17.47	17.44	.010	.013
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20	20	17.37	15.59	.079	.163	23.35	20.68	.021	.058
40	20	14.56	14.29	.055	.092	19.45	19.06	.011	.024
80	20	13.54	13.48	.051	.069	18.09	18.00	.010	.016
160	20	13.06	13.05	.050	.059	17.43	17.42	.010	.013
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20	40	17.09	14.81	.090	.154	23.07	19.58	.026	.054
40	40	14.46	14.00	.058	.089	19.31	18.66	.013	.023
80	40	13.49	13.39	.052	.068	18.00	17.87	.010	.015
160	40	13.05	13.02	.050	.059	17.40	17.38	.010	.013
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$(p_1, p_2, p_3) = (2, 4, 2)$									
20	10	17.24	16.22	.065	.159	23.21	21.62	.015	.056
40	10	14.58	14.45	.052	.092	19.47	19.30	.011	.024
80	10	13.57	13.53	.051	.070	18.12	18.06	.010	.016
160	10	13.09	13.06	.050	.060	17.51	17.44	.010	.013
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20	20	16.74	15.32	.073	.146	22.55	20.36	.019	.049
40	20	14.44	14.19	.054	.089	19.27	18.94	.011	.023
80	20	13.50	13.45	.051	.068	18.04	17.96	.010	.016
160	20	13.08	13.04	.051	.059	17.49	17.41	.010	.013
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20	40	16.34	14.51	.081	.134	22.11	19.25	.022	.044
40	40	14.23	13.87	.056	.084	19.02	18.49	.012	.021
80	40	13.43	13.34	.051	.067	17.92	17.81	.010	.015
160	40	13.02	13.01	.050	.058	17.41	17.37	.010	.013

Note : $\chi_6^2(0.05) = 12.59$, $\chi_6^2(0.01) = 16.81$

TABLE 3 : $p_2 = p_3$, and $\alpha = 0.05, 0.01$

		$\alpha=0.05$				$\alpha=0.01$			
n_1	n_2	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 2)$									
20	40	11.93	10.88	.070	.109	16.78	15.12	.018	.032
40	80	10.51	10.17	.057	.073	14.72	14.21	.012	.018
80	160	9.96	9.82	.053	.060	13.96	13.74	.011	.013
160	320	9.70	9.65	.051	.055	13.58	13.51	.010	.011
20	20	12.10	11.31	.064	.113	16.98	15.75	.015	.034
40	40	10.57	10.34	.055	.075	14.78	14.45	.011	.018
80	80	10.01	9.90	.052	.061	14.02	13.85	.011	.013
160	160	9.75	9.69	.051	.056	13.61	13.56	.010	.012
20	10	12.29	11.76	.059	.119	17.30	16.42	.013	.037
40	20	10.66	10.51	.053	.077	14.97	14.70	.011	.019
80	40	10.05	9.97	.052	.062	14.06	13.95	.010	.014
160	80	9.75	9.72	.050	.056	13.60	13.61	.010	.011
$(p_1, p_2, p_3) = (4, 2, 2)$									
20	40	13.35	11.23	.091	.146	18.87	15.57	.026	.051
40	80	10.98	10.35	.062	.084	15.39	14.45	.014	.022
80	160	10.13	9.91	.054	.064	14.19	13.86	.011	.014
160	320	9.81	9.70	.052	.057	13.70	13.57	.011	.012
20	20	13.59	11.96	.079	.155	19.19	16.62	.021	.055
40	40	11.06	10.63	.058	.086	15.49	14.85	.013	.022
80	80	10.20	10.03	.053	.066	14.29	14.03	.011	.015
160	160	9.82	9.75	.051	.057	13.75	13.65	.010	.012
20	10	13.94	12.78	.069	.165	19.60	17.82	.017	.060
40	20	11.22	10.91	.056	.091	15.72	15.26	.012	.024
80	40	10.27	10.15	.052	.067	14.40	14.21	.011	.015
160	80	9.86	9.81	.051	.058	13.78	13.73	.010	.012
$(p_1, p_2, p_3) = (8, 2, 2)$									
20	40	18.24	11.99	.170	.271	26.27	16.63	.068	.132
40	80	12.01	10.71	.076	.111	16.91	14.92	.019	.033
80	160	10.55	10.09	.059	.074	14.79	14.10	.013	.018
160	320	9.99	9.78	.054	.061	13.99	13.69	.011	.013
20	20	18.67	13.47	.137	.289	26.83	18.71	.050	.142
40	40	12.28	11.23	.069	.118	17.28	15.67	.017	.036
80	80	10.64	10.30	.057	.076	14.91	14.41	.012	.018
160	160	10.03	9.88	.053	.062	14.03	13.83	.011	.014
20	10	19.33	15.38	.106	.311	27.60	21.42	.033	.158
40	20	12.47	11.81	.062	.125	17.48	16.50	.014	.039
80	40	10.77	10.53	.055	.079	15.06	14.73	.011	.020
160	80	10.07	9.98	.052	.063	14.14	13.97	.011	.014

Note : $\chi_4^2(0.05) = 9.49$, $\chi_4^2(0.01) = 13.28$

TABLE 4 : n_2 is fixed, $p_2 = p_3$, and $\alpha = 0.05, 0.01$

		$\alpha=0.05$				$\alpha=0.01$			
n_1	n_2	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c	$q_{\text{sim}}(\alpha)$	$q_{\text{M}}^*(\alpha)$	P_{q^*}	P_c
$(p_1, p_2, p_3) = (2, 2, 2)$									
20	10	12.29	11.76	.059	.119	17.30	16.42	.013	.037
40	10	10.73	10.65	.052	.078	15.04	14.90	.011	.020
80	10	10.11	10.07	.051	.063	14.16	14.10	.010	.014
160	10	9.79	9.78	.050	.056	13.69	13.69	.010	.012
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20	20	12.10	11.31	.064	.113	16.98	15.75	.015	.034
40	20	10.66	10.51	.053	.077	14.97	14.70	.011	.019
80	20	10.06	10.03	.050	.062	14.10	14.04	.010	.014
160	20	9.77	9.77	.050	.056	13.68	13.67	.010	.012
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20	40	11.93	10.88	.070	.109	16.78	15.12	.018	.032
40	40	10.57	10.34	.055	.075	14.78	14.45	.011	.018
80	40	10.05	9.97	.052	.062	14.06	13.95	.010	.014
160	40	9.79	9.75	.051	.056	13.70	13.65	.010	.012
<hr/>									
$(p_1, p_2, p_3) = (4, 2, 2)$									
20	10	13.94	12.78	.069	.165	19.60	17.82	.017	.060
40	10	11.34	11.15	.054	.094	15.86	15.60	.011	.025
80	10	10.34	10.32	.050	.069	14.49	14.44	.010	.016
160	10	9.90	9.90	.050	.059	13.83	13.86	.010	.013
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20	20	13.59	11.96	.079	.155	19.19	16.62	.021	.055
40	20	11.22	10.91	.056	.091	15.72	15.26	.012	.024
80	20	10.32	10.25	.051	.068	14.44	14.34	.010	.016
160	20	9.89	9.88	.050	.059	13.84	13.83	.010	.013
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20	40	13.35	11.23	.091	.146	18.87	15.57	.026	.051
40	40	11.06	10.63	.058	.086	15.49	14.85	.013	.022
80	40	10.27	10.15	.052	.067	14.40	14.21	.011	.015
160	40	9.88	9.85	.051	.059	13.82	13.79	.010	.012
<hr/>									
$(p_1, p_2, p_3) = (8, 2, 2)$									
20	10	19.33	15.38	.106	.311	27.60	21.42	.033	.158
40	10	12.75	12.30	.057	.132	17.88	17.21	.012	.043
80	10	10.92	10.84	.051	.083	15.30	15.17	.011	.021
160	10	10.16	10.15	.050	.065	14.23	14.21	.010	.015
<hr/>									
20	20	18.67	13.47	.137	.289	26.83	18.71	.050	.142
40	20	12.47	11.81	.062	.125	17.48	16.50	.014	.039
80	20	10.84	10.72	.052	.081	15.19	14.99	.011	.020
160	20	10.15	10.12	.051	.064	14.18	14.16	.010	.015
<hr/>									
20	40	18.24	11.99	.170	.271	26.27	16.63	.068	.132
40	40	12.28	11.23	.069	.118	17.28	15.67	.017	.036
80	40	10.77	10.53	.055	.079	15.06	14.73	.011	.020
160	40	10.13	10.06	.051	.064	14.21	14.08	.010	.014

Note : $\chi_4^2(0.05) = 9.49$, $\chi_4^2(0.01) = 13.28$

§6. Numerical example

Now, we illustrate the results of this study using an example given in Wei and Lachin (1984). The sample data set consists of serum cholesterol values that were measured under the treatment group at five different time points: the baseline and at months 6, 12, 20, and 24. The original data set contains 36 complete observations, and we create two-step monotone missing data by randomly selecting 30 observations and deleting the values for 10 observations for each of the months 20 and 24. Thus, we have $n = 30$, $n_1 = 20$, $n_2 = 10$, $p = 5$, $p_1 = 1$, and $p_2 = p_3 = 2$. We are interested in the change from the baseline at each post-baseline time point. It is known that the mean for all baseline value was 220. We consider the hypothesis $H : (\mu_2, \mu_3, \mu_4, \mu_5)' = (220, 220, 220, 220)'$, given $\mu_1 = 220$. Then, we compute $-2\log \lambda_M = 10.92$. Because we have $q_{\text{sim}}(0.05) = 11.63$ from the simulation study, we do not reject the null hypothesis at the 0.05 significance level. Moreover, when we use $q_M^*(0.05) = 11.30$, the null hypothesis is not rejected. When we use $\chi_{4(0.05)}^2 = 9.49$, the null hypothesis is rejected. However, $q_{\text{sim}}(0.01) = 16.32$ from the simulation study, and the null hypothesis is rejected at the 0.01 significance level. When we use $q_M^*(0.01) = 15.79$ or $\chi_{4(0.01)}^2 = 13.28$, the null hypothesis is also rejected.

§7. Concluding remarks

In this paper, we have considered the one-sample problem of testing for the subvector of a mean vector with two-step monotone missing data. First, we provided an introduction to two-step monotone missing data. Then, we reviewed the test for the subvector of a mean vector with non-missing data. In the case that the data set consists of complete data with p dimensions and incomplete data with $(p_1 + p_2)$ dimensions, we derived the likelihood ratio criterion for testing the $(p_2 + p_3)$ mean vector under the given mean vector of p_1 dimensions, which is given by (1.1). This test procedure only treats the $(p_2 + p_3)$ -components as if observations are present. Next, we derived the MLEs, and provided the LRT statistic and the approximate upper 100α percentiles of the LRT, $q_M^*(\alpha)$, for a subvector. The approximate values can easily be calculated, and the simulation results suggest that the type I error rates are close to α when the sample size n_1 is large. In all cases, it appears that the approximate upper 100α percentiles $q_M^*(\alpha)$ are preferable to $\chi_{p_2+p_3}^2$.

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