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Factorization technique for the modified Korteweg-de Vries equation

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Abstract. We study the large time asymptotics of solutions to the Cauchy problem for the modified Korteweg-de Vries equation

$$\begin{cases} \partial_t u - \frac{1}{3}\partial_x^3 u = \partial_x(u^3),\ t > 0,\ x \in \mathbf{R}, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}. \end{cases}$$

We develop the factorization technique to obtain the large time asymptotics of solutions in the neighborhood of the self-similar solution in the case of nonzero total mass initial data. Our result is an improvement of the previous work [18].

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§1. Introduction

We study the large time asymptotics of solutions to the Cauchy problem for the modified Korteweg-de Vries (KdV) equation

$$\begin{cases} \partial_t u - \frac{1}{3}\partial_x^3 u = \partial_x\left(u^3\right),\ t > 0,\ x \in \mathbf{R}, \\ u(0,x) = u_0(x),\ x \in \mathbf{R} \end{cases} \tag{1.1}$$

with nonzero total mass initial data $\int_{\mathbf{R}} u_0(x)\,dx \neq 0$.

Cauchy problem (1.1) was intensively studied by many authors. The existence and uniqueness of solutions to (1.1) were proved in [10], [12], [19], [20], [21], [22], [25], [27], [30], [33] and the smoothing properties of solutions were studied in [3], [5], [6], [19], [20], [21], [22], [30]. The blow-up effect for a special class of slowly decaying solutions of the Cauchy problem (1.1) was found in [2].

The large time asymptotics of solutions to the generalized Korteweg-de Vries equation $u_t - \frac{1}{3}u_{xxx} = \partial_x |u|^{\rho-1} u$ was studied in [4], [16], [21], [23], [24], [26], [28], [29], [31], [32] for different values of ρ in the super critical region $\rho > 3$. For the special cases of the KdV equation itself $u_t - \frac{1}{3}u_{xxx} = \partial_x \left(u^2\right)$ and of the modified KdV equation (1.1), the Cauchy problem was solved by the Inverse Scattering Transform (IST) method and thus the large time asymptotic behavior of solutions was studied (see [1], [7]). Thus the solutions of the modified KdV equation (1.1) decay with the same speed as in the corresponding linear case, i.e. $\|u(t)\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{3}}$ as $t \to \infty$ (see [7]). The IST method depends essentially on the nonlinearity in the equation. It is not applicable if we slightly change the nonlinear term. So it is very important to develop alternative methods for studying the large time asymptotics of solutions to the Cauchy problem (1.1).

In [17] we obtained the large time asymptotics of solutions to (1.1) in the case of small real-valued initial data $u_0 \in \mathbf{H}^{1,1}$ with zero total mass assumption $\int_{\mathbf{R}} u_0(x)dx = 0$. More precisely we have the asymptotics

$$
\begin{aligned}
u(t,x) &= \sqrt{2\pi}t^{-\frac{1}{3}}\mathrm{Re}Ai\left(xt^{-\frac{1}{3}}\right)W\left(xt^{-1}\right)\exp\left(-3i\pi\left|W\left(xt^{-1}\right)\right|^2\log t\right) \\
&\quad +O\left(t^{-\frac{1}{2}-\lambda}\right) \qquad (1.2)
\end{aligned}
$$

for large time t, where $0 < \lambda < \frac{1}{21}$, $W \in \mathbf{L}^\infty$ is uniquely defined by the data u_0, is such that $W(0) = 0$, and $Ai(x) = \frac{1}{\pi}\int_0^\infty e^{ix\xi - \frac{i}{3}\xi^3}d\xi$ is the Airy function.

It is known by [8], that the following asymptotics for Airy function

$$
Ai(\eta) = C|\eta|^{-\frac{1}{4}}\exp\left(-\frac{2}{3}i|\eta|^{\frac{3}{2}} + i\frac{\pi}{4}\right) + O\left(|\eta|^{-7/4}\right) \text{ as } \eta = xt^{-\frac{1}{3}} \to \infty
$$

is valid. Airy function oscillates rapidly and decays slowly as $x \to \infty$, When $x \to -\infty$, $Ai(\eta)$ decays exponentially as

$$
Ai(\eta) = C|\eta|^{-\frac{1}{4}}e^{-\frac{2}{3}|\eta|^{\frac{3}{2}}} + O\left(|\eta|^{-7/4}e^{-\frac{2}{3}|\eta|^{\frac{3}{2}}}\right) \text{ as } \eta = xt^{-\frac{1}{3}} \to -\infty.
$$

Therefore, the first term of the right-hand side of (1.2) decays in time like $t^{-\frac{1}{2}}$ since $W(0) = 0$ and the estimate

$$
\begin{aligned}
|u(t,x)| &\le Ct^{-\frac{1}{3}}\left|Ai(\eta)W\left(\eta t^{-\frac{2}{3}}\right)\right| \\
&= Ct^{-\frac{1}{3}}\left|Ai(\eta)\left(W\left(\eta t^{-\frac{2}{3}}\right) - W(0)\right)\right| \\
&\le Ct^{-\frac{1}{3}}|\eta|^{-\frac{1}{4}}\left|\eta t^{-\frac{2}{3}}\right|^{\frac{1}{4}} \le Ct^{-\frac{1}{2}}
\end{aligned}
$$

is true.

In the previous work [18] we showed global existence of small solution:

Proposition 1.1. *Assume hat the initial data* $u_0 \in \mathbf{H^{1,1}}$ *are real - valued functions with sufficiently small norm* $\|u_0\|_{\mathbf{H^{1,1}}} = \varepsilon$. *Then there exists a unique global solution* $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{1,1}\right)$ *of the Cauchy problem for (1.1) such that*

$$\langle t\rangle^{\frac{1}{3}-\frac{1}{3\beta}} \|u(t)\|_{\mathbf{L}^p} \le C\varepsilon$$

for all $t \in \mathbf{R}$, *where* $4 < \beta \le \infty$.

To state the result on the asymptotics of solutions, we denote by $v_m(t,x) = t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right)$ the self-similar solution of (1.1). Note that if the function $f_m(\eta)$ satisfies the second Peinleve equation $f_m'' = \eta f_m + 3f_m^3$, then v_m obeys (1.1).

The next result from [18] says the asymptotic stability of solutions in the neighborhood of the self-similar solution.

Proposition 1.2. *Let* $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{1,1}\right)$ *be the solution of (1.1) constructed in Proposition 1.1 and* $\frac{1}{\sqrt{2\pi}}\int f_m(x)\,dx = \frac{1}{\sqrt{2\pi}}\int u_0(x)\,dx$. *Then for any* $u_0 \in \mathbf{H^{1,1}}$, *there exist unique functions* H_j *and* $B_j \in \mathbf{L}^\infty$ *(*B_j *are real-valued),* $j = 1,2$, *such that the following asymptotic formula is valid for large time* $t \ge 1$

$$\begin{aligned} u(t,x) &= t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right) \\ &\quad +\sqrt{2\pi}t^{-\frac{1}{3}} ReAi\left(xt^{-\frac{1}{3}}\right)\left(H_1\left(xt^{-1}\right)\exp\left(iB_1\left(xt^{-1}\right)\log|x|\,t^{-\frac{1}{3}}\right)\right. \\ &\quad \left.+H_2\left(xt^{-1}\right)\exp\left(iB_2\left(xt^{-1}\right)\log|x|\,t^{-\frac{1}{3}}\right)\right) \\ &\quad +O\left(\varepsilon t^{4\gamma-\frac{5}{12}}\left(1+|x|\,t^{-\frac{1}{3}}\right)^{-1/4}\right), \end{aligned} \tag{1.3}$$

where $\gamma \in \left(0, \frac{1}{50}\right)$.

Since H_j in the second term of the right-hand side of (1.3) are not necessarily zero at the origin, and asymptotic property of solutions to the second Peinleve equation is not stated explicitly in [18], therefore it is not determined which one is the leading term $f_m(\eta)$ or $Ai(\eta)$ from the previous work. Below, we prove that the leading term of $f_m(\eta)$ as $\eta = xt^{-\frac{1}{3}} \to \infty$ is similar to the leading term of $Ai(\eta)$ for $\eta > 0$. Thus the previous work says that the main term consists of the first and the second terms of the right-hand side of (1.3).

We note that the large time behavior of solution to (1.1) was also studied in [9], [11] and their methods involve the estimates for the multi-linear oscillatory integrals, which make proofs very complicated. In the present paper we develop the factorization technique for (1.1) to obtain the sharp time decay estimate of solutions and make an improvement of the previous result from [18]. We evaluate the nonlinear term by various factorization formulas for the solution of Airy equation which involves one dimensional oscillatory integral only. Therefore our proof is simpler than [9], [11].

To state our results precisely we introduce *Notation and Function Spaces.* We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int |\phi(x)|^p \, dx\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x \in \mathbf{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{k,s} = \left\{\varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{k,s}} = \left\| \langle x \rangle^s \langle i\partial_x \rangle^k \phi \right\|_{\mathbf{L}^p} < \infty \right\},$$

$k, s \in \mathbf{R}, 1 \leq p \leq \infty$, $\langle x \rangle = \sqrt{1+x^2}$, $\langle i\partial_x \rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{k,s} = \mathbf{H}_2^{k,s}$, $\mathbf{H}^k = \mathbf{H}^{k,0}$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I}; \mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C.

We are now in a position to state our first result.

Theorem 1.3. *Assume that the initial data $u_0 \in \mathbf{H}^s \cap \mathbf{H}^{0,1}, s > \frac{3}{4}$ are real-valued with a sufficiently small norm $\|u_0\|_{\mathbf{H}^s \cap \mathbf{H}^{0,1}} \leq \varepsilon$. Then there exists a unique global solution $\mathcal{F} e^{-\frac{it}{3}\partial_x^3} u \in \mathbf{C}\left([0,\infty); \mathbf{L}^\infty \cap \mathbf{H}^{0,1}\right)$ of the Cauchy problem (1.1). Furthermore the estimate*

$$\sup_{t>0} \left(\left\| \mathcal{F} e^{-\frac{it}{3}\partial_x^3} u(t) \right\|_{\mathbf{L}^\infty} + \langle t \rangle^{-\frac{1}{6}} \left\| x e^{-\frac{it}{3}\partial_x^3} u(t) \right\|_{\mathbf{L}^2} + \langle t \rangle^{\frac{1}{3}\left(1-\frac{1}{p}\right)} \|u(t)\|_{\mathbf{L}^p} \right) \leq C\varepsilon$$

is true, where $p > 4$.

In order to state the stability of global solutions in the neighborhood of the self-similar solution $v_m(t,x) = t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right)$, we prove

Theorem 1.4. *Assume that m is sufficiently small real number and*

$$m = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f_m(x) \, dx \neq 0.$$

Then there exists a unique real-valued solution of the Cauchy problem (1.1) in the form $v_m(t,x) = t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right)$, such that

$$\mathcal{F} e^{-\frac{it}{3}\partial_x^3} v_m \in \mathbf{C}\left([1,\infty); \mathbf{L}^\infty\right), \ x e^{-\frac{it}{3}\partial_x^3} v_m \in \mathbf{C}\left([1,\infty); \mathbf{L}^2\right).$$

Furthermore the estimate

$$\sup_{t>1} \left(\left\| \mathcal{F} e^{-\frac{it}{3}\partial_x^3} v_m(t) \right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{6}} \left\| x e^{-\frac{it}{3}\partial_x^3} v_m(t) \right\|_{\mathbf{L}^2} + t^{\frac{1}{3}\left(1-\frac{1}{p}\right)} \|v_m(t)\|_{\mathbf{L}^p} \right) \leq C|m|$$

is true, where $p > 4$.

Theorem 1.5. *Suppose that*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f_m(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} u_0(x)\,dx = m \neq 0.$$

Let $u(t,x)$ and $v_m(t,x)$ be the solutions constructed in Theorem 1.3 and Theorem 1.4, respectively. Then there exists a $\gamma > 0$ such that the asymptotics

$$|u(t,x) - v_m(t,x)| \le C\varepsilon t^{-\frac{1}{2}+\gamma} \tag{1.4}$$

for $x > 0$ and

$$|u(t,x) - v_m(t,x)| \le C\varepsilon t^{-\frac{1}{2}+\gamma}\left\langle xt^{-\frac{1}{3}}\right\rangle^{-\frac{3}{4}} \tag{1.5}$$

for $x \le 0$ are true for large $t \ge 1$. Also the sharp time decay estimate of solutions is valid, namely there exist positive constants C_1, C_2 such that

$$C_1\varepsilon t^{-\frac{1}{3}\left(1-\frac{1}{q}\right)} \le \|u(t)\|_{\mathbf{L}^q} \le C_2\varepsilon t^{-\frac{1}{3}\left(1-\frac{1}{q}\right)}$$

for $4 < q < \infty$.

Remark. It is expected that the asymptotic behavior of solutions to (1.1) is similar to the sum of a self-similar solution and the Airy function (see [1], [7]). Therefore the asymptotic behavior of solutions presented in (1.5) is not necessarily sharp in the domain $x < 0$, since the Airy function decays exponentially with respect to $|x|\,t^{-\frac{1}{3}}$ for $x < 0$.

We now introduce the factorization formula for the case of the modified KdV equation (1.1). We define the free evolution group $\mathcal{U}(t) = \mathcal{F}^{-1}E\mathcal{F}$, where the multiplication factor $E(t,\xi) = e^{-\frac{it}{3}\xi^3}$. Then we find

$$\begin{aligned}\mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{F}^{-1}E\phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\xi-\frac{it}{3}\xi^3}\phi(\xi)\,d\xi \\ &= \frac{1}{\sqrt{2\pi}}\mathcal{D}_t\,|t|^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{it\left(x\xi-\frac{1}{3}\xi^3\right)}\phi(\xi)\,d\xi = \mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\left(x|x|\xi-\frac{1}{3}\xi^3\right)}\phi(\xi)\,d\xi,\end{aligned}$$

where $\mathcal{D}_t\phi = |t|^{-\frac{1}{2}}\phi\left(xt^{-1}\right)$ and we introduce the operator

$$(\mathcal{B}\phi)(x) = \phi\left(x\,|x|^{-\frac{1}{2}}\right).$$

We define the cut off function $\chi(\xi) \in \mathbf{C}^2(\mathbf{R})$ such that $\chi(\xi) = 0$ for $\xi \leq -\frac{1}{3}$, $\chi(\xi) = 1$ for $\xi \geq \frac{1}{3}$, and such that $\chi(\xi) + \chi(-\xi) \equiv 1$. Then we write

$$\begin{aligned}
&\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(x^2\xi - \frac{1}{3}\xi^3)}\phi(\xi)\chi(\xi x^{-1})\,d\xi \\
&+\mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(x^2\xi - \frac{1}{3}\xi^3)}\phi(\xi)\chi(-\xi x^{-1})\,d\xi \\
= \;& \mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(x^2\xi - \frac{1}{3}\xi^3)}\phi(\xi)\chi(\xi x^{-1})\,d\xi \\
&+\mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it(x^2\xi - \frac{1}{3}\xi^3)}\phi(-\xi)\chi(\xi x^{-1})\,d\xi
\end{aligned}$$

for $x > 0$. Also we have

$$\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it(x^2\xi + \frac{1}{3}\xi^3)}\phi(\xi)\,d\xi$$

for $x \leq 0$. Since $u = \mathcal{U}(t)\mathcal{F}^{-1}\phi$ is a real-valued function, we have $\phi(-\xi) = \overline{\phi(\xi)}$, hence

$$\begin{aligned}
&\theta(x)\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(x^2\xi - \frac{1}{3}\xi^3)}\phi(\xi)\chi(\xi x^{-1})\,d\xi \\
&+\mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it(x^2\xi - \frac{1}{3}\xi^3)}\overline{\phi(\xi)}\chi(\xi x^{-1})\,d\xi \\
= \;& \mathcal{D}_t\mathcal{B}\left(M\mathcal{V}\phi + \overline{M\mathcal{V}\phi}\right)
\end{aligned}$$

with $\theta(x) = 0$ for $x \leq 0$, and $\theta(x) = 1$ for $x > 0$, where the multiplication factor $M(t,x) = e^{\frac{2it}{3}x^3}$, the phase function $S(x,\xi) = \frac{2}{3}x^3 - x^2\xi + \frac{1}{3}\xi^3$, and the operator

$$\mathcal{V}\phi = \frac{|t|^{\frac{1}{2}}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itS(x,\xi)}\phi(\xi)\chi(\xi x^{-1})\,d\xi.$$

Also we have

$$\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}\mathcal{W}\phi$$

for $x \leq 0$, where the operator

$$\mathcal{W}\phi = \frac{|t|^{\frac{1}{2}}(1-\theta(x))}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itS_0(x,\xi)}\phi(\xi)\,d\xi,$$

and the phase function $S_0(x,\xi) = x^2\xi + \frac{1}{3}\xi^3$. If we define the new dependent variable $\widehat{\varphi} = \mathcal{FU}(-t)\,u$, then we obtain the factorization representation

$$u(t) = \mathcal{U}(t)\,\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}\left(M\mathcal{V}\widehat{\varphi} + \overline{M\mathcal{V}\widehat{\varphi}}\right) + \mathcal{D}_t\mathcal{B}\mathcal{W}\widehat{\varphi} \tag{1.6}$$

We will prove below that first summand of the right-hand side of (1.6) is the main term of the large time asymptotics, namely, solutions decay in time faster in the negative region $x \le 0$ comparing with the positive region $x > 0$.

We also need the representation for the inverse evolution group $\mathcal{FU}(-t)$

$$\begin{aligned}
\mathcal{FU}(-t)\,\phi &= \overline{E}\mathcal{F}\phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{it}{3}\xi^3 - ix\xi}\phi(x)\,dx \\
&= \frac{t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\left(\frac{1}{3}\xi^3 - x\xi\right)}\mathcal{D}_t^{-1}\phi(x)\,dx \\
&= \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\left(\frac{1}{3}\xi^3 - x|x|\xi\right)}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi(x)\,|x|\,dx \\
&= \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_{0}^{\infty} e^{itS(x,\xi)}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi(x)\,x dx \\
&\quad + \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{itS_0(x,\xi)}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi(x)\,|x|\,dx \\
&= \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi + \mathcal{R}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi, \qquad (1.7)
\end{aligned}$$

where $\mathcal{D}_t^{-1}\phi = |t|^{\frac{1}{2}}\,\phi(xt)\,,\ \left(\mathcal{B}^{-1}\phi\right)(x) = \phi(x\,|x|)$ and the operators

$$\mathcal{Q}\phi = \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_{0}^{\infty} e^{itS(x,\xi)}\phi(x)\,x dx,$$

and

$$\mathcal{R}\phi = -\frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{itS_0(x,\xi)}\phi(x)\,x dx.$$

Since $\mathcal{FU}(-t)\,\mathcal{L} = \partial_t\mathcal{FU}(-t)\,,$ with $\mathcal{L} = \partial_t - \frac{1}{3}\partial_x^3$, applying the operator $\mathcal{FU}(-t)$ to equation (1.1) we get

$$\partial_t\widehat{\varphi} = \partial_t\mathcal{FU}(-t)\,u = \mathcal{FU}(-t)\,\mathcal{L}u = \mathcal{FU}(-t)\,\partial_x\left(u^3\right) = 3\mathcal{FU}(-t)\left(u^2u_x\right).$$

Then by (1.6) we find the following factorization property

$$\begin{aligned}
&\partial_t\widehat{\varphi} = 3\mathcal{F}\mathcal{U}(-t)\left(u^2u_x\right) = 3\mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(u^2u_x\right) + 3\mathcal{R}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(u^2u_x\right)\\
=\ & 3\mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(\mathcal{D}_t\mathcal{B}\left(M\mathcal{V}\widehat{\varphi}+\overline{M\mathcal{V}\widehat{\varphi}}\right)\right)^2\left(\mathcal{D}_t\mathcal{B}\left(M\mathcal{V}i\xi\widehat{\varphi}+\overline{M\mathcal{V}i\xi\widehat{\varphi}}\right)\right)\\
& +3\mathcal{R}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(\mathcal{D}_t\mathcal{B}\mathcal{W}\widehat{\varphi}\right)^2\left(\mathcal{D}_t\mathcal{B}\mathcal{W}i\xi\widehat{\varphi}\right)\\
=\ & 3t^{-1}\mathcal{Q}\overline{M}\mathcal{B}^{-1}\left(\mathcal{B}\left(M\mathcal{V}\widehat{\varphi}+\overline{M\mathcal{V}\widehat{\varphi}}\right)\right)^2\left(\mathcal{B}\left(M\mathcal{V}i\xi\widehat{\varphi}+\overline{M\mathcal{V}i\xi\widehat{\varphi}}\right)\right)\\
& +3t^{-1}\mathcal{R}\mathcal{B}^{-1}\left(\mathcal{B}\mathcal{W}\widehat{\varphi}\right)^2\left(\mathcal{B}\mathcal{W}i\xi\widehat{\varphi}\right)\\
=\ & 3t^{-1}\mathcal{Q}\overline{M}\left(M\mathcal{V}\widehat{\varphi}+\overline{M\mathcal{V}\widehat{\varphi}}\right)^2\left(M\mathcal{V}i\xi\widehat{\varphi}+\overline{M\mathcal{V}i\xi\widehat{\varphi}}\right)\\
& +3t^{-1}\mathcal{R}\left(\mathcal{W}\widehat{\varphi}\right)^2\left(\mathcal{W}i\xi\widehat{\varphi}\right)
\end{aligned}$$

and similarly we have another representation

$$\begin{aligned}
&\partial_t\widehat{\varphi} = i\xi\mathcal{F}\mathcal{U}(-t)\left(u^3\right) = i\xi\mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(u^3\right) + i\xi\mathcal{R}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(u^3\right)\\
=\ & i\xi t^{-1}\mathcal{Q}\overline{M}\left(M\mathcal{V}\widehat{\varphi}+\overline{M\mathcal{V}\widehat{\varphi}}\right)^3 + i\xi t^{-1}\mathcal{R}\left(\mathcal{W}\widehat{\varphi}\right)^3,
\end{aligned}$$

which will be used in the domain $0\le\xi\le t^{-\frac{1}{3}}$. Note that

$$\begin{aligned}
\overline{M}\left(M\mathcal{V}\widehat{\varphi}+\overline{M\mathcal{V}\widehat{\varphi}}\right)^3 &= M^2\left(\mathcal{V}\widehat{\varphi}\right)^3 + 3\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\\
&\quad +3\overline{M}^2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2 + \overline{M}^4\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3
\end{aligned}$$

and for $\alpha\neq-1$

$$\begin{aligned}
\mathcal{Q}(t)M^{\alpha}\phi &= \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{it\left(\frac{2(1+\alpha)}{3}x^3-x^2\xi+\frac{1}{3}\xi^3\right)}\phi(x)\,xdx\\
&= \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}e^{\frac{\alpha(2+\alpha)}{(1+\alpha)^2}\frac{it}{3}\xi^3}\int_0^\infty e^{it(1+\alpha)\left(\frac{2}{3}x^3-x^2\frac{\xi}{1+\alpha}+\frac{1}{3}\left(\frac{\xi}{1+\alpha}\right)^3\right)}\phi(x)\,xdx\\
&= E^{-\frac{\alpha(2+\alpha)}{(1+\alpha)^2}}\mathcal{D}_{1+\alpha}\mathcal{Q}\left(t(1+\alpha)\right)\phi.
\end{aligned}$$

Thus we obtain the equation for the new dependent variable $\widehat{\varphi}(t,\xi)$

$$\begin{aligned}
\partial_t\widehat{\varphi}(t,\xi) &= 3t^{-1}E^{-\frac{8}{9}}\mathcal{D}_3\mathcal{Q}(3t)\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right)\\
&\quad +3t^{-1}\mathcal{Q}(t)\left(2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\left(\mathcal{V}i\xi\widehat{\varphi}\right)+\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right)\\
&\quad +3t^{-1}\mathcal{D}_{-1}\mathcal{Q}(-t)\left(\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right)+2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right)\\
&\quad +3t^{-1}E^{-\frac{8}{9}}\mathcal{D}_{-3}\mathcal{Q}(-3t)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\\
&\quad +3t^{-1}\mathcal{R}\left(\mathcal{W}\widehat{\varphi}\right)^2\left(\mathcal{W}i\xi\widehat{\varphi}\right). \qquad (1.8)
\end{aligned}$$

Now we explain how to use equation (1.8) for estimating $|\widehat{\varphi}(t,\xi)|$ uniformly with respect to ξ. For the real-valued solution u, we have $\overline{\widehat{\varphi}(t,\xi)} = \widehat{\varphi}(t,-\xi)$, hence it is sufficient to consider the case $\xi > 0$ only. In Lemma 2.2 below we state the property that the main term of $\mathcal{Q}(t)\phi$ lies in the positive region $\xi > 0$. Therefore the third and fourth terms of the right-hand side of (1.8) are remainder terms. From Lemma 2.2 and Lemma 2.3, we find that the last term of the right-hand side of (1.8) is also the remainder. We need to consider the first and second terms of the right-hand side of (1.8). Due to the oscillating factor $E^{-\frac{8}{9}}$, we will show that the first term of (1.8) is also the remainder. For the second term, by Lemma 2.2, we get

$$\begin{aligned} & 3t^{-1}\mathcal{Q}(t)\left(2(\mathcal{V}\widehat{\varphi})\left(\overline{\mathcal{V}\widehat{\varphi}}\right)(\mathcal{V}i\xi\widehat{\varphi}) + (\mathcal{V}\widehat{\varphi})^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right) \\ \simeq\ & 3t^{-1}t^{\frac{1}{6}}\widetilde{A_0}\left(t^{\frac{1}{3}}\xi\right)\xi\left(2(\mathcal{V}\widehat{\varphi})\left(\overline{\mathcal{V}\widehat{\varphi}}\right)(\mathcal{V}i\xi\widehat{\varphi}) + (\mathcal{V}\widehat{\varphi})^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right), \end{aligned}$$

where $\widetilde{A_0}(x) = \frac{2\theta(x)}{\sqrt{2\pi}}\int_{\frac{x}{3}}^{\infty} e^{iS(\xi,x)}\widetilde{\chi}\left(\xi x^{-1}\right)d\xi$ for $x > 0$ with the phase function $S(x,\xi) = \frac{2}{3}x^3 - x^2\xi + \frac{1}{3}\xi^3$ and the cut off function $\widetilde{\chi}(z) \in \mathbf{C}^2(\mathbf{R})$ such that $\widetilde{\chi}(z) = 0$ for $z \leq \frac{1}{3}$ and $\widetilde{\chi}(z) = 1$ for $z \geq \frac{2}{3}$. The asymptotics of $\widetilde{A_0}(x)$ is obtained in Lemma 2.1. By Lemma 2.3 we find the main term of right-hand side of the above is

$$\begin{aligned} & 3it^{-1}t^{\frac{1}{3}}\xi\widetilde{A_0}\left(t^{\frac{1}{3}}\xi\right) \\ & \times\left(2A_0\left(\xi t^{\frac{1}{3}}\right)\overline{A_0\left(\xi t^{\frac{1}{3}}\right)}A_1\left(\xi t^{\frac{1}{3}}\right)|\widehat{\varphi}|^2\widehat{\varphi}(\xi)\right. \\ & \left.-\left(A_0\left(\xi t^{\frac{1}{3}}\right)\right)^2\overline{A_1\left(\xi t^{\frac{1}{3}}\right)}|\widehat{\varphi}|^2\widehat{\varphi}(\xi)\right), \end{aligned}$$

where $A_j(x) = \frac{\theta(x)}{\sqrt{2\pi}}\int_{\frac{x}{3}}^{\infty} e^{-iS(x,\xi)}\widetilde{\chi}\left(\xi x^{-1}\right)\xi^j d\xi$. Then using the asymptotics of $A_j(x)$ from Lemma 2.1 we get the main term for the second summand of the right-hand side of (1.8) which is

$$\frac{3}{2}it^{-1}|\widehat{\varphi}|^2\widehat{\varphi}(\xi)$$

for $\xi t^{\frac{1}{3}} > 1$. In the domain $\xi t^{\frac{1}{3}} \leq 1$, the main term for the second summand of the right-hand side of (1.8) is

$$\frac{3}{2}it^{-1}\xi t^{\frac{1}{3}}\left\langle \xi t^{\frac{1}{3}}\right\rangle^{-1}|\widehat{\varphi}|^2\widehat{\varphi}(\xi)\ .$$

To justify the above procedure, we need the estimates of the derivatives $\partial_\xi\mathcal{W}$ and $\partial_\xi\mathcal{V}$ which are considered in the next section.

It is known that the operator $\mathcal{J} = x + t\partial_x^2 = \mathcal{U}(t)\, x\mathcal{U}(-t)$ is a useful tool for obtaining the $\mathbf{L}^\infty$ - time decay estimates of solutions and has been used widely for the studying the asymptotic behavior of solutions to various nonlinear dispersive equations (see [14], [15], [16]). However, the operator $\mathcal{J}$ does not work well on the nonlinear terms. Then, instead of using the operator $\mathcal{J}$ we apply the following operator $\mathcal{P} = \partial_x x + 3t\partial_t$ in the same way as in [13]. Note that $\mathcal{P}$ acts well on the nonlinear terms as the first order differential operator. Also $\mathcal{J}$ and $\mathcal{P}$ are related via the identity

$$\partial_x^{-1}\mathcal{P} - \mathcal{J} = 3t\partial_x^{-1}\mathcal{L}$$

with $\mathcal{L} = \partial_t - \frac{1}{3}\partial_x^3$.

We organize the rest of our paper as follows. In Section 2, we state main estimates for the decomposition operators related to the Korteweg-de Vries evolution group $\mathcal{U}$. We prove a-priori estimates of solutions in Section 3. Section 4 is devoted to the proof of Theorem 1.3. Finally, we show Theorems 1.4 and 1.5 in Section 5.

§2. Preliminaries

2.1. Estimates for two kernels A_α and $\widetilde{A_\alpha}$

Define two kernels

$$A_\alpha(x) = \frac{\theta(x)}{\sqrt{2\pi}} \int_{\frac{x}{3}}^{\infty} e^{-iS(x,\xi)} \widetilde{\chi}\left(\xi x^{-1}\right) \xi^\alpha d\xi$$

and

$$\widetilde{A_\alpha}(x) = \frac{2\theta(x)}{\sqrt{2\pi}} \int_{\frac{x}{3}}^{\infty} e^{iS(\xi,x)} \widetilde{\chi}\left(\xi x^{-1}\right) \xi^\alpha d\xi$$

for $x > 0$, where the phase function $S(x,\xi) = \frac{2}{3}x^3 - x^2\xi + \frac{1}{3}\xi^3$, and the cut off function $\widetilde{\chi}(z) \in \mathbf{C}^2(\mathbf{R})$ is such that $\widetilde{\chi}(z) = 0$ for $z \le \frac{1}{3}$ and $\widetilde{\chi}(z) = 1$ for $z \ge \frac{2}{3}$.

Lemma 2.1. *Let $\alpha \in [0,2)$. Then the estimate*

$$\left|\langle x\rangle^{\frac{1}{2}-\alpha} A_\alpha(x)\right| + \left|\langle x\rangle^{\frac{1}{2}-\alpha} \widetilde{A_\alpha}(x)\right| \le C$$

is true for any $x > 0$. Moreover the asymptotics takes place

$$A_\alpha(x) = \frac{1}{\sqrt{2i}} x^{\alpha-\frac{1}{2}} + O\left(\langle x\rangle^{\alpha-\frac{7}{2}}\right)$$

and

$$\widetilde{A_\alpha}(x) = \sqrt{2i}\, x^{\alpha-\frac{1}{2}} + O\left(\langle x\rangle^{\alpha-\frac{7}{2}}\right)$$

as $x \to +\infty$.

Proof. For the case of $0 < x < 1$, changing the contour of integration $\xi = re^{-i\delta}$ with a small $\delta > 0$ we get

$$|A_\alpha(x)| \le C\left|\int_{\frac{x}{3}}^{1} e^{-iS(x,\xi)}\widetilde{\chi}\left(\xi x^{-1}\right)\xi^\alpha d\xi\right|$$
$$+C\left|\int_{\mathbf{C}_\delta} e^{-iS(x,\xi)}\xi^\alpha d\xi\right| + C\int_1^\infty e^{-Cr^3+Cr}r^\alpha dr \le C,$$

where $\mathbf{C}_\delta = \left\{\xi \in \mathbf{C} : \xi = e^{-i\phi}, 0 \le \phi \le \delta\right\}$. In the same manner in the case of $0 < x < 1$, changing the contour of integration $\xi = re^{i\delta}$ with a small $\delta > 0$ we get

$$\left|\widetilde{A_\alpha}(x)\right| \le C\left|\int_{\frac{x}{3}}^{1} e^{iS(\xi,x)}\widetilde{\chi}\left(\xi x^{-1}\right)\xi^\alpha d\xi\right|$$
$$+C\left|\int_{\overline{\mathbf{C}_\delta}} e^{iS(\xi,x)}\xi^\alpha d\xi\right| + C\int_1^\infty e^{-Cr^3+Cr^2}r^\alpha dr \le C.$$

For $x \ge 1$ we use the identities

$$e^{-iS(x,\xi)} = H_1\partial_\xi\left((\xi - x)\, e^{-iS(x,\xi)}\right)$$

with $H_1 = \left(1 - i(\xi + x)(\xi - x)^2\right)^{-1}$ and

$$e^{iS(\xi,x)} = H_2\partial_\xi\left((\xi - x)\, e^{iS(\xi,x)}\right)$$

with $H_2 = \left(1 + 2i\xi(\xi - x)^2\right)^{-1}$ to integrate by parts to have

$$A_\alpha(x) = -\frac{1}{\sqrt{2\pi}}\int_{\frac{x}{3}}^{\infty} e^{-iS(x,\xi)}(\xi - x)\,\partial_\xi\left(H_1\widetilde{\chi}\left(\xi x^{-1}\right)\xi^\alpha\right)d\xi$$

and

$$\widetilde{A_\alpha}(x) = -\frac{2}{\sqrt{2\pi}}\int_{\frac{x}{3}}^{\infty} e^{iS(\xi,x)}(\xi - x)\,\partial_\xi\left(H_2\widetilde{\chi}\left(\xi x^{-1}\right)\xi^\alpha\right)d\xi.$$

Hence in view of the estimate

$$\left|(\xi - x)\,\partial_\xi\left(H_1\widetilde{\chi}\left(\xi x^{-1}\right)\xi^\alpha\right)\right| + \left|(\xi - x)\,\partial_\xi\left(H_2\widetilde{\chi}\left(\xi x^{-1}\right)\xi^\alpha\right)\right| \le \frac{C\xi^\alpha}{1 + \xi(x - \xi)^2}$$

for $\xi \ge \frac{x}{3}$, we obtain

$$|A_\alpha(x)| + \left|\widetilde{A_\alpha}(x)\right| \le Cx^\alpha\int_{\frac{x}{3}}^{2x}\frac{d\xi}{1 + x(\xi - x)^2} + C\int_{2x}^{\infty}\xi^{\alpha-3}d\xi \le Cx^{\alpha-\frac{1}{2}}$$

for all $x \geq 1$. To compute the asymptotics of the functions $A_\alpha(x)$ and $\widetilde{A_\alpha}(x)$ for large x we apply the stationary phase method (see [8], p. 163). We have the asymptotics

$$\int_{\mathbf{R}} e^{irG(\eta)} f(\eta)\, d\eta = e^{irG(\eta_0)} f(\eta_0) \sqrt{\frac{2\pi}{r\,|G''(\eta_0)|}} e^{i\frac{\pi}{4}\operatorname{sgn} G''(\eta_0)} + O\left(r^{-\frac{3}{2}}\right) \tag{2.1}$$

for $r \to +\infty$, where the stationary point η_0 is defined by $G'(\eta_0) = 0$. We change $\xi = x\eta$, then we get

$$A_\alpha(x) = \frac{x^{1+\alpha}\theta(x)}{\sqrt{2\pi}} \int_{\frac{1}{3}}^{\infty} e^{-\frac{i}{3}x^3\left(2-3\eta+\eta^3\right)} \widetilde{\chi}(\eta)\, \eta^\alpha d\eta.$$

By virtue of formula (2.1) with $r = \frac{1}{3}x^3$, $G(\eta) = -\left(2-3\eta+\eta^3\right)$, $f(\eta) = \widetilde{\chi}(\eta)\,\eta^\alpha$, $\eta_0 = 1$, we get

$$A_\alpha(x) = \frac{x^{1+\alpha}}{\sqrt{2\pi}} \left(\sqrt{\frac{\pi}{x^3}} e^{-i\frac{\pi}{4}} + O\left(x^{-\frac{9}{2}}\right)\right) = \frac{x^{\alpha-\frac{1}{2}}}{\sqrt{2i}} + O\left(\langle x\rangle^{\alpha-\frac{7}{2}}\right)$$

for $x \to +\infty$. We also have

$$\widetilde{A_\alpha}(x) = \frac{2x^{1+\alpha}\theta(x)}{\sqrt{2\pi}} \int_{\frac{1}{3}}^{\infty} e^{\frac{i}{3}x^3\left(2\eta^3-3\eta^2+1\right)} \widetilde{\chi}(\eta)\, \eta^\alpha d\eta,$$

then in the same way as above

$$\widetilde{A_\alpha}(x) = \frac{2x^{1+\alpha}}{\sqrt{2\pi}} \left(\sqrt{\frac{\pi}{x^3}} e^{i\frac{\pi}{4}} + O\left(x^{-\frac{9}{2}}\right)\right) = x^{\alpha-\frac{1}{2}}\sqrt{2i} + O\left(\langle x\rangle^{\alpha-\frac{7}{2}}\right)$$

as $x \to +\infty$. Lemma 2.1 is proved. □

2.2. Estimates for the operators $\mathcal{Q}$ and $\mathcal{R}$

In this subsection, we consider the operators

$$\mathcal{Q}\phi = \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{itS(x,\xi)} \phi(x)\, x dx$$

for $\xi \in \mathbf{R}$ and

$$\mathcal{R}\phi = -\frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^0 e^{itS_0(x,\xi)} \phi(x)\, x dx$$

for $\xi \neq 0$, where $S(x,\xi) = \frac{2}{3}x^3 - x^2\xi + \frac{1}{3}\xi^3$ and $S_0(x,\xi) = x^2\xi + \frac{1}{3}\xi^3$. In the next lemma we obtain the estimates of the operators $\mathcal{Q}$ and $\mathcal{R}$.

Lemma 2.2. *Let* $\alpha \in \left[0, \frac{3}{4}\right]$, $0 \leq \beta \leq \frac{3}{4} - \alpha$. *Then the estimates*

$$\begin{aligned} & \left|\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\frac{3}{4}-\alpha-\beta}\left(\mathcal{Q}\phi(\xi) - t^{\frac{1-2\alpha}{6}}\widetilde{A_\alpha}\left(t^{\frac{1}{3}}\xi\right)\xi^{1-\alpha}\phi(\xi)\right)\right| \\ \leq\ & Ct^{-\frac{\alpha}{3}}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \end{aligned}$$

for $\xi > 0$,

$$\left|\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\frac{3}{4}-\alpha-\beta}\mathcal{Q}\phi(\xi)\right| \leq Ct^{-\frac{\alpha}{3}}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

for $\xi \leq 0$, *and*

$$\left|\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\frac{3}{4}}\mathcal{R}\phi(\xi)\right| \leq C\|\phi\|_{\mathbf{L}^2(\mathbf{R}_-)} + C\|x\phi_x\|_{\mathbf{L}^2(\mathbf{R}_-)}$$

for $\xi \neq 0$ *are valid for all* $t \geq 1$.

Proof. We write

$$t^{\frac{1}{6}-\frac{\alpha}{3}}\widetilde{A_\alpha}\left(t^{\frac{1}{3}}\xi\right) = \frac{2t^{\frac{1}{2}}\theta(\xi)}{\sqrt{2\pi}}\int_{\frac{\xi}{3}}^{\infty} e^{itS(x,\xi)}\widetilde{\chi}\left(x\xi^{-1}\right)x^\alpha dx,$$

therefore

$$\begin{aligned} & \mathcal{Q}\phi - t^{\frac{1-2\alpha}{6}}\widetilde{A_\alpha}\left(t^{\frac{1}{3}}\xi\right)\xi^{1-\alpha}\phi(\xi) \\ =\ & \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}\left(x^{1-\alpha}\phi(x) - \xi^{1-\alpha}\phi(\xi)\right)\widetilde{\chi}\left(x\xi^{-1}\right)x^\alpha dx \\ & + \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}\phi(x)\chi_2\left(x\xi^{-1}\right)x dx \\ =\ & I_1 + I_2 \end{aligned}$$

for $\xi > 0$, where $\chi_2\left(x\xi^{-1}\right) = 1 - \widetilde{\chi}\left(x\xi^{-1}\right)$. In the first integral I_1 using the identity

$$e^{itS(x,\xi)} = H_3\partial_x\left((x-\xi)e^{itS(x,\xi)}\right)$$

with $H_3 = \left(1 + 2itx(x-\xi)^2\right)^{-1}$ we integrate by parts

$$\begin{aligned} I_1 =\ & \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}\left(x^{1-\alpha}\phi(x) - \xi^{1-\alpha}\phi(\xi)\right)\widetilde{\chi}\left(x\xi^{-1}\right)x^\alpha dx \\ =\ & -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}\left(x^{1-\alpha}\phi(x) - \xi^{1-\alpha}\phi(\xi)\right) \\ \times\ & (x-\xi)\partial_x\left(x^\alpha H_3\widetilde{\chi}\left(x\xi^{-1}\right)\right)dx \\ & -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}(x-\xi)x^\alpha H_3\widetilde{\chi}\left(x\xi^{-1}\right)\partial_x\left(x^{1-\alpha}\phi(x)\right)dx. \end{aligned}$$

Then using the estimates

$$\begin{aligned}&\left|x^{1-\alpha}\phi(x)-\xi^{1-\alpha}\phi(\xi)\right|\\ \le\ & C\left|x-\xi\right|^{\frac12}\left\langle t^{\frac13}x\right\rangle^{\beta}\left\|\left\langle t^{\frac13}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi(x)\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)},\end{aligned}$$

$$|H_3|\le C\left(1+tx\left(x-\xi\right)^2\right)^{-1}$$

and

$$\left|\left(x-\xi\right)\partial_x\left(x^{\alpha}H_3\widetilde{\chi}\left(x\xi^{-1}\right)\right)\right|\le Cx^{\alpha}\left(1+tx\left(x-\xi\right)^2\right)^{-1}$$

for $x\ge\frac13\xi$, we find

$$\begin{aligned}|I_1|&\le Ct^{\frac12}\left\|\left\langle t^{\frac13}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\int_{\frac13\xi}^{\infty}\frac{x^{\alpha}\left|x-\xi\right|^{\frac12}\left\langle t^{\frac13}x\right\rangle^{\beta}}{1+tx\left(x-\xi\right)^2}dx\\ &+Ct^{\frac12}\int_{\frac13\xi}^{\infty}\frac{x^{\alpha}\left|x-\xi\right|\left\langle t^{\frac13}x\right\rangle^{\beta}}{1+tx\left(x-\xi\right)^2}\left|\left\langle t^{\frac13}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi(x)\right)\right|dx\\ &\le Ct^{\frac12}\left\|\left\langle t^{\frac13}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\ &\times\left(\int_{\frac13\xi}^{\infty}\frac{x^{\alpha}\left|x-\xi\right|^{\frac12}\left\langle t^{\frac13}x\right\rangle^{\beta}}{1+tx\left(x-\xi\right)^2}dx+\left(\int_{\frac13\xi}^{\infty}\frac{\left\langle t^{\frac13}x\right\rangle^{2\beta}x^{2\alpha}\left(x-\xi\right)^2dx}{\left(1+tx\left(x-\xi\right)^2\right)^2}\right)^{\frac12}\right)\\ &\le Ct^{-\frac{\alpha}{3}}\left\langle t^{\frac13}\xi\right\rangle^{\alpha+\beta-\frac34}\left\|\left\langle t^{\frac13}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)},\end{aligned}$$

since changing $xt^{\frac13}=y$ we get for $\zeta=\xi t^{\frac13}>1$

$$\begin{aligned}&\int_{\frac13\xi}^{\infty}\frac{x^{\alpha}\left|x-\xi\right|^{\frac12}\left\langle t^{\frac13}x\right\rangle^{\beta}}{1+tx\left(x-\xi\right)^2}dx=t^{-\frac{\alpha}{3}-\frac12}\int_{\frac13\zeta}^{\infty}\frac{y^{\alpha}\left|y-\zeta\right|^{\frac12}\left\langle y\right\rangle^{\beta}}{1+y\left(y-\zeta\right)^2}dy\\ \le\ & Ct^{-\frac{\alpha}{3}-\frac12}\zeta^{\alpha+\beta}\int_{\frac13\zeta}^{2\zeta}\frac{\left|y-\zeta\right|^{\frac12}dy}{1+\zeta\left(y-\zeta\right)^2}+Ct^{-\frac{\alpha}{3}-\frac12}\int_{2\zeta}^{\infty}y^{\alpha+\beta-\frac52}dy\\ \le\ & Ct^{-\frac{\alpha}{3}-\frac12}\left\langle t^{\frac13}\xi\right\rangle^{\alpha+\beta-\frac34}\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{3}\xi}^{\infty} \frac{x^{2\alpha}\left\langle t^{\frac{1}{3}}x\right\rangle^{2\beta}(x-\xi)^2\,dx}{\left(1+tx(x-\xi)^2\right)^2} = t^{-\frac{2\alpha}{3}-1}\int_{\frac{1}{3}\zeta}^{\infty}\frac{y^{2\alpha}\langle y\rangle^{2\beta}(y-\zeta)^2}{\left(1+y(y-\zeta)^2\right)^2}dy\\
\leq\;& Ct^{-\frac{2\alpha}{3}-1}\zeta^{2\alpha+2\beta}\int_{\frac{1}{3}\zeta}^{2\zeta}\frac{(y-\zeta)^2\,dy}{\left(1+\zeta(y-\zeta)^2\right)^2}+Ct^{-\frac{2\alpha}{3}-1}\int_{2\zeta}^{\infty}y^{2\alpha+2\beta-4}dy\\
\leq\;& Ct^{-\frac{2\alpha}{3}-1}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{2\alpha+2\beta-\frac{3}{2}}.
\end{aligned}$$

In the second integral I_2, using the identity $e^{itS(x,\xi)} = H_4\partial_x\left(xe^{itS(x,\xi)}\right)$ with $H_4 = \left(1+2itx^2(x-\xi)\right)^{-1}$ we integrate by parts

$$\begin{aligned}
I_2 &= \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^{\infty} e^{itS(x,\xi)}\phi(x)\,\chi_2\left(x\xi^{-1}\right)x\,dx\\
&= -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^{\infty} e^{itS(x,\xi)}x^{1-\alpha}\phi(x)\,x\partial_x\left(x^{\alpha}H_4\chi_2\left(x\xi^{-1}\right)\right)dx\\
&\quad -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^{\infty} e^{itS(x,\xi)}x^{1+\alpha}H_4\chi_2\left(x\xi^{-1}\right)\partial_x\left(x^{1-\alpha}\phi(x)\right)dx.
\end{aligned}$$

Then using the estimates $\left|H_4\chi_2\left(x\xi^{-1}\right)\right| \leq C\left(1+t\xi x^2\right)^{-1}$,

$$\left|x^{1-\alpha}\phi(x)\right| \leq Cx^{\frac{1}{2}}\left\langle t^{\frac{1}{3}}x\right\rangle^{\beta}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

and

$$\left|x\partial_x\left(x^{\alpha}H_4\chi_2\left(x\xi^{-1}\right)\right)\right| \leq Cx^{\alpha}\left(1+t\xi x^2\right)^{-1}$$

for $0 < x < \frac{2}{3}\xi$, we get

$$\begin{aligned} |I_2| &\le Ct^{\frac{1}{2}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \left(x^{1-\alpha} \phi \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \int_0^{\frac{2}{3}\xi} \frac{x^{\alpha+\frac{1}{2}} \left\langle t^{\frac{1}{3}} x \right\rangle^{\beta}}{1+t\xi x^2} dx \\ &+ Ct^{\frac{1}{2}} \int_0^{\frac{2}{3}\xi} \frac{x^{1+\alpha} \left\langle t^{\frac{1}{3}} x \right\rangle^{\beta}}{1+t\xi x^2} \left| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \left(x^{1-\alpha} \phi(x) \right) \right| dx \\ &\le Ct^{\frac{1}{2}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \left(x^{1-\alpha} \phi \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\times \left(\int_0^{\frac{2}{3}\xi} \frac{x^{\alpha+\frac{1}{2}} \left\langle t^{\frac{1}{3}} x \right\rangle^{\beta}}{1+t\xi x^2} dx + \left(\int_0^{\frac{2}{3}\xi} \frac{x^{2+2\alpha} \left\langle t^{\frac{1}{3}} x \right\rangle^{2\beta} dx}{\left(1+t\xi x^2\right)^2} \right)^{\frac{1}{2}} \right) \\ &\le Ct^{-\frac{\alpha}{3}} \left\langle t^{\frac{1}{3}} \xi \right\rangle^{\alpha+\beta-\frac{3}{4}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \left(x^{1-\alpha} \phi \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)}, \end{aligned}$$

since changing $xt^{\frac{1}{3}} = y$ we get for $\zeta = \xi t^{\frac{1}{3}} > 1$

$$\begin{aligned} &\int_0^{\frac{2}{3}\xi} \frac{x^{\alpha+\frac{1}{2}} \left\langle t^{\frac{1}{3}} x \right\rangle^{\beta}}{1+t\xi x^2} dx = t^{-\frac{\alpha}{3}-\frac{1}{2}} \int_0^{\frac{2}{3}\zeta} \frac{y^{\alpha+\frac{1}{2}} \langle y \rangle^{\beta}}{1+\zeta y^2} dy \\ &\le Ct^{-\frac{\alpha}{3}-\frac{1}{2}} \zeta^{\alpha+\beta} \int_0^{\frac{2}{3}\zeta} \frac{y^{\frac{1}{2}} dy}{1+\zeta y^2} \le Ct^{-\frac{\alpha}{3}-\frac{1}{2}} \left\langle t^{\frac{1}{3}} \xi \right\rangle^{\alpha+\beta-\frac{3}{4}} \end{aligned}$$

and

$$\begin{aligned} &\int_0^{\frac{2}{3}\xi} \frac{x^{2+2\alpha} \left\langle t^{\frac{1}{3}} x \right\rangle^{2\beta} dx}{\left(1+t\xi x^2\right)^2} = t^{-\frac{2\alpha}{3}-1} \int_0^{\frac{2}{3}\zeta} \frac{y^{2+2\alpha} \langle y \rangle^{2\beta}}{\left(1+\zeta y^2\right)^2} dy \\ &\le Ct^{-\frac{2\alpha}{3}-1} \zeta^{2\alpha+2\beta} \int_0^{\frac{2}{3}\zeta} \frac{y^2 dy}{\left(1+\zeta y^2\right)^2} \le Ct^{-\frac{2\alpha}{3}-1} \left\langle t^{\frac{1}{3}} \xi \right\rangle^{2\alpha+2\beta-\frac{3}{2}}. \end{aligned}$$

Next consider

$$\mathcal{Q}\phi = \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_0^{\infty} e^{itS(x,\xi)} \phi(x) \, x dx$$

for $\xi \le 0$. Using the identity $e^{itS(x,\xi)} = H_4 \partial_x \left(x e^{itS(x,\xi)} \right)$ with H_4 which is the same as defined in the above, we integrate by parts

$$\begin{aligned}\mathcal{Q}\phi &= \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}\phi(x)\,xdx \\ &= -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}x^{2-\alpha}\phi(x)\,\partial_x\left(H_4x^\alpha\right)dx \\ &\quad -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}x^{\alpha+1}H_4\partial_x\left(x^{1-\alpha}\phi(x)\right)dx.\end{aligned}$$

Hence

$$\begin{aligned}|\mathcal{Q}\phi| &\le Ct^{\frac{1}{2}}\int_0^\infty \frac{x^\alpha\left|x^{1-\alpha}\phi(x)\right|dx}{1+tx^2(x+|\xi|)} + Ct^{\frac{1}{2}}\int_0^\infty \frac{x^{\alpha+1}\left|\partial_x\left(x^{1-\alpha}\phi(x)\right)\right|dx}{1+tx^2(x+|\xi|)} \\ &\le Ct^{\frac{1}{2}}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi(x)\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\int_0^\infty \frac{x^{\alpha+\frac{1}{2}}\left\langle t^{\frac{1}{3}}x\right\rangle^{\beta}dx}{1+tx^2(x+|\xi|)} \\ &+ Ct^{\frac{1}{2}}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi(x)\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\left(\int_0^\infty \frac{\left\langle t^{\frac{1}{3}}x\right\rangle^{2\beta}x^{2+2\alpha}dx}{\left(1+tx^2(x+|\xi|)\right)^2}\right)^{\frac{1}{2}} \\ &\le Ct^{-\frac{\alpha}{3}}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\alpha+\beta-\frac{3}{4}}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x^{1-\alpha}\phi(x)\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.\end{aligned}$$

Finally we estimate

$$\mathcal{R}\phi = -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^0 e^{itS_0(x,\xi)}\phi(x)\,xdx$$

for $\xi \neq 0$. Using the identity

$$e^{itS_0(x,\xi)} = H_8\partial_x\left(xe^{itS_0(x,\xi)}\right)$$

with $H_8 = \left(1+2itx^2\xi\right)^{-1}$ we integrate by parts

$$\begin{aligned}\mathcal{R}\phi &= -\frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^0 e^{itS_0(x,\xi)}\phi(x)\,xdx \\ &= \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^0 e^{itS_0(x,\xi)}\phi(x)\,x\partial_x\left(xH_8\right)dx \\ &+ \frac{2t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^0 e^{itS_0(x,\xi)}xH_8x\phi_x(x)\,dx.\end{aligned}$$

Hence we find

$$\begin{aligned} |\mathcal{R}\phi| &\leq C\left(\|\phi\|_{\mathbf{L}^2(\mathbf{R}_-)}+\|x\phi_x\|_{\mathbf{L}^2(\mathbf{R}_-)}\right)\left(t\int_{-\infty}^{0}\frac{x^2dx}{(1+tx^2|\xi|)^2}\right)^{\frac12} \\ &\leq C\left\langle \xi t^{\frac13}\right\rangle^{-\frac34}\left(\|\phi\|_{\mathbf{L}^2(\mathbf{R}_-)}+\|x\phi_x\|_{\mathbf{L}^2(\mathbf{R}_-)}\right) \end{aligned}$$

for all $\xi \neq 0$. Lemma 2.2 is proved. □

2.3. Estimates for the operators $\mathcal{V}$ and $\mathcal{W}$

In the next lemma we obtain the estimates for the operators

$$\mathcal{V}\phi=\frac{|t|^{\frac12}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\phi(\xi)\chi\left(\xi x^{-1}\right)d\xi$$

for $x>0$, and

$$\mathcal{W}\phi=\frac{|t|^{\frac12}(1-\theta(x))}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS_0(x,\xi)}\phi(\xi)\,d\xi,$$

for $x\leq 0$. Here the phase functions $S(x,\xi)=\frac23 x^3-x^2\xi+\frac13\xi^3$ and $S_0(x,\xi)=x^2\xi+\frac13\xi^3$.

Lemma 2.3. *Let $j=0,1$. Then the estimate*

$$\left|\left\langle xt^{\frac13}\right\rangle^{\frac34-j}\left(\mathcal{V}\xi^j\phi(x)-t^{\frac{1-2j}{6}}A_j\left(xt^{\frac13}\right)\phi(x)\right)\right|\leq Ct^{-\frac{j}{3}}\left(t^{\frac16}|\phi(0)|+\|\phi_\xi\|_{\mathbf{L}^2}\right)$$

for $x>0$ and

$$\left|\left\langle xt^{\frac13}\right\rangle^{\frac32-j}\mathcal{W}\xi^j\phi(x)\right|\leq Ct^{-\frac{j}{3}}\left(t^{\frac16}|\phi(0)|+\|\phi_\xi\|_{\mathbf{L}^2}\right)$$

for $x\leq 0$ are valid for all $t\geq 1$.

Remark. By Lemma 2.1 and Lemma 2.3 we have the following estimate for $x>0$

$$\begin{aligned} &\left|\left\langle xt^{\frac13}\right\rangle^{\frac12-j}\mathcal{V}\xi^j\phi(x)\right| \\ \leq\ & t^{\frac16-\frac{j}{3}}\left|\left\langle xt^{\frac13}\right\rangle^{\frac12-j}A_j\left(xt^{\frac13}\right)\right||\phi(x)| \\ &+Ct^{\frac16-\frac{j}{3}}\left\langle xt^{\frac13}\right\rangle^{-\frac14}\left(|\phi(0)|+t^{-\frac16}\|\phi_\xi\|_{\mathbf{L}^2}\right) \\ \leq\ & Ct^{\frac16-\frac{j}{3}}\left(\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}+t^{-\frac16}\|\phi_\xi\|_{\mathbf{L}^2}\right) \end{aligned}$$

for $j=0,1$.

Proof. Since

$$t^{\frac{1}{6}-\frac{j}{3}}A_j\left(xt^{\frac{1}{3}}\right)=\frac{t^{\frac{1}{2}}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\widetilde{\chi}\left(\xi x^{-1}\right)\xi^j d\xi,$$

we can write

$$\begin{aligned}&\mathcal{V}\xi^j\phi-t^{\frac{1-2j}{6}}A_j\left(xt^{\frac{1}{3}}\right)\phi(x)\\ =\;&\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\left(\phi(\xi)-\phi(x)\right)\widetilde{\chi}\left(\xi x^{-1}\right)\xi^j d\xi\\ &+\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\phi(\xi)\chi_2\left(\xi x^{-1}\right)\xi^j d\xi=I_3+I_4\end{aligned}$$

for $x>0$, where $\chi_2\left(\xi x^{-1}\right)=\chi\left(\xi x^{-1}\right)-\widetilde{\chi}\left(\xi x^{-1}\right)$. We integrate by parts via the identity

$$e^{-itS(x,\xi)}=H_5\partial_\xi\left(\left(\xi-x\right)e^{-itS(x,\xi)}\right)$$

with $H_5=\left(1-it\left(\xi-x\right)^2\left(\xi+x\right)\right)^{-1}$ to get

$$\begin{aligned}I_3=&-\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^{\infty}e^{-itS(x,\xi)}\left(\xi-x\right)\partial_\xi\left(\xi^jH_5\widetilde{\chi}\left(\xi x^{-1}\right)\right)\left(\phi(\xi)-\phi(x)\right)d\xi\\ &-\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}}\int_0^{\infty}e^{-itS(x,\xi)}\left(\xi-x\right)H_5\xi^j\widetilde{\chi}\left(\xi x^{-1}\right)\phi_\xi(\xi)\,d\xi.\end{aligned}$$

Using the estimates

$$|H_5|\le C\left(1+t\left(\xi-x\right)^2\left(\xi+x\right)\right)^{-1}$$

and

$$\left|\left(\xi-x\right)\partial_\xi\left(\xi^jH_5\widetilde{\chi}\left(\xi x^{-1}\right)\right)\right|\le\frac{C\xi^j}{1+t\left(\xi-x\right)^2\left(\xi+x\right)}$$

for $\xi>\frac{1}{3}x$, we obtain

$$\begin{aligned}|I_3|\le\;&Ct^{\frac{1}{2}}\int_{\frac{x}{3}}^{\infty}\frac{\left|\phi(\xi)-\phi(x)\right|\xi^j d\xi}{1+t\left(\xi-x\right)^2\left(\xi+x\right)}+Ct^{\frac{1}{2}}\int_{\frac{x}{3}}^{\infty}\frac{\left|\xi-x\right|\left|\phi_\xi(\xi)\right|\xi^j d\xi}{1+t\left(\xi-x\right)^2\left(\xi+x\right)}\\ \le\;&Ct^{\frac{1}{2}}\left\|\phi_\xi\right\|_{\mathbf{L}^2}\int_{\frac{x}{3}}^{\infty}\frac{\left|\xi-x\right|^{\frac{1}{2}}\xi^j d\xi}{1+t\left(\xi-x\right)^2\left(\xi+x\right)}\\ &+C\left\|\phi_\xi\right\|_{\mathbf{L}^2}\left(t\int_{\frac{x}{3}}^{\infty}\frac{\left(\xi-x\right)^2\xi^{2j}d\xi}{\left(1+t\left(\xi-x\right)^2\left(\xi+x\right)\right)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Changing $y = xt^{\frac{1}{3}}$ and $\zeta = \xi t^{\frac{1}{3}}$ we get

$$\begin{aligned} & t^{\frac{1}{2}} \int_{\frac{x}{3}}^{\infty} \frac{|\xi - x|^{\frac{1}{2}} \xi^j d\xi}{1 + t(\xi - x)^2 (\xi + x)} = t^{-\frac{j}{3}} \int_{\frac{y}{3}}^{\infty} \frac{|\zeta - y|^{\frac{1}{2}} \zeta^j d\zeta}{1 + (\zeta - y)^2 (\zeta + y)} \\ \leq \;& Ct^{-\frac{j}{3}} \langle y \rangle^j \int_{\frac{y}{3}}^{2\langle y \rangle} \frac{|\zeta - y|^{\frac{1}{2}} d\zeta}{1 + y(\zeta - y)^2} + Ct^{-\frac{j}{3}} \int_{2\langle y \rangle}^{\infty} |\zeta|^{j - \frac{5}{2}} d\zeta \leq Ct^{-\frac{j}{3}} \langle y \rangle^{j - \frac{3}{4}} \end{aligned}$$

and

$$\begin{aligned} & t \int_{\frac{x}{3}}^{\infty} \frac{(\xi - x)^2 \xi^{2j} d\xi}{\left(1 + t(\xi - x)^2 (\xi + x)\right)^2} = t^{-\frac{2j}{3}} \int_{\frac{y}{3}}^{\infty} \frac{(\zeta - y)^2 \zeta^{2j} d\zeta}{\left(1 + (\zeta - y)^2 (\zeta + y)\right)^2} \\ \leq \;& Ct^{-\frac{2j}{3}} \langle y \rangle^{2j} \int_{\frac{y}{3}}^{2\langle y \rangle} \frac{(\zeta - y)^2 d\zeta}{\left(1 + y(\zeta - y)^2\right)^2} + Ct^{-\frac{2j}{3}} \int_{2\langle y \rangle}^{\infty} |\zeta|^{2j-4} d\zeta \\ \leq \;& Ct^{-\frac{2j}{3}} \langle y \rangle^{2j - \frac{3}{2}}. \end{aligned}$$

Thus we have

$$|I_3| \leq Ct^{-\frac{j}{3}} \left\langle xt^{\frac{1}{3}} \right\rangle^{j - \frac{3}{4}} \|\phi_\xi\|_{\mathbf{L}^2}$$

for all $x > 0$, $t \geq 1$. To estimate

$$I_4 = \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS(x,\xi)} \phi(\xi) \chi_2(\xi x^{-1}) \xi^j d\xi$$

for $x > 0$, we integrate by parts via the identity

$$e^{-itS(x,\xi)} = H_6 \partial_\xi \left(\xi e^{-itS(x,\xi)}\right)$$

with $H_6 = \left(1 - it\xi(\xi^2 - x^2)\right)^{-1}$ to get

$$\begin{aligned} I_4 = & -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS(x,\xi)} \xi \partial_\xi \left(\xi^j H_6 \chi_2(\xi x^{-1})\right) \phi(\xi) d\xi \\ & -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS(x,\xi)} \xi^{j+1} H_6 \chi_2(\xi x^{-1}) \phi_\xi(\xi) d\xi. \end{aligned}$$

Using the estimates $|H_6| \leq C\left(1 + t|\xi| x^2\right)^{-1}$ and

$$\left|\xi \partial_\xi \left(\xi^j H_6 \chi_2(\xi x^{-1})\right)\right| \leq \frac{C\xi^j}{1 + t\xi x^2}$$

for $|\xi| \leq \frac{2}{3}x$, we obtain

$$
\begin{aligned}
& |I_4| \leq Ct^{\frac{1}{2}} |\phi(0)| \int_0^{\frac{2}{3}x} \frac{\xi^j d\xi}{1+t\xi x^2} + Ct^{\frac{1}{2}} \int_0^{\frac{2}{3}x} \frac{|\phi(\xi)-\phi(0)|\,\xi^j}{1+t\xi x^2} d\xi \\
& +Ct^{\frac{1}{2}} \int_0^{\frac{2}{3}x} \frac{|\xi|^{1+j} |\phi_\xi(\xi)|}{1+t\xi x^2} d\xi \leq Ct^{\frac{1}{2}} |\phi(0)| \int_0^{\frac{2}{3}x} \frac{\xi^j d\xi}{1+t\xi x^2} \\
& +Ct^{\frac{1}{2}} \|\phi_\xi\|_{\mathbf{L}^2} \left(\int_0^{\frac{2}{3}x} \frac{\xi^{\frac{1}{2}+j} d\xi}{1+t\xi x^2} + \left(\int_0^{\frac{2}{3}x} \frac{\xi^{2+2j} d\xi}{(1+t\xi x^2)^2} \right)^{\frac{1}{2}} \right) \\
\leq \;& C|\phi(0)|\, t^{\frac{1}{2}} x^{1+j} \left\langle xt^{\frac{1}{3}} \right\rangle^{\gamma-3} + C\|\phi_\xi\|_{\mathbf{L}^2} t^{\frac{1}{2}} x^{\frac{3}{2}+j} \left\langle xt^{\frac{1}{3}} \right\rangle^{-3} \\
\leq \;& Ct^{-\frac{j}{3}} \left\langle xt^{\frac{1}{3}} \right\rangle^{j-\frac{3}{2}} \left(t^{\frac{1}{6}} |\phi(0)| + \|\phi_\xi\|_{\mathbf{L}^2} \right)
\end{aligned}
$$

for all $x > 0$, where $\gamma \in \left(0, \frac{1}{2}\right)$.

Finally we estimate

$$
\mathcal{W}\xi^j \phi = \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} \phi(\xi)\, \xi^j d\xi
$$

for $x \leq 0$. We integrate by parts via the identity

$$
e^{-itS_0(x,\xi)} = H_7 \partial_\xi \left(\xi e^{-itS_0(x,\xi)} \right)
$$

with $H_7 = \left(1 - it\xi\left(x^2+\xi^2\right)\right)^{-1}$ to get

$$
\begin{aligned}
& \mathcal{W}\xi^j \phi = -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \phi(0) \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} \xi \partial_\xi \left(\xi^j H_7\right) d\xi \\
& -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} \left(\phi(\xi)-\phi(0)\right) \xi \partial_\xi \left(\xi^j H_7\right) d\xi \\
& -\frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} H_7 \xi^{j+1} \phi_\xi(\xi)\, d\xi.
\end{aligned}
$$

Using the estimates $|H_7| \leq C\left(1 + t|\xi|\left(x^2+\xi^2\right)\right)^{-1}$ and

$$
\left|\xi \partial_\xi \left(\xi^j H_7\right)\right| \leq C|\xi|^j \left(1 + t|\xi|\left(x^2+\xi^2\right)\right)^{-1},
$$

we obtain

$$\left|\mathcal{W}\xi^{j}\phi\right| \leq C\left|\phi\left(0\right)\right| t^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\left|\xi\right|^{j} d\xi}{1+t\left|\xi\right|\left(x^{2}+\xi^{2}\right)}$$
$$+C\left\|\phi_{\xi}\right\|_{\mathbf{L}^{2}} t^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\left|\xi\right|^{j+\frac{1}{2}} d\xi}{1+t\left|\xi\right|\left(x^{2}+\xi^{2}\right)}$$
$$+C\left\|\phi_{\xi}\right\|_{\mathbf{L}^{2}} t^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} \frac{\left|\xi\right|^{2j+2} d\xi}{\left(1+t\left|\xi\right|\left(x^{2}+\xi^{2}\right)\right)^{2}}\right)^{\frac{1}{2}}.$$

Changing $y = xt^{\frac{1}{3}}$ and $\zeta = \xi t^{\frac{1}{3}}$ we find

$$t^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\left|\xi\right|^{j} d\xi}{1+t\left|\xi\right|\left(x^{2}+\xi^{2}\right)} = t^{\frac{1}{6}-\frac{j}{3}} \int_{0}^{\infty} \frac{\zeta^{j} d\zeta}{1+\zeta\left(y^{2}+\zeta^{2}\right)} \leq Ct^{\frac{1}{6}-\frac{j}{3}} \left\langle y\right\rangle^{j-\frac{3}{2}},$$

$$t^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\left|\xi\right|^{j+\frac{1}{2}} d\xi}{1+t\left|\xi\right|\left(x^{2}+\xi^{2}\right)} = t^{-\frac{j}{3}} \int_{0}^{\infty} \frac{\zeta^{j+\frac{1}{2}} d\zeta}{1+\zeta\left(y^{2}+\zeta^{2}\right)} \leq Ct^{-\frac{j}{3}} \left\langle y\right\rangle^{j-\frac{3}{2}}$$

and

$$t \int_{-\infty}^{\infty} \frac{\left|\xi\right|^{2j+2} d\xi}{\left(1+t\left|\xi\right|\left(x^{2}+\xi^{2}\right)\right)^{2}} = t^{-\frac{2j}{3}} \int_{0}^{\infty} \frac{\zeta^{2j+2} d\zeta}{\left(1+\zeta\left(y^{2}+\zeta^{2}\right)\right)^{2}} \leq Ct^{-\frac{2j}{3}} \left\langle y\right\rangle^{2j-3}.$$

Thus we get

$$\left|\mathcal{W}\xi^{j}\phi\right| \leq Ct^{-\frac{j}{3}} \left\langle xt^{\frac{1}{3}}\right\rangle^{j-\frac{3}{2}} \left(t^{\frac{1}{6}}\left|\phi\left(0\right)\right| + \left\|\phi_{\xi}\right\|_{\mathbf{L}^{2}}\right)$$

for all $x \leq 0$, $t > 0$. Lemma 2.3 is proved. □

2.4. Estimates for derivatives $\partial_x \mathcal{V}$ and $\partial_x \mathcal{W}$

In the next lemma we estimate the derivative $\partial_x \mathcal{V}\phi$.

Lemma 2.4. *Let $0 \leq \alpha < \frac{1}{2}$, $\frac{1}{2} < \alpha+\beta < \frac{3}{2}$, $j = 0, 1$. Then the estimate*

$$\left\|\left\langle xt^{\frac{1}{3}}\right\rangle^{-\beta} x^{1-\alpha-j} \partial_x \mathcal{V}\xi^{j}\phi\right\|_{\mathbf{L}^{2}(\mathbf{R}_{+})} \leq Ct^{\frac{\alpha}{3}} \left(\left|\phi\left(0\right)\right| + t^{-\frac{1}{6}}\left\|\phi_{\xi}\right\|_{\mathbf{L}^{2}}\right)$$

is true for all $t \geq 1$, provided that the right-hand side is finite.

Proof. We have $\partial_x e^{-itS(x,\xi)} = -\frac{2x}{\xi+x}\partial_\xi e^{-itS(x,\xi)}$. Thus we get

$$\begin{aligned}\partial_x \mathcal{V}\xi^j\phi &= -\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itS(x,\xi)}\frac{\chi\left(\xi x^{-1}\right)\xi^j}{\xi+x}\phi_\xi\left(\xi\right)d\xi \\ &\quad -\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itS(x,\xi)}\psi_1\left(x,\xi\right)\phi\left(\xi\right)d\xi \\ &\quad -\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itS(x,\xi)}\psi_2\left(x,\xi\right)\phi\left(\xi\right)d\xi = I + J_1 + J_2\end{aligned}$$

for $x > 0$, where

$$\psi_1\left(x,\xi\right) = \psi\left(x,\xi\right)\widetilde{\chi}\left(\xi x^{-1}\right), \psi_2\left(x,\xi\right) = \psi\left(x,\xi\right)\left(1-\widetilde{\chi}\left(\xi x^{-1}\right)\right)$$

and

$$\psi\left(x,\xi\right) = \frac{1}{2}\xi^{1+j}x^{-3}\chi'\left(\xi x^{-1}\right) + \partial_\xi\frac{\chi\left(\xi x^{-1}\right)\xi^j}{\xi+x}.$$

Then we have

$$\begin{aligned}&\left\|\left\langle xt^{\frac{1}{3}}\right\rangle^{-\beta} x^{1-\alpha-j} I\right\|^2_{\mathbf{L}^2(\mathbf{R}_+)} \\ = \ & Ct\int_0^{\infty}\left\langle xt^{\frac{1}{3}}\right\rangle^{-2\beta} x^{4-2\alpha-2j}dx\int_{-\infty}^{\infty} e^{it\left(\frac{2}{3}x^3-x^2\xi+\frac{1}{3}\xi^3\right)}\frac{\chi\left(\xi x^{-1}\right)\xi^j}{\xi+x}\overline{\phi_\xi\left(\xi\right)}d\xi \\ & \times\int_{-\infty}^{\infty} e^{-it\left(\frac{2}{3}x^3-x^2\eta+\frac{1}{3}\eta^3\right)}\frac{\chi\left(\eta x^{-1}\right)\eta^j}{\eta+x}\phi_\eta\left(\eta\right)d\eta \\ = \ & C\int_{-\infty}^{\infty} d\xi e^{\frac{it}{3}\xi^3}\overline{\phi_\xi\left(\xi\right)}\int_{-\infty}^{\infty} d\eta e^{-\frac{it}{3}\eta^3}\phi_\eta\left(\eta\right)K_j\left(t,\xi,\eta\right),\end{aligned}$$

where

$$K_j\left(t,\xi,\eta\right) = t\int_0^{\infty} e^{-itx^2(\xi-\eta)}\frac{\chi\left(\xi x^{-1}\right)\xi^j}{\xi+x}\frac{\chi\left(\eta x^{-1}\right)\eta^j}{\eta+x}\left\langle xt^{\frac{1}{3}}\right\rangle^{-2\beta} x^{4-2\alpha-2j}dx.$$

Changing $y = x^2$ we get

$$\begin{aligned}&K_j\left(t,\xi,\eta\right) \\ = \ & Ct\int_0^{\infty} e^{-ity(\xi-\eta)}\frac{\chi\left(\xi y^{-\frac{1}{2}}\right)\xi^j}{\xi+y^{\frac{1}{2}}}\frac{\chi\left(\eta y^{-\frac{1}{2}}\right)\eta^j}{\eta+y^{\frac{1}{2}}}\left\langle y^{\frac{1}{2}}t^{\frac{1}{3}}\right\rangle^{-2\beta} y^{\frac{3}{2}-\alpha-j}dy.\end{aligned}$$

We can rotate the contour of integration $y = re^{-i\frac{\pi}{8}sgn(\xi-\eta)}$, then

$$\begin{aligned} &|K_j(t,\xi,\eta)| \\ \le\ & Ct\int_0^\infty e^{-Ct|\xi-\eta|r}\frac{|\xi|^j|\eta|^j r^{\frac{3}{2}-\alpha-j}\left\langle r^{\frac{1}{2}}t^{\frac{1}{3}}\right\rangle^{-2\beta}dr}{\left(r^{\frac{1}{2}}+|\xi|\right)\left(r^{\frac{1}{2}}+|\eta|\right)} \\ \le\ & Ct^{\frac{2\alpha}{3}}\int_0^\infty e^{-C|\xi-\eta|t^{\frac{1}{3}}r}\langle r\rangle^{-\beta}r^{\frac{1}{2}-\alpha}dr \\ \le\ & Ct^{\frac{2\alpha}{3}}\left(|\xi-\eta|t^{\frac{1}{3}}\right)^{\alpha+\beta-\frac{3}{2}}\left\langle(\xi-\eta)t^{\frac{1}{3}}\right\rangle^{-\beta}. \end{aligned}$$

Then by the Young inequality we obtain

$$\begin{aligned} &\left\|\left\langle xt^{\frac{1}{3}}\right\rangle^{-\beta}x^{1-\alpha-j}I\right\|^2_{\mathbf{L}^2(\mathbf{R}_+)} \\ \le\ & Ct^{\frac{2\alpha}{3}}\|\phi_\xi\|_{\mathbf{L}^2}\left\|\int_{\mathbf{R}}\left(|\xi-\eta|t^{\frac{1}{3}}\right)^{\alpha+\beta-\frac{3}{2}}\left\langle(\xi-\eta)t^{\frac{1}{3}}\right\rangle^{-\beta}|\phi_\eta(\eta)|d\eta\right\|_{\mathbf{L}^2} \\ \le\ & Ct^{\frac{2\alpha}{3}}\|\phi_\xi\|^2_{\mathbf{L}^2}\left\|\left(|\xi|t^{\frac{1}{3}}\right)^{\alpha+\beta-\frac{3}{2}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-\beta}\right\|_{\mathbf{L}^1}\le Ct^{\frac{2\alpha-1}{3}}\|\phi_\xi\|^2_{\mathbf{L}^2} \end{aligned}$$

if $\alpha+\beta-\frac{3}{2}>-1$ and $\alpha-\frac{3}{2}<-1$. To estimate the integral

$$J_1 = -\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\psi_1(x,\xi)\phi(\xi)d\xi$$

for $x>0$ as in the proof of Lemma 2.3 we use the identity

$$e^{-itS(x,\xi)} = H_5\partial_\xi\left((\xi-x)e^{-itS(x,\xi)}\right)$$

with $H_5 = \left(1-it(\xi-x)^2(\xi+x)\right)^{-1}$ and integrate by parts to find

$$\begin{aligned} J_1 &= \frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\phi(0)\int_{-\infty}^{\infty}e^{-itS(x,\xi)}(\xi-x)\partial_\xi(H_5\psi_1(x,\xi))d\xi \\ &+\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}(\xi-x)(\phi(\xi)-\phi(0))\partial_\xi(H_5\psi_1(x,\xi))d\xi \\ &+\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}(\xi-x)H_5\psi_1(x,\xi)\phi_\xi(\xi)d\xi. \end{aligned}$$

Using the estimates

$$|(\xi-x)H_5\psi_1(x,\xi)|\le\frac{C|\xi-x|\xi^{j-2}}{1+t(\xi-x)^2(\xi+x)},$$

$$\left|(\xi-x)\,\partial_\xi\left(H_5\psi_1(x,\xi)\right)\right|\le C\frac{\xi^{j-2}}{1+t\,(\xi-x)^2\,(\xi+x)}$$

for $\xi>\frac{x}{3}$ and $|\phi(\xi)-\phi(0)|\le C\,|\xi|^{\frac12}\,\|\phi_\xi\|_{\mathbf{L}^2}$ we get

$$\begin{aligned}
&x^{1-j}\,|J_1|\le Cx^{2-j}t^{\frac12}\,|\phi(0)|\left(\int_{\frac{x}{3}}^{2x}\frac{\xi^{j-2}d\xi}{1+tx\,(\xi-x)^2}+\int_{2x}^{\infty}\frac{\xi^{j-2}d\xi}{1+t\xi^3}\right)\\
&+Cx^{2-j}t^{\frac12}\,\|\phi_\xi\|_{\mathbf{L}^2}\left(\int_{\frac{x}{3}}^{2x}\frac{\xi^{j-\frac32}d\xi}{1+tx\,(\xi-x)^2}+\int_{2x}^{\infty}\frac{\xi^{j-\frac32}d\xi}{1+t\xi^3}\right)\\
&+Ct^{\frac12}\,\|\phi_\xi\|_{\mathbf{L}^2}\left(\int_0^{\infty}\frac{\xi^2d\xi}{(1+t\xi^3)^2}\right)^{\frac12}\\
\le\;& Ct^{\frac16}\,|\phi(0)|\left\langle xt^{\frac13}\right\rangle^{-\frac12}+C\,\|\phi_\xi\|_{\mathbf{L}^2}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\|\left\langle xt^{\frac13}\right\rangle^{-\beta}x^{1-\alpha-j}J_1\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
\le\;& Ct^{\frac16}\,|\phi(0)|\left\|\left\langle xt^{\frac13}\right\rangle^{-\beta-\frac12}x^{-\alpha}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}+\|\phi_\xi\|_{\mathbf{L}^2}\left\|\left\langle xt^{\frac13}\right\rangle^{-\beta}x^{-\alpha}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
\le\;& Ct^{\frac{\alpha}{3}}\,|\phi(0)|+Ct^{\frac{\alpha}{3}-\frac16}\,\|\phi_\xi\|_{\mathbf{L}^2}
\end{aligned}$$

if $\beta+\alpha>\frac12$ and $0\le\alpha<\frac12$.

To estimate

$$J_2(x)=-\frac{2xt^{\frac12}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\psi_2(x,\xi)\,\phi(\xi)\,d\xi$$

for $x>0$, as in the proof of Lemma 2.3 we integrate by parts via the identity

$$e^{-itS(x,\xi)}=H_6\partial_\xi\left(\xi e^{-itS(x,\xi)}\right)$$

with $H_6=\left(1-it\xi\left(\xi^2-x^2\right)\right)^{-1}$ to get

$$\begin{aligned}
J_2(x)&=-\frac{2xt^{\frac12}}{\sqrt{2\pi}}\phi(0)\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\xi\partial_\xi\left(H_6\psi_2(x,\xi)\right)d\xi\\
&-\frac{2xt^{\frac12}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\left(\phi(\xi)-\phi(0)\right)\xi\partial_\xi\left(H_6\psi_2(x,\xi)\right)d\xi\\
&-\frac{2xt^{\frac12}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS(x,\xi)}\xi H_6\psi_2(x,\xi)\,\phi_\xi(\xi)\,d\xi.
\end{aligned}$$

Using the estimates $|H_6| \le C\left(1+t|\xi|x^2\right)^{-1}$ and

$$|\xi\partial_\xi\left(H_6\psi_2(x,\xi)\right)| \le \frac{Cx^{j-2}}{1+t\xi x^2}$$

for $-\frac{x}{3} \le \xi \le \frac{2x}{3}$, we obtain

$$\begin{aligned}
&\left|x^{1-j}J_2(x)\right| \le Ct^{\frac{1}{2}}|\phi(0)|\int_{-\frac{x}{3}}^{\frac{2x}{3}}\frac{d\xi}{1+t\xi x^2} + Ct^{\frac{1}{2}}\int_{-\frac{x}{3}}^{\frac{2x}{3}}\frac{|\phi(\xi)-\phi(0)|}{1+t\xi x^2}d\xi \\
&+Ct^{\frac{1}{2}}x\int_{-\frac{x}{3}}^{\frac{2x}{3}}\frac{|\phi_\xi(\xi)|}{1+t\xi x^2}d\xi \le Ct^{\frac{1}{2}}|\phi(0)|\int_{-\frac{x}{3}}^{\frac{2x}{3}}\frac{d\xi}{1+t\xi x^2} \\
&+Ct^{\frac{1}{2}}\|\phi_\xi\|_{\mathbf{L}^2}\left(\int_{-\frac{x}{3}}^{\frac{2x}{3}}\frac{\xi^{\frac{1}{2}}d\xi}{1+t\xi x^2} + \left(\int_{-\frac{x}{3}}^{\frac{2x}{3}}\frac{\xi^2 d\xi}{(1+t\xi x^2)^2}\right)^{\frac{1}{2}}\right) \\
\le\ & C\left\langle xt^{\frac{1}{3}}\right\rangle^{-\frac{3}{2}}\left(t^{\frac{1}{6}}|\phi(0)| + \|\phi_\xi\|_{\mathbf{L}^2}\right)
\end{aligned}$$

for all $x > 0$. Hence

$$\begin{aligned}
&\left\|\left\langle xt^{\frac{1}{3}}\right\rangle^{-\beta}x^{1-\alpha-j}J_2\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\le\ & C\left(t^{\frac{1}{6}}|\phi(0)| + \|\phi_\xi\|_{\mathbf{L}^2}\right)\left\|\left\langle xt^{\frac{1}{3}}\right\rangle^{-\beta-\frac{3}{2}}x^{-\alpha}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\le\ & Ct^{\frac{\alpha}{3}}\left(|\phi(0)| + t^{-\frac{1}{6}}\|\phi_\xi\|_{\mathbf{L}^2}\right).
\end{aligned}$$

Lemma 2.4 is proved. □

In the next lemma we estimate the derivative $\partial_x\mathcal{W}$.

Lemma 2.5. *Let $0 \le \alpha < \frac{1}{2}$, $\frac{1}{2} < \alpha+\beta < \frac{3}{2}$, $j = 0, 1$. Then the estimate*

$$\left\|\left\langle xt^{\frac{1}{3}}\right\rangle^{-\beta}|x|^{1-\alpha-j}\partial_x\mathcal{W}\xi^j\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le Ct^{\frac{\alpha}{3}}\left(|\phi(0)| + t^{-\frac{1}{6}}\|\phi_\xi\|_{\mathbf{L}^2}\right)$$

is true for all $t \ge 1$, provided that the right-hand side is finite.

Proof. Applying the identity $\partial_x e^{-itS_0(x,\xi)} = \frac{2x\xi}{x^2+\xi^2}\partial_\xi e^{-itS_0(x,\xi)}$ we get

$$\begin{aligned}
\partial_x\mathcal{W}\xi^j\phi &= -\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS_0(x,\xi)}\frac{\xi^{j+1}}{x^2+\xi^2}\phi_\xi(\xi)\,d\xi \\
&-\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-itS_0(x,\xi)}\psi_3(x,\xi)\,\phi(\xi)\,d\xi = \widetilde{I} + \widetilde{J}
\end{aligned}$$

for $x \leq 0$, where $\psi_3(x,\xi) = \partial_\xi \frac{\xi^{j+1}}{x^2+\xi^2}$. Then

$$\begin{aligned} &\left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-\beta} |x|^{1-\alpha-j} \widetilde{I} \right\|^2_{\mathbf{L}^2(\mathbf{R}_-)} \\ = \ & Ct \int_{-\infty}^0 \left\langle xt^{\frac{1}{3}} \right\rangle^{-2\beta} x^{4-2\alpha-2j} dx \int_{-\infty}^{\infty} e^{it\left(x^2\xi + \frac{1}{3}\xi^3\right)} \frac{\xi^{j+1}}{x^2+\xi^2} \overline{\phi_\xi(\xi)} d\xi \\ & \times \int_{-\infty}^{\infty} e^{-it\left(x^2\eta + \frac{1}{3}\eta^3\right)} \frac{\eta^{j+1}}{x^2+\eta^2} \phi_\eta(\eta)\, d\eta \\ = \ & C \int_{-\infty}^{\infty} d\xi e^{\frac{it}{3}\xi^3} \overline{\phi_\xi(\xi)} \int_{-\infty}^{\infty} d\eta e^{-\frac{it}{3}\eta^3} \phi_\eta(\eta) \widetilde{K_j}(t,\xi,\eta), \end{aligned}$$

where

$$\widetilde{K_j}(t,\xi,\eta) = t \int_{-\infty}^0 e^{itx^2(\xi-\eta)} \frac{\xi^{j+1}}{x^2+\xi^2} \frac{\eta^{j+1}}{x^2+\eta^2} \left\langle xt^{\frac{1}{3}} \right\rangle^{-2\beta} |x|^{4-2\alpha-2j} dx.$$

Changing $y = x^2$ we get

$$\widetilde{K_j}(t,\xi,\eta) = Ct \int_0^{\infty} e^{ity(\xi-\eta)} \frac{\xi^{j+1}}{y+\xi^2} \frac{\eta^{j+1}}{y+\eta^2} \left\langle y^{\frac{1}{2}} t^{\frac{1}{3}} \right\rangle^{-2\beta} y^{\frac{3}{2}-\alpha-j} dy.$$

We rotate the contour of integration $y = re^{i\frac{\pi}{8}sgn(\xi-\eta)}$ to get

$$\begin{aligned} & \left| \widetilde{K_j}(t,\xi,\eta) \right| \leq Ct \int_0^{\infty} e^{-Ct|\xi-\eta|r} \frac{\xi^{j+1}\eta^{j+1} r^{\frac{3}{2}-\alpha-j} \left\langle r^{\frac{1}{2}} t^{\frac{1}{3}} \right\rangle^{-2\beta} dr}{(r+\xi^2)(r+\eta^2)} \\ \leq \ & Ct^{\frac{2\alpha}{3}} \int_0^{\infty} e^{-C|\xi-\eta|t^{\frac{1}{3}} r} \langle r \rangle^{-\beta} r^{\frac{1}{2}-\alpha} dr \\ \leq \ & Ct^{\frac{2\alpha}{3}} \left(|\xi-\eta| t^{\frac{1}{3}} \right)^{\alpha+\beta-\frac{3}{2}} \left\langle (\xi-\eta) t^{\frac{1}{3}} \right\rangle^{-\beta}. \end{aligned}$$

Then by the Young inequality we obtain

$$\begin{aligned} & \left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-\beta} x^{1-\alpha-j} \widetilde{I} \right\|^2_{\mathbf{L}^2(\mathbf{R}_-)} \\ \leq \ & Ct^{\frac{2\alpha}{3}} \|\phi_\xi\|_{\mathbf{L}^2} \left\| \int_{\mathbf{R}} \left(|\xi-\eta| t^{\frac{1}{3}} \right)^{\alpha+\beta-\frac{3}{2}} \left\langle (\xi-\eta) t^{\frac{1}{3}} \right\rangle^{-\beta} |\phi_\eta(\eta)|\, d\eta \right\|_{\mathbf{L}^2} \\ \leq \ & Ct^{\frac{2\alpha}{3}} \|\phi_\xi\|^2_{\mathbf{L}^2} \left\| \left(|\xi| t^{\frac{1}{3}} \right)^{\alpha+\beta-\frac{3}{2}} \left\langle \xi t^{\frac{1}{3}} \right\rangle^{-\beta} \right\|_{\mathbf{L}^1} \leq Ct^{\frac{2\alpha-1}{3}} \|\phi_\xi\|^2_{\mathbf{L}^2} \end{aligned}$$

if $\alpha + \beta - \frac{3}{2} > -1$ and $\alpha - \frac{3}{2} < -1$. To estimate the integral

$$\widetilde{J} = -\frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} \psi_3(x,\xi)\, \phi(\xi)\, d\xi$$

for $x \leq 0$, as in the proof of Lemma 2.3 we integrate by parts using the identity

$$e^{-itS_0(x,\xi)} = H_7 \partial_\xi \left(\xi e^{-itS_0(x,\xi)} \right)$$

with $H_7 = \left(1 - it\xi\left(x^2+\xi^2\right)\right)^{-1}$ to get

$$\begin{aligned}\widetilde{J} &= \frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}} \phi(0) \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} \xi \partial_\xi \left(\psi_3 H_7\right) d\xi \\ &+ \frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} \left(\phi(\xi) - \phi(0)\right) \xi \partial_\xi \left(\psi_3 H_7\right) d\xi \\ &+ \frac{2xt^{\frac{1}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itS_0(x,\xi)} \xi \psi_3 H_7 \phi_\xi(\xi) \, d\xi.\end{aligned}$$

Using the estimates

$$|\xi \psi_3 H_7| \leq \frac{C|\xi|^{j+1}}{x^2+\xi^2} \left(1 + t|\xi|\left(x^2+\xi^2\right)\right)^{-1}$$

and

$$|\xi \partial_\xi (\psi_3 H_7)| \leq \frac{C|\xi|^j}{x^2+\xi^2} \left(1 + t|\xi|\left(x^2+\xi^2\right)\right)^{-1},$$

we obtain

$$\begin{aligned}\left| x^{1-j} \widetilde{J} \right| &= Ct^{\frac{1}{2}} |\phi(0)| \int_{-\infty}^{\infty} \frac{d\xi}{1 + t|\xi|(x^2+\xi^2)} \\ &+ Ct^{\frac{1}{2}} \|\phi_\xi\|_{\mathbf{L}^2} \int_{-\infty}^{\infty} \frac{|\xi|^{\frac{1}{2}} d\xi}{1 + t|\xi|(x^2+\xi^2)} \\ &+ Ct^{\frac{1}{2}} \|\phi_\xi\|_{\mathbf{L}^2} \left(\int_{-\infty}^{\infty} \frac{\xi^2 d\xi}{\left(1 + t|\xi|(x^2+\xi^2)\right)^2} \right)^{\frac{1}{2}}.\end{aligned}$$

Changing $y = xt^{\frac{1}{3}}$ and $\zeta = \xi t^{\frac{1}{3}}$ we have

$$t^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{d\xi}{1 + t|\xi|(x^2+\xi^2)} = t^{\frac{1}{6}} \int_0^{\infty} \frac{d\zeta}{1 + \zeta(y^2+\zeta^2)} \leq Ct^{\frac{1}{6}} \langle y \rangle^{-\frac{3}{2}},$$

$$t^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{|\xi|^{\frac{1}{2}} d\xi}{1 + t|\xi|(x^2+\xi^2)} = \int_0^{\infty} \frac{\zeta^{\frac{1}{2}} d\zeta}{1 + \zeta(y^2+\zeta^2)} \leq C \langle y \rangle^{-\frac{3}{2}}$$

and

$$t \int_{-\infty}^{\infty} \frac{|\xi|^2 d\xi}{\left(1 + t|\xi|(x^2+\xi^2)\right)^2} = \int_0^{\infty} \frac{\zeta^2 d\zeta}{\left(1 + \zeta(y^2+\zeta^2)\right)^2} \leq C \langle y \rangle^{-3}.$$

Thus we get

$$\begin{aligned}
& \left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-\beta} |x|^{1-\alpha-j} \widetilde{J} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\
\leq & \; C \left(t^{\frac{1}{6}} |\phi(0)| + \|\phi_\xi\|_{\mathbf{L}^2} \right) \left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-\beta-\frac{3}{2}} |x|^{-\alpha} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\
\leq & \; Ct^{\frac{\alpha}{3}} \left(|\phi(0)| + t^{-\frac{1}{6}} \|\phi_\xi\|_{\mathbf{L}^2} \right).
\end{aligned}$$

Lemma 2.5 is proved. □

2.5. Estimate for the nonlinearity

In the next lemma we estimate the large time behavior of $\mathcal{FU}(-t)\partial_x(u^3)$. Define the norm $\|\phi\|_{\mathbf{W}} = \|\phi\|_{\mathbf{L}^\infty} + t^{-\frac{1}{6}} \|\phi_\xi\|_{\mathbf{L}^2}$.

Lemma 2.6. *The asymptotics*

$$\begin{aligned}
\mathcal{FU}(-t)\partial_x(u^3) = & \; \frac{\sqrt{3}}{2it} \xi t^{\frac{1}{3}} \left\langle t^{\frac{1}{3}} \xi \right\rangle^{-1} e^{-\frac{8it}{27}\xi^3} \widehat{\varphi}^3 \left(t, \frac{\xi}{3}\right) \\
& + \frac{3i}{2t} \xi t^{\frac{1}{3}} \left\langle t^{\frac{1}{3}} \xi \right\rangle^{-1} |\widehat{\varphi}(t,\xi)|^2 \widehat{\varphi}(t,\xi) \\
& + O\left(t^{-1} \xi t^{\frac{1}{3}} \left\langle \xi t^{\frac{1}{3}} \right\rangle^{-1-\gamma} \|\widehat{\varphi}\|_{\mathbf{W}}^3 \right)
\end{aligned}$$

is true for all $t \geq 1$ and $\xi > 0$, where $\widehat{\varphi}(t) = \mathcal{FU}(-t)u(t)$, γ is small.

Proof. In view of the factorization property (1.8), we have for the case of $\xi > t^{-\frac{1}{3}}$

$$\begin{aligned}
& \mathcal{FU}(-t)\partial_x(u^3) \\
= & \; 3t^{-1} E^{-\frac{8}{9}} \mathcal{D}_3 \mathcal{Q}(3t) (\mathcal{V}\widehat{\varphi})^2 (\mathcal{V} i\xi\widehat{\varphi}) \\
& + 3it^{-1} \mathcal{Q}(t) \left(2(\mathcal{V}\widehat{\varphi}) \left(\overline{\mathcal{V}\widehat{\varphi}}\right) (\mathcal{V}\xi\widehat{\varphi}) - (\mathcal{V}\widehat{\varphi})^2 \left(\overline{\mathcal{V}\xi\widehat{\varphi}}\right) \right) \\
& + R_1 + R_2 + R_3,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= 3t^{-1} \mathcal{D}_{-1} \mathcal{Q}(-t) \left(\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2 (\mathcal{V} i\xi\widehat{\varphi}) + 2(\mathcal{V}\widehat{\varphi}) \left(\overline{\mathcal{V}\widehat{\varphi}}\right) \left(\overline{\mathcal{V} i\xi\widehat{\varphi}}\right) \right), \\
R_2 &= 3t^{-1} E^{-\frac{8}{9}} \mathcal{D}_{-3} \mathcal{Q}(-3t) \left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2 \left(\overline{\mathcal{V} i\xi\widehat{\varphi}}\right), \\
R_3 &= 3t^{-1} \mathcal{R} (\mathcal{W}\widehat{\varphi})^2 (\mathcal{W} i\xi\widehat{\varphi}).
\end{aligned}$$

To estimate R_1 and R_2 we apply Lemma 2.2 with $\alpha = 0$, $\beta \in \left(\frac{1}{2}, \frac{3}{4}\right)$ to get

$$\begin{aligned}
&\left\| \left\langle t^{\frac{1}{3}} \xi \right\rangle^{\frac{3}{4}-\beta} R_1 \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
= \;& Ct^{-1} \left\| \left\langle t^{\frac{1}{3}} \xi \right\rangle^{\frac{3}{4}-\beta} \mathcal{D}_{-1} \mathcal{Q}(-t)\, \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
= \;& Ct^{-1} \left\| \left\langle t^{\frac{1}{3}} \xi \right\rangle^{\frac{3}{4}-\beta} \mathcal{Q}(-t)\, \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \\
\leq \;& Ct^{-1} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{\frac{1}{2}} \mathcal{V}\widehat{\varphi} \right\|^2_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta-1} \partial_x \left(x \mathcal{V} i\xi \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& + Ct^{-1} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{\frac{1}{2}} \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\frac{1}{2}} \mathcal{V} i\xi \widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
& \times \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \left(x \overline{\mathcal{V}\widehat{\varphi}} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left\| \left\langle t^{\frac{1}{3}} \xi \right\rangle^{\frac{3}{4}-\beta} R_1 \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
\leq \;& Ct^{-1} t^{\frac{1}{3}} \left\| \widehat{\varphi} \right\|^2_{\mathbf{W}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta-1} \partial_x \left(x \mathcal{V} i\xi \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ \;& Ct^{-1} \left\| \widehat{\varphi} \right\|^2_{\mathbf{W}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \left(x \overline{\mathcal{V}\widehat{\varphi}} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq \;& Ct^{-1} t^{\frac{1}{6}} \left\| \widehat{\varphi} \right\|^3_{\mathbf{W}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta-\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ \;& Ct^{-1} \left\| \widehat{\varphi} \right\|^2_{\mathbf{W}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \mathcal{V} i\xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ \;& Ct^{-1} \left\| \widehat{\varphi} \right\|^2_{\mathbf{W}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} x \partial_x \overline{\mathcal{V}\widehat{\varphi}} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq Ct^{-1} \left\| \widehat{\varphi} \right\|^3_{\mathbf{W}},
\end{aligned}$$

since by Remark 2.3 we have

$$\left\langle t^{\frac{1}{3}} x \right\rangle^{\frac{1}{2}-j} \left| \mathcal{V} \xi^j \phi \right| \leq Ct^{\frac{1}{6}-\frac{j}{3}} \left\| \phi \right\|_{\mathbf{W}}$$

and by Lemma 2.4 we find

$$\left\| \left\langle x t^{\frac{1}{3}} \right\rangle^{-\beta} x^{1-j} \partial_x \mathcal{V} \xi^j \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| \phi \right\|_{\mathbf{W}}.$$

Similarly

$$\left\| \left\langle t^{\frac{1}{3}}\xi \right\rangle^{\frac{3}{4}-\beta} R_2 \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \le Ct^{-1} \|\widehat{\varphi}\|_{\mathbf{W}}^3 .$$

Consider the estimate for R_3. Using Lemma 2.2 we obtain

$$\begin{aligned} \left\| \left\langle t^{\frac{1}{3}}\xi \right\rangle^{\frac{3}{4}} R_3 \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} &= Ct^{-1} \left\| \left\langle t^{\frac{1}{3}}\xi \right\rangle^{\frac{3}{4}} \mathcal{R}\left(\mathcal{W}\widehat{\varphi}\right)^2 \left(\mathcal{W}i\xi\widehat{\varphi}\right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ \le \quad & Ct^{-1} \left\| \left(\mathcal{W}\widehat{\varphi}\right)^2 \left(\mathcal{W}i\xi\widehat{\varphi}\right) \right\|_{\mathbf{L}^2(\mathbf{R}_-)} + Ct^{-1} \left\| \left(\mathcal{W}\widehat{\varphi}\right) \left(\mathcal{W}i\xi\widehat{\varphi}\right) x\partial_x \mathcal{W}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\ & + Ct^{-1} \left\| x \left(\mathcal{W}\widehat{\varphi}\right)^2 \left(\partial_x \mathcal{W}i\xi\widehat{\varphi}\right) \right\|_{\mathbf{L}^2(\mathbf{R}_-)} . \end{aligned}$$

Then by Lemma 2.3

$$\left| \mathcal{W}\xi^j \phi \right| \le Ct^{-\frac{j}{3}} t^{\frac{1}{6}} \left\langle xt^{\frac{1}{3}} \right\rangle^{j-\frac{3}{2}} \|\phi\|_{\mathbf{W}}$$

and by Lemma 2.5 with $\alpha = 0$ and $\beta \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$$\left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-\beta} x^{1-j} \partial_x \mathcal{W}\xi^j \phi \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C \|\phi\|_{\mathbf{W}} .$$

Hence

$$\begin{aligned} & \left\| \left\langle t^{\frac{1}{3}}\xi \right\rangle^{\frac{3}{4}} R_3 \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \le Ct^{-1} t^{\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{W}}^3 \left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-\frac{7}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\ & + Ct^{-1} \|\widehat{\varphi}\|_{\mathbf{W}}^2 \left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-2} x\partial_x \mathcal{W}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\ & + Ct^{-1} \|\widehat{\varphi}\|_{\mathbf{W}}^2 \left\| \left\langle xt^{\frac{1}{3}} \right\rangle^{-2} \partial_x \mathcal{W}i\xi\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le Ct^{-1} \|\widehat{\varphi}\|_{\mathbf{W}}^3 . \end{aligned}$$

Next we calculate the asymptotics of

$$I_1 = 3t^{-1} E^{-\frac{8}{9}} \mathcal{D}_3 \mathcal{Q}\left(3t\right) \left(\mathcal{V}\widehat{\varphi}\right)^2 \left(\mathcal{V}i\xi\widehat{\varphi}\right)$$

and

$$I_2 = 3it^{-1} \mathcal{Q}\left(t\right) \left(2\left(\mathcal{V}\widehat{\varphi}\right) \left(\overline{\mathcal{V}\widehat{\varphi}}\right) \left(\mathcal{V}\xi\widehat{\varphi}\right) - \left(\mathcal{V}\widehat{\varphi}\right)^2 \left(\overline{\mathcal{V}\xi\widehat{\varphi}}\right) \right) .$$

By Lemma 2.2 with $\alpha = 0$, $\beta \in \left(\frac{1}{2}, \frac{3}{4}\right)$ we find

$$\begin{aligned} \mathcal{Q}\left(t\right) \phi \quad &= \quad t^{\frac{1}{6}} \widetilde{A_0} \left(t^{\frac{1}{3}}\xi \right) \xi\phi\left(\xi\right) \\ & + O\left(\left\langle t^{\frac{1}{3}}\xi \right\rangle^{\beta-\frac{3}{4}} \left\| \left\langle t^{\frac{1}{3}} x \right\rangle^{-\beta} \partial_x \left(x\phi\left(x\right) \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \right) . \end{aligned}$$

Hence

$$\begin{aligned} I_1 &= 3t^{-1}E^{-\frac{8}{9}}\mathcal{D}_3 (3t)^{\frac{1}{6}}\widetilde{A_0}\left((3t)^{\frac{1}{3}}\xi\right)\xi\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right) \\ &\quad +O\left(t^{-1}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\beta-\frac{3}{4}}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right)\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\right) \end{aligned}$$

and

$$\begin{aligned} I_2 &= 3it^{-1}t^{\frac{1}{6}}\widetilde{A_0}\left(t^{\frac{1}{3}}\xi\right)\xi\left(2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\left(\mathcal{V}\xi\widehat{\varphi}\right)-\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}\xi\widehat{\varphi}}\right)\right) \\ &\quad +O\left(t^{-1}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\beta-\frac{3}{4}}\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\right). \end{aligned}$$

The remainder can be estimated as follows

$$\begin{aligned} &\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\left(x\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right)\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \le\ & C\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & +C\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\left(\mathcal{V}\widehat{\varphi}\right)\left(\mathcal{V}i\xi\widehat{\varphi}\right)x\partial_x\mathcal{V}\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & +C\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\left(\mathcal{V}\widehat{\varphi}\right)^2x\partial_x\mathcal{V}i\xi\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \le\ & Ct^{\frac{1}{6}}\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^3\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta-\frac{1}{2}}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}+C\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^2\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}x\partial_x\mathcal{V}\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & +C\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^2\left\|\left\langle t^{\frac{1}{3}}x\right\rangle^{-\beta}\partial_x\mathcal{V}i\xi\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\le C\left\|\widehat{\varphi}\right\|_{\mathbf{W}}^3. \end{aligned}$$

Next by Lemma 2.3

$$\mathcal{V}\xi^j\phi = t^{\frac{1}{6}-\frac{j}{3}}A_j\left(xt^{\frac{1}{3}}\right)\phi(x)+O\left(t^{\frac{1}{6}-\frac{j}{3}}\left\|\phi\right\|_{\mathbf{W}}\left\langle xt^{\frac{1}{3}}\right\rangle^{j-\frac{3}{4}}\right)$$

and so we get

$$\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right) = t^{\frac{1}{6}}A_0^2\left(xt^{\frac{1}{3}}\right)A_1\left(xt^{\frac{1}{3}}\right)\widehat{\varphi}^3(x)+O\left(t^{\frac{1}{6}}\left\|\phi\right\|_{\mathbf{W}}^3\left\langle xt^{\frac{1}{3}}\right\rangle^{-\frac{3}{4}}\right)$$

and

$$\begin{aligned} & 2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\left(\mathcal{V}\xi\widehat{\varphi}\right)-\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}\xi\widehat{\varphi}}\right) \\ =\ & 2t^{\frac{1}{6}}A_0\left(xt^{\frac{1}{3}}\right)\overline{A_0\left(xt^{\frac{1}{3}}\right)}A_1\left(xt^{\frac{1}{3}}\right)\left|\widehat{\varphi}\right|^2\widehat{\varphi}(x) \\ & -t^{\frac{1}{6}}\left(A_0\left(xt^{\frac{1}{3}}\right)\right)^2\overline{A_1\left(xt^{\frac{1}{3}}\right)}\left|\widehat{\varphi}\right|^2\widehat{\varphi}(x)+O\left(t^{\frac{1}{6}}\left\|\phi\right\|_{\mathbf{W}}^3\left\langle xt^{\frac{1}{3}}\right\rangle^{-\frac{3}{4}}\right). \end{aligned}$$

By Lemma 2.1

$$A_0\left(\xi t^{\frac{1}{3}}\right) = \frac{1}{\sqrt{2i}}\left(\xi t^{\frac{1}{3}}\right)^{-\frac{1}{2}} + O\left(\left\langle \xi t^{\frac{1}{3}}\right\rangle^{-\frac{7}{2}}\right),$$

$$\widetilde{A_0}\left(\xi t^{\frac{1}{3}}\right) = \sqrt{2i}\left(\xi t^{\frac{1}{3}}\right)^{-\frac{1}{2}} + O\left(\left\langle \xi t^{\frac{1}{3}}\right\rangle^{-\frac{7}{2}}\right),$$

hence

$$I_1 = \frac{\sqrt{3}}{2it}E^{-\frac{8}{9}}\widehat{\varphi}^3\left(\frac{\xi}{3}\right) + O\left(t^{-1}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\beta-\frac{3}{4}}\|\widehat{\varphi}\|^3_{\mathbf{W}}\right)$$

and

$$I_2 = \frac{3}{2t}i\left|\widehat{\varphi}\right|^2\widehat{\varphi} + O\left(t^{-1}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{\beta-\frac{3}{4}}\|\widehat{\varphi}\|^3_{\mathbf{W}}\right).$$

Consider the case of $0 \le \xi \le t^{-\frac{1}{3}}$. In view of the factorization property we have

$$\begin{aligned}
&\mathcal{FU}(-t)\,\partial_x\left(u^3\right)\\
=\;& i\xi t^{-1}E^{-\frac{8}{9}}\mathcal{D}_3\mathcal{Q}(3t)\left(\mathcal{V}\widehat{\varphi}\right)^3 + 3i\xi t^{-1}\mathcal{Q}(t)\left(\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\right)\\
&+R_1 + R_2 + R_3,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= i\xi t^{-1}\mathcal{D}_{-1}\mathcal{Q}(-t)\left(\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2\left(\mathcal{V}\widehat{\varphi}\right)\right),\\
R_2 &= i\xi t^{-1}E^{-\frac{8}{9}}\mathcal{D}_{-3}\mathcal{Q}(-3t)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2\left(\overline{\mathcal{V}\widehat{\varphi}}\right),\\
R_3 &= i\xi t^{-1}\mathcal{R}\left(\mathcal{W}\widehat{\varphi}\right)^2\left(\mathcal{W}\widehat{\varphi}\right).
\end{aligned}$$

As above we get

$$\begin{aligned}
\mathcal{FU}(-t)\,\partial_x\left(u^3\right) &= \frac{\sqrt{3}}{2it}\xi t^{\frac{1}{3}}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{-1}e^{-\frac{8it}{27}\xi^3}\widehat{\varphi}^3\left(t,\frac{\xi}{3}\right)\\
&\quad+\frac{3}{2t}i\xi t^{\frac{1}{3}}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{-1}\left|\widehat{\varphi}(t,\xi)\right|^2\widehat{\varphi}(t,\xi)\\
&\quad+O\left(t^{-1}\xi t^{\frac{1}{3}}\left\langle \xi t^{\frac{1}{3}}\right\rangle^{-1-\gamma}\|\widehat{\varphi}\|^3_{\mathbf{W}}\right),
\end{aligned}$$

where $\gamma > 0$ is small. Lemma 2.6 is proved. □

§3. A priori estimates

Local existence and uniqueness of solutions to the Cauchy problem (1.1) was shown in [22] and [20] when $u_0 \in \mathbf{H}^s$, $s > \frac{3}{4}$ and the estimate of solutions such that $\int_0^T \|\partial_x u(t)\|_{\mathbf{L}^\infty}^4 dt \le C$ for some time T was also shown in these papers. By using the local existence result, we have (for the result $\mathcal{U}(-t)u \in \mathbf{C}\left([0,T];\cap \mathbf{H}^{0,1}\right)$, see the estimate of $\mathcal{J}u(t)$ in Lemma 3.2 below).

Theorem 3.1. *Assume that the initial data* $u_0 \in \mathbf{H}^s \cap \mathbf{H}^{0,1}$, $s > \frac{3}{4}$. *Then there exists a unique local solution* u *of the Cauchy problem (1.1) such that* $\mathcal{U}(-t)u \in \mathbf{C}\left([0,T];\mathbf{H}^s \cap \mathbf{H}^{0,1}\right)$.

We can take $T > 1$ if the data are small in $\mathbf{H}^s \cap \mathbf{H}^{0,1}$ and we may assume that

$$\|\mathcal{F}\mathcal{U}(-t)u(1)\|_{\mathbf{L}^\infty} + \|\mathcal{J}u(1)\|_{\mathbf{L}^2} + \|u(1)\|_{\mathbf{L}^p} \le \varepsilon, \tag{3.1}$$

where $p > 4$. To get the desired results, we prove a priori estimates of solutions uniformly in time. Define the following norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[1,T]} \left(\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{L}^\infty} + t^{-\frac{1}{6}} \|\mathcal{J}u(t)\|_{\mathbf{L}^2} + t^{\frac{1}{3}\left(1-\frac{1}{p}\right)} \|u(t)\|_{\mathbf{L}^p} \right),$$

where $\mathcal{J} = x - t\partial_x^2 = \mathcal{U}(t)x\mathcal{U}(-t)$.

Lemma 3.2. *Assume that (3.1) holds. Then there exists an* ε *such that the estimate*

$$\|u\|_{\mathbf{X}_T} < C\varepsilon$$

is true for all $T > 1$.

Proof. By continuity of the norm $\|u\|_{\mathbf{X}_T}$ with respect to T, arguing by the contradiction we can find the first time $T > 0$ such that $\|u\|_{\mathbf{X}_T} = C\varepsilon$. Define as above $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)u(t)$, then applying estimate of Lemma 2.3 we find for $x > 0$

$$\begin{aligned}
& \left|\partial_x^j u(t,x)\right| \\
= & \left|\mathcal{D}_t\mathcal{B}\left(M\mathcal{V}i\xi^j\widehat{\varphi} + \overline{M\mathcal{V}i\xi^j\widehat{\varphi}}\right)\right| \\
\le & Ct^{-\frac{1}{3}-\frac{j}{3}}\left\langle x^{\frac{1}{2}}t^{-\frac{1}{6}}\right\rangle^{j-\frac{1}{2}}\left|\widehat{\varphi}\left(x^{\frac{1}{2}}t^{-\frac{1}{2}}\right)\right| \\
& + Ct^{-\frac{1}{3}-\frac{j}{3}}\left\langle x^{\frac{1}{2}}t^{-\frac{1}{6}}\right\rangle^{j-\frac{3}{4}}\left(|\widehat{\varphi}(0)| + \langle t\rangle^{-\frac{1}{6}}\|\widehat{\varphi}_\xi\|_{\mathbf{L}^2}\right) \\
\le & Ct^{-\frac{1}{3}-\frac{j}{3}}\left\langle x^{\frac{1}{2}}t^{-\frac{1}{6}}\right\rangle^{j-\frac{1}{2}}\left(\|\widehat{\varphi}\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{1}{6}}\|\widehat{\varphi}_\xi\|_{\mathbf{L}^2}\right) \\
\le & C\|u\|_{\mathbf{X}_T}t^{-\frac{1}{3}-\frac{j}{3}}\left\langle x^{\frac{1}{2}}t^{-\frac{1}{6}}\right\rangle^{j-\frac{1}{2}} \le C\varepsilon\left\langle x^{\frac{1}{2}}t^{-\frac{1}{6}}\right\rangle^{j-\frac{1}{2}}t^{-\frac{1}{3}-\frac{j}{3}}
\end{aligned} \tag{3.2}$$

and for $x \leq 0$

$$\begin{aligned} \left|\partial_x^j u(t,x)\right| &= \left|\mathcal{D}_t \mathcal{B} \mathcal{W} i \xi^j \widehat{\varphi}\right| \\ &\leq Ct^{-\frac{1}{3}-\frac{j}{3}} \left\langle x|x|^{-\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{j-\frac{3}{2}} \left(|\widehat{\varphi}(0)| + t^{-\frac{1}{6}} \|\widehat{\varphi}_\xi\|_{\mathbf{L}^2}\right) \\ &\leq C\varepsilon t^{-\frac{1}{3}-\frac{j}{3}} \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{j-\frac{3}{2}}. \qquad (3.3) \end{aligned}$$

Hence

$$\left|\partial_x^j u(t,x)\right| \leq C\varepsilon \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{j-\frac{1}{2}} t^{-\frac{1}{3}-\frac{j}{3}},$$

$$|u(t,x)\,\partial_x u(t,x)| \leq C\varepsilon^2 t^{-1}$$

and

$$\|u\|_{\mathbf{L}^p} \leq Ct^{-\frac{1}{3}} \|u\|_{\mathbf{X}_T} \left\| \left\langle x^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-\frac{1}{2}} \right\|_{\mathbf{L}^p} \leq C\varepsilon t^{-\frac{1}{3}\left(1-\frac{1}{p}\right)} \qquad (3.4)$$

for $p > 4$. Consider a-priori estimates of $\|\mathcal{J}u(t)\|_{\mathbf{L}^2}$. Using the identity

$$\partial_x^{-1}\mathcal{P}u - \mathcal{J}u = 3t\partial_x^{-1}\mathcal{L}u$$

with $\mathcal{J} = x + t\partial_x^2$, $\mathcal{L} = \partial_t - \frac{1}{3}\partial_x^3$, $\mathcal{P} = \partial_x x + 3t\partial_t$, we get

$$\|\mathcal{J}u\|_{\mathbf{L}^2} \leq C\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} + Ct\|u\|_{\mathbf{L}^6}^3 \leq C\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} + C\varepsilon^3 t^{\frac{1}{6}}.$$

We apply the operator $\partial_x^{-1}\mathcal{P}$ to equation (1.1). In view of the commutators $[\mathcal{L},\mathcal{P}] = 3\mathcal{L}$, $[\mathcal{P},\partial_x] = -\partial_x$, we get

$$\mathcal{L}\partial_x^{-1}\mathcal{P}u = \partial_x^{-1}(\mathcal{P}+3)\mathcal{L}u = \partial_x^{-1}(\mathcal{P}+3)\partial_x\left(u^3\right) = (\mathcal{P}+2)\left(u^3\right).$$

Then by the energy method we obtain

$$\begin{aligned} \frac{d}{dt}\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2}^2 &= 2\int_{\mathbf{R}} \left(\partial_x^{-1}\mathcal{P}u\right)(\mathcal{P}+2)\left(u^3\right)dx \\ &= 6\int_{\mathbf{R}} u^2\left(\partial_x^{-1}\mathcal{P}u\right)\mathcal{P}u\,dx + 4\int_{\mathbf{R}}\left(u^3\right)\partial_x^{-1}\mathcal{P}u\,dx \\ &= -6\int_{\mathbf{R}} uu_x\left(\partial_x^{-1}\mathcal{P}u\right)^2 dx + 4\int_{\mathbf{R}}\left(u^3\right)\partial_x^{-1}\mathcal{P}u\,dx \\ &\leq C\|uu_x\|_{\mathbf{L}^\infty}\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2}^2 + \|u\|_{\mathbf{L}^6}^3\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \\ &\leq C\varepsilon^2 t^{-1}\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2}^2 + C\varepsilon^3 t^{-\frac{5}{6}}\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \end{aligned}$$

from which it follows

$$\frac{d}{dt}e^{-C\varepsilon^2\int_1^t t^{-1}dt}\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-\frac{5}{6}} e^{-C\varepsilon^2\int_1^t t^{-1}dt}.$$

Then integrating we get

$$\begin{aligned}\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} &\leq \left\|\partial_x^{-1}\mathcal{P}u(1)\right\|_{\mathbf{L}^2} e^{C\varepsilon^2\int_1^t t^{-1}dt} \\ &\quad + C\varepsilon^3 e^{C\varepsilon^2\int_1^t t^{-1}dt}\int_1^t s^{-\frac{5}{6}} e^{-C\varepsilon^2\int_1^s \tau^{-1}d\tau}ds \\ &\leq \left\|\partial_x^{-1}\mathcal{P}u(1)\right\|_{\mathbf{L}^2} t^{C\varepsilon^2} + C\varepsilon^3 t^{\frac{1}{6}}.\end{aligned}$$

Since

$$\left\|\partial_x^{-1}\mathcal{P}u(1)\right\|_{\mathbf{L}^2} \leq \|\mathcal{J}u(1)\|_{\mathbf{L}^2} + C\|u(1)\|_{\mathbf{L}^6}^3 \leq \varepsilon + C\varepsilon^3$$

we have

$$\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} \leq \varepsilon t^{C\varepsilon^2} + C\varepsilon^3 t^{\frac{1}{6}}.$$

Therefore

$$\|\mathcal{J}u\|_{\mathbf{L}^2} \leq C\left\|\partial_x^{-1}\mathcal{P}u\right\|_{\mathbf{L}^2} + Ct\|u\|_{\mathbf{L}^6}^3 \leq C\varepsilon^3 t^{\frac{1}{6}}$$

for all $t \in [1,T]$. Finally we need the estimate for $\|\widehat{\varphi}\|_{\mathbf{L}^\infty}$. By equation (1.8) for $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$, using Lemma 2.6 we get

$$\partial_t\widehat{\varphi} = O\left(\xi t^{-\frac{2}{3}}\|\widehat{\varphi}\|_{\mathbf{W}}^3\right)$$

for $0 < \xi < t^{-\frac{1}{3}}$ and

$$\begin{aligned}\partial_t\widehat{\varphi} &= \frac{\sqrt{3}}{2it}e^{-\frac{8it}{27}\xi^3}\widehat{\varphi}^3\left(t,\frac{\xi}{3}\right) + \frac{3i}{2t}\left|\widehat{\varphi}(t,\xi)\right|^2\widehat{\varphi}(t,\xi) \\ &\quad + O\left(t^{-1}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-\gamma}\|\widehat{\varphi}\|_{\mathbf{W}}^3\right)\end{aligned}$$

for $\xi > t^{-\frac{1}{3}}$. For the case of $0 < \xi < t^{-\frac{1}{3}}$ we can integrate

$$|\widehat{\varphi}(t,\xi)| \leq |\widehat{\varphi}(1,\xi)| + C\xi\|\widehat{\varphi}\|_{\mathbf{W}}^3\int_1^t t^{-\frac{2}{3}}dt \leq \varepsilon + C\xi t^{\frac{1}{3}}\varepsilon^3 \leq \varepsilon + C\varepsilon^3.$$

For the case of $\xi \geq t^{-\frac{1}{3}}$ choosing

$$\Psi(t,\xi) = \exp\left(\frac{3i}{2}\int_{\xi^{-3}}^t |\widehat{\varphi}(t,\xi)|^2\frac{d\tau}{\tau}\right)$$

we get

$$\begin{aligned}&\partial_t\left(\widehat{\varphi}(t,\xi)\Psi(t,\xi)\right) \\ &= \frac{\sqrt{3}}{2it}e^{-\frac{8it}{27}\xi^3}\widehat{\varphi}^3\left(t,\frac{\xi}{3}\right)\Psi(t,\xi) + O\left(\varepsilon^3 t^{-1}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-\gamma}\right).\end{aligned}$$

Integrating in time , we obtain

$$\begin{aligned} & \left|\widehat{\varphi}\left(t,\xi\right)\Psi\left(t,\xi\right)\right| \\ \leq\ & \widehat{\varphi}\left(\xi^{-3},\xi\right)+C\int_{\xi^{-3}}^{t}e^{-\frac{8i\tau}{27}\xi^{3}}\widehat{\varphi}^{3}\left(\tau,\frac{\xi}{3}\right)\Psi\left(\tau,\xi\right)\frac{d\tau}{\tau} \\ & +C\varepsilon^{3}\int_{\xi^{-3}}^{t}\left\langle \xi\tau^{\frac{1}{3}}\right\rangle^{-\gamma}\frac{d\tau}{\tau} \end{aligned}$$

Integrating by parts we get

$$\begin{aligned} & \left|\widehat{\varphi}\left(t\right)\right| \\ \leq\ & C\varepsilon+C\xi^{-3}\left|\int_{\xi^{-3}}^{t}e^{-\frac{8i\tau}{27}\xi^{3}}\partial_{\tau}\left(\tau^{-1}\widehat{\varphi}^{3}\left(\tau,\frac{\xi}{3}\right)\Psi\left(\tau,\xi\right)\right)d\tau\right| \\ & +C\varepsilon^{3}\int_{1}^{t\xi^{3}}\frac{dz}{z^{1+\frac{\gamma}{3}}}\leq C\varepsilon+\frac{C}{\xi^{3}t}\varepsilon^{3}+C\varepsilon^{3}<C\varepsilon \end{aligned}$$

for $\xi>0$. Since the solution u is real, we have $\overline{\widehat{\varphi}\left(t,\xi\right)}=\widehat{\varphi}\left(t,-\xi\right)$. Therefore $\left\|\mathcal{FU}\left(-t\right)u\left(t\right)\right\|_{\mathbf{L}^{\infty}}<C\varepsilon$. Thus we obtain $\left\|u\right\|_{\mathbf{X}_{T}}<C\varepsilon$. Lemma 3.2 is proved. □

Also we consider the estimates for the difference of two solutions u_j with the same mass. Define

$$\left\|u_{1}-u_{2}\right\|_{\mathbf{Y}_{T}}=\sup_{t\in[1,T]}\left(t^{\frac{1}{2}-\gamma}\left\|u_{1}-u_{2}\right\|_{\mathbf{L}^{\infty}}+t^{-\gamma}\left\|\mathcal{J}\left(u_{1}-u_{2}\right)\right\|_{\mathbf{L}^{2}}\right)$$

with a small $\gamma>0$.

Lemma 3.3. *Suppose that* $\left\|u_{j}\right\|_{\mathbf{X}_{T}}\leq C\varepsilon$, $j=1,2$, *where* ε *is sufficiently small. Let* $\widehat{\varphi_{1}}\left(t,0\right)=\widehat{\varphi_{2}}\left(t,0\right)$ *for* $j=1,2$, $t\geq1$, *where* $\widehat{\varphi_{j}}\left(t,\xi\right)=\mathcal{FU}\left(-t\right)u_{j}$. *Let* $u_{2}=t^{-\frac{1}{3}}f\left(xt^{-\frac{1}{3}}\right)$ *be a self-similar solution. Then the estimate*

$$\left\|u_{1}-u_{2}\right\|_{\mathbf{Y}_{T}}<C\varepsilon$$

is true for all $T>1$.

Proof. By the continuity of the norm $\left\|u_{1}-u_{2}\right\|_{\mathbf{Y}_{T}}$ with respect to T, arguing by the contradiction we can find the first time $T>0$ such that $\left\|u_{1}-u_{2}\right\|_{\mathbf{Y}_{T}}=C\varepsilon$. We let $\widehat{w}=\widehat{\varphi_{1}}-\widehat{\varphi_{2}}, y=u_{1}-u_{2}$. Applying estimate of Lemma 2.3 we find for $x>0$

$$\begin{aligned} \left|y\left(t,x\right)\right| &= \left|\mathcal{D}_{t}\mathcal{B}\left(M\mathcal{V}\widehat{w}+\overline{M\mathcal{V}\widehat{w}}\right)\right| \\ &\leq Ct^{-\frac{1}{3}}\left\langle x^{\frac{1}{2}}t^{-\frac{1}{6}}\right\rangle^{-\frac{1}{2}}\left|\widehat{w}\left(t,x^{\frac{1}{2}}t^{-\frac{1}{2}}\right)\right| \\ & \quad +Ct^{-\frac{1}{2}}\left\langle x^{\frac{1}{2}}t^{-\frac{1}{6}}\right\rangle^{-\frac{3}{4}}\left\|\partial_{\xi}\widehat{w}\right\|_{\mathbf{L}^{2}} \\ &\leq Ct^{-\frac{1}{2}}\left\|\partial_{\xi}\widehat{w}\right\|_{\mathbf{L}^{2}} \qquad (3.5) \end{aligned}$$

and for $x \leq 0$

$$\begin{aligned} |y(t,x)| &= |\mathcal{D}_t \mathcal{B} \mathcal{W} \widehat{w}| \\ &\leq Ct^{-\frac{1}{2}} \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-\frac{3}{2}} \|\partial_\xi \widehat{w}\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{2}} \|\partial_\xi \widehat{w}\|_{\mathbf{L}^2} \end{aligned} \tag{3.6}$$

since

$$\left| \widehat{w}\left(|x|^{\frac{1}{2}} t^{-\frac{1}{2}} \right) \right| \leq C \left| |x|^{\frac{1}{2}} t^{-\frac{1}{2}} \right|^{\frac{1}{2}} \|\partial_\xi \widehat{w}\|_{\mathbf{L}^2}.$$

Thus we need to estimate the norm $\|\mathcal{J} y\|_{\mathbf{L}^2}$. From equation (1.1) we get for the difference y

$$\mathcal{L} \partial_x^{-1} \mathcal{P} y = (\mathcal{P} + 2) \left(u_1^3 - u_2^3 \right).$$

Hence by the energy method

$$\begin{aligned} &\frac{d}{dt} \left\| \partial_x^{-1} \mathcal{P} y \right\|_{\mathbf{L}^2}^2 \\ &= 6 \int_{\mathbf{R}} \partial_x^{-1} \mathcal{P} y \left(u_1^2 \mathcal{P} u_1 - u_2^2 \mathcal{P} u_2 \right) dx + 4 \int_{\mathbf{R}} \partial_x^{-1} \mathcal{P} y \left(u_1^3 - u_2^3 \right) dx. \end{aligned}$$

Next we get

$$\begin{aligned} & 6 \int_{\mathbf{R}} \partial_x^{-1} \mathcal{P} y \left(\left(u_1^2 \mathcal{P} u_1 - u_2^2 \mathcal{P} u_2 \right) \right) dx \\ &= 6 \int_{\mathbf{R}} u_1^2 \partial_x^{-1} \mathcal{P} y \mathcal{P} y dx + 6 \int_{\mathbf{R}} \left(u_1^2 - u_2^2 \right) \mathcal{P} u_2 \partial_x^{-1} \mathcal{P} y dx \\ &= 6 \int_{\mathbf{R}} u_1 u_{1x} \left(\partial_x^{-1} \mathcal{P} y \right)^2 dx + 6 \int_{\mathbf{R}} \left(u_1^2 - u_2^2 \right) \mathcal{P} u_2 \partial_x^{-1} \mathcal{P} y dx \\ &\leq C \|u_1 u_{1x}\|_{\mathbf{L}^\infty} \left\| \partial_x^{-1} \mathcal{P} y \right\|_{\mathbf{L}^2}^2 \\ &\quad + C \left\| \partial_x^{-1} \mathcal{P} y \right\|_{\mathbf{L}^2} \left\| \left(u_1^2 - u_2^2 \right) \mathcal{P} u_2 \right\|_{\mathbf{L}^2}. \end{aligned}$$

Note that

$$\mathcal{P} u_2 = \partial_x x t^{-\frac{1}{3}} f\left(x t^{-\frac{1}{3}} \right) + 3t \partial_t t^{-\frac{1}{3}} f\left(x t^{-\frac{1}{3}} \right) = 0$$

for the case of self-similar solution $u_2 = t^{-\frac{1}{3}} f\left(x t^{-\frac{1}{3}} \right)$. Hence

$$\begin{aligned} &\frac{d}{dt} \left\| \partial_x^{-1} \mathcal{P} y \right\|_{\mathbf{L}^2}^2 \\ &\leq C \|u_1 u_{1x}\|_{\mathbf{L}^\infty} \left\| \partial_x^{-1} \mathcal{P} y \right\|_{\mathbf{L}^2}^2 + C \left\| \partial_x^{-1} \mathcal{P} y \right\|_{\mathbf{L}^2} \left\| u_1^3 - u_2^3 \right\|_{\mathbf{L}^2}. \end{aligned} \tag{3.7}$$

By (3.2) and (3.3) we have for $x > 0$

$$\left\langle x^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{\frac{1}{2}} |u(t,x)| \leq C \varepsilon t^{-\frac{1}{3}}$$

and for $x \leq 0$

$$\left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{\frac{3}{2}} |u(t,x)| \leq C\varepsilon t^{-\frac{1}{3}}.$$

To estimate $\left\| u_1^3 - u_2^3 \right\|_{\mathbf{L}^2}$ we use the above estimates to get

$$\begin{aligned} & \left\| u_1^3 - u_2^3 \right\|_{\mathbf{L}^2} \\ \leq\ & C \sum_{j=1}^{2} \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{\frac{1}{2}} u_j \right\|_{\mathbf{L}^\infty}^2 \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-1} y \right\|_{\mathbf{L}^2} \\ \leq\ & C\varepsilon^2 t^{-\frac{2}{3}} \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-1} y \right\|_{\mathbf{L}^2}. \end{aligned} \tag{3.8}$$

In view of Lemma 2.3

$$\begin{aligned} \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-1} y \right\|_{\mathbf{L}^2} \leq\ & Ct^{-\frac{1}{3}} \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-\frac{3}{2}} \widehat{w}\left(t, x^{\frac{1}{2}} t^{-\frac{1}{2}}\right) \right\|_{\mathbf{L}^2_x(0,\infty)} \\ & + Ct^{-\frac{1}{2}} \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-\frac{7}{4}} \right\|_{\mathbf{L}^2} \left\| \partial_\xi \widehat{w} \right\|_{\mathbf{L}^2}. \end{aligned}$$

Since $\widehat{w}(0) = 0$, we get

$$\begin{aligned} & \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-\frac{3}{2}} \widehat{w}\left(t, x^{\frac{1}{2}} t^{-\frac{1}{2}}\right) \right\|_{\mathbf{L}^2_x(0,\infty)}^2 \\ \leq\ & C \left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-\frac{3}{2}} x^{\frac{1}{2}} t^{-\frac{1}{2}} \left(\partial_\xi \widehat{w}\right)\left(t, \theta x^{\frac{1}{2}} t^{-\frac{1}{2}}\right) \right\|_{\mathbf{L}^2_x(0,\infty)}^2 \\ =\ & C \int_0^\infty \left\langle x^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-3} x t^{-1} \left| \left(\partial_\xi \widehat{w}\right)\left(t, \theta x^{\frac{1}{2}} t^{-\frac{1}{2}}\right) \right|^2 dx \\ \leq\ & C \int_0^\infty \left\langle y t^{\frac{1}{3}} \right\rangle^{-3} y^2 \left| \left(\partial_\xi \widehat{w}\right)(t, \theta y) \right|^2 t y dy \leq C \left\| \partial_\xi \widehat{w} \right\|_{\mathbf{L}^2}^2 \end{aligned}$$

and by a direct calculation

$$\left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-\frac{7}{4}} \right\|_{\mathbf{L}^2} \leq Ct^{\frac{1}{6}} \left(\int \langle y \rangle^{-\frac{7}{4}} dy \right)^{\frac{1}{2}} \leq Ct^{\frac{1}{6}}$$

Hence

$$\left\| \left\langle |x|^{\frac{1}{2}} t^{-\frac{1}{6}} \right\rangle^{-1} y \right\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{3}} \left\| \partial_\xi \widehat{w} \right\|_{\mathbf{L}^2}.$$

Therefore by (3.8)

$$\left\| u_1^3 - u_2^3 \right\|_{\mathbf{L}^2} \leq C\varepsilon^2 t^{-1} \left\| \partial_\xi \widehat{w} \right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1+\gamma}$$

and by (3.7)

$$\frac{d}{dt}\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{-1+\gamma}$$

which implies

$$\left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{\gamma}.$$

Then

$$\|\mathcal{J}y\|_{\mathbf{L}^2} \leq \left\|\partial_x^{-1}\mathcal{P}y\right\|_{\mathbf{L}^2} + Ct\left\|u_1^3 - u_2^3\right\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{\gamma}.$$

Lemma 3.3 is proved. □

§4. Proof of Theorem 1.3

By Lemma 3.2 we see that a priori estimate $\|u\|_{\mathbf{X}_T} \leq C\varepsilon$ is true for all $T > 0$. Therefore the global existence of solutions of the Cauchy problem (1.1) satisfying the estimate

$$\|u\|_{\mathbf{X}_\infty} \leq C\varepsilon$$

follows by a standard continuation argument by the local existence Theorem 3.1.

§5. Proofs of Theorems 1.4 and 1.5

5.1. Proof of Theorem 1.4

In this section we prove the existence of a unique self-similar solution

$$v_m(t,x) = t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right) = \mathcal{D}_t\left(t^{\frac{1}{6}} f_m\left(xt^{\frac{2}{3}}\right)\right)$$

of equation (1.1), which is uniquely determined by the total mass condition $m = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} v_m(t,x)\,dx \neq 0$. Define the operators

$$\mathcal{V}_\alpha\phi = \frac{\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha S(x,\xi)}\phi(\xi)\,\chi\left(\xi x^{-1}\right)d\xi$$

for $x > 0$, and

$$\mathcal{W}_\alpha\phi = \frac{(1-\theta(x))}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha S_0(x,\xi)}\phi(\xi)\,d\xi$$

for $x \leq 0$. Here the phase functions $S(x,\xi) = \frac{2}{3}x^3 - x^2\xi + \frac{1}{3}\xi^3$ and $S_0(x,\xi) = x^2\xi + \frac{1}{3}\xi^3$. Also let

$$\mathcal{Q}_\alpha\phi = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{i\alpha S(x,\xi)}\phi(x)\,xdx$$

for $\xi \in \mathbf{R}$ and

$$\mathcal{R}_{\alpha}\phi = -\frac{2}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{i\alpha S_0(x,\xi)}\phi(x)\,xdx.$$

Then by the factorization formula (1.6)

$$\begin{aligned} v_m(t,x) &= t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right) = \mathcal{D}_t\left(t^{\frac{1}{6}} f_m\left(xt^{\frac{2}{3}}\right)\right) \\ &= \mathcal{D}_t\mathcal{B}\left(M\mathcal{V}\widehat{\varphi_m} + \overline{M\mathcal{V}\widehat{\varphi_m}}\right) + \mathcal{D}_t\mathcal{B}\mathcal{W}\widehat{\varphi_m}, \end{aligned}$$

where $\widehat{\varphi_m}(t,\xi) = \mathcal{F}\mathcal{U}(-t)\,v_m(t)$. Hence

$$f_m\left(xt^{\frac{2}{3}}\right) = t^{-\frac{1}{6}}\mathcal{B}\left(M\mathcal{V}\widehat{\varphi_m} + \overline{M\mathcal{V}\widehat{\varphi_m}}\right) + t^{-\frac{1}{6}}\mathcal{B}\mathcal{W}\widehat{\varphi_m}.$$

Then for the self-similar form $\widehat{\varphi_m}(t,\xi) = \phi_m(\eta)$, $\eta = \xi t^{\frac{1}{3}}$, we get with $y = x|x|^{-\frac{1}{2}} t^{\frac{1}{3}}$

$$\begin{aligned} &\mathcal{B}M\mathcal{V}\phi_m \\ = &\frac{|t|^{\frac{1}{2}}\theta\left(x|x|^{-\frac{1}{2}}\right)}{\sqrt{2\pi}} e^{\frac{2it}{3}x^3|x|^{-\frac{3}{2}}} \\ &\times\int_{-\infty}^{\infty} e^{-itS\left(x|x|^{-\frac{1}{2}},\eta t^{-\frac{1}{3}}\right)}\phi_m(\eta)\,\chi\left(\eta t^{-\frac{1}{3}}|x|^{\frac{1}{2}}x^{-1}\right)d\xi \\ = &\frac{|t|^{\frac{1}{6}}\theta(y)}{\sqrt{2\pi}} e^{\frac{2i}{3}y^3}\int_{-\infty}^{\infty} e^{-iS(y,\eta)}\phi_m(\eta)\,\chi\left(\eta y^{-1}\right)d\eta \end{aligned}$$

and similarly

$$\mathcal{B}M\mathcal{W}\phi_m = \frac{|t|^{\frac{1}{6}}(1-\theta(y))}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iS_0(y,\eta)}\phi_m(\eta)\,d\eta.$$

Hence

$$\begin{aligned} f_m\left(xt^{\frac{2}{3}}\right) &= 2\mathrm{Re}e^{\frac{2i}{3}y^3}\frac{\theta(y)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iS(y,\eta)}\phi_m(\eta)\,\chi\left(\eta y^{-1}\right)d\eta \\ &\quad +2\mathrm{Re}\frac{(1-\theta(y))}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-iS_0(y,\eta)}\phi_m(\eta)\,d\eta. \end{aligned}$$

Using the relation $\partial_t\widehat{\varphi_m}(t,\xi) = \frac{1}{3}t^{-1}\eta\phi_m'(\eta)$ we get from equation (1.8)

$$\begin{aligned} \frac{1}{3}\eta\phi_m'(\eta) &= i\eta E_1^{-\frac{8}{9}}\mathcal{D}_3\mathcal{Q}_3\left(\mathcal{V}_1\phi_m\right)^3 + i\eta\mathcal{Q}_1\left(\mathcal{V}_1\phi_m\right)^2\overline{\mathcal{V}_1\phi_m} \\ &\quad + i\eta\mathcal{D}_{-1}\mathcal{Q}_{-1}\left(\overline{\mathcal{V}_1\phi_m}\right)^2\mathcal{V}_1\phi_m + i\eta E_1^{-\frac{8}{9}}\mathcal{D}_{-3}\mathcal{Q}_{-3}\left(\overline{\mathcal{V}_1\phi_m}\right)^3 \\ &\quad + i\eta\mathcal{R}_1\left(\mathcal{W}_1\phi_m\right)^3. \end{aligned}$$

Therefore

$$
\begin{aligned}
\phi_m'(\eta) &= 3iE_1^{-\frac{8}{9}}\mathcal{D}_3\mathcal{Q}_3(\mathcal{V}_1\phi_m)^3 + 3i\mathcal{Q}_1(\mathcal{V}_1\phi_m)^2\overline{\mathcal{V}_1\phi_m} \\
&\quad +3i\mathcal{D}_{-1}\mathcal{Q}_{-1}\left(\overline{\mathcal{V}_1\phi_m}\right)^2\mathcal{V}_1\phi_m + 3iE_1^{-\frac{8}{9}}\mathcal{D}_{-3}\mathcal{Q}_{-3}\left(\overline{\mathcal{V}_1\phi_m}\right)^3 \\
&\quad +3i\mathcal{R}_1(\mathcal{W}_1\phi_m)^3 \equiv F(\phi_m). \qquad (5.1)
\end{aligned}
$$

Note that $F(\phi_m(\eta))$ is not in $\mathbf{L}^2$. Hence we need the approximate equation. Denote $\Theta_R(\eta) = 1$ for $|\eta| \le R$ and $\Theta_R(\eta) = 0$ for $|\eta| > R$. And define the approximate equation

$$
\begin{aligned}
\phi_{m,R}'(\eta) &= 3iE_1^{-\frac{8}{9}}\mathcal{D}_3\mathcal{Q}_3\Theta_R(\mathcal{V}_1\Theta_R\phi_{m,R})^3 \\
&\quad +3i\mathcal{Q}_1\Theta_R(\mathcal{V}_1\Theta_R\phi_{m,R})^2\overline{\mathcal{V}_1\Theta_R\phi_{m,R}} \\
&\quad +3i\mathcal{D}_{-1}\mathcal{Q}_{-1}\Theta_R\left(\overline{\mathcal{V}_1\Theta_R\phi_{m,R}}\right)^2\mathcal{V}_1\Theta_R\phi_{m,R} \\
&\quad +3iE_1^{-\frac{8}{9}}\mathcal{D}_{-3}\mathcal{Q}_{-3}\Theta_R\left(\overline{\mathcal{V}_1\Theta_R\phi_{m,R}}\right)^3 \\
&\quad +3i\mathcal{R}_1\Theta_R(\mathcal{W}_1\Theta_R\phi_{m,R})^3 \equiv F_R(\phi_{m,R}(\eta)). \qquad (5.2)
\end{aligned}
$$

Let us consider the linearized equation

$$
\phi_{m,R}(\eta) = m + \int_0^R F_R(\psi_{m,R}(\eta))\,d\eta,
$$

where $\psi_{m,R}$ is a given function satisfying

$$
\|\psi_{m,R}\|_{\mathbf{Z}} = \|\psi_{m,R}\|_{\mathbf{L}^\infty} + \left\|\psi_{m,R}'\right\|_{\mathbf{L}^2} \le 3|m|.
$$

We have

$$
\|\phi_{m,R}\|_{\mathbf{Z}} \le 2|m| + C|m|^3 R \le 3|m|
$$

if $|m| \le CR^{\frac{1}{2}}$ which implies the existence of solutions to (5.2). Let us show a priori estimate

$$
\|\phi_{m,R}\|_{\mathbf{Z}} \le 3|m|
$$

uniformly in R. Applying Lemma 2.5 with $t = 1$ we get equation

$$
\begin{aligned}
\phi_{m,R}'(\eta) &= \frac{\sqrt{3}}{2i}e^{-\frac{8i}{27}\eta^3}\langle\eta\rangle^{-1}(\Theta_R\phi_{m,R})^3\left(\frac{\eta}{3}\right) \\
&\quad +\frac{3i}{2}\langle\eta\rangle^{-1}|\Theta_R\phi_{m,R}(\eta)|^2\Theta_R\phi_{m,R}(\eta) \\
&\quad +O\left(\Theta_R\langle\eta\rangle^{-1-\gamma}\|\Theta_R\phi_{m,R}\|_{\mathbf{Z}}^3\right). \qquad (5.3)
\end{aligned}
$$

Integrating in η, we obtain

$$
\left\|\phi_{m,R}'\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\|\phi_{m,R}\|_{\mathbf{Z}}^3. \qquad (5.4)
$$

To exclude the second term of the right-hand side of (5.3) we change $\psi(\eta) = \phi_m(\eta)\Psi(\eta)$ with

$$\Psi(\eta) = \exp\left(-\frac{3i}{2}\int_0^{\eta} \langle\eta\rangle^{-1}\left|\Theta_R\phi_{m,R}(\eta)\right|^2 \Theta_R\phi_{m,R}(\eta)\,d\eta\right).$$

Hence we get

$$\begin{aligned}\psi'(\eta) &= \frac{\sqrt{3}}{2i}\langle\eta\rangle^{-1} e^{-\frac{8i}{27}\eta^3}\left(\Theta_R\phi_{m,R}\right)^3\left(\frac{\eta}{3}\right)\Psi(\eta)\\ &\quad +O\left(\Theta_R\langle\eta\rangle^{-1-\gamma}\left\|\Theta_R\phi_{m,R}\right\|_{\mathbf{Z}}^3\right).\end{aligned}$$

Integrating we get

$$\begin{aligned}\psi(\eta) &= m+\frac{\sqrt{3}}{2i}\int_0^{\eta} e^{-\frac{8i}{27}\eta^3}\left(\Theta_R\phi_{m,R}\right)^3\left(\frac{\eta}{3}\right)\Psi(\eta)\frac{d\eta}{\langle\eta\rangle}\\ &\quad +O\left(\int_0^{\eta}\langle\eta\rangle^{-1-\gamma}\left\|\phi_{m,R}\right\|_{\mathbf{Z}}^3 d\eta\right).\end{aligned}$$

The first integral can be estimated by integrating by parts. Hence

$$\left\|\phi_{m,R}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \le \left\|\psi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \le |m| + C\left\|\phi_{m,R}\right\|_{\mathbf{Z}}^3 \tag{5.5}$$

By (5.4) and (5.5) we get

$$\left\|\phi_{m,R}\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \left\|\phi'_{m,R}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le |m| + C\left\|\phi_{m,R}\right\|_{\mathbf{Z}}^3$$

Since the solution v_m is real, we have $\overline{\phi_m(\eta)} = \phi_m(-\eta)$. Therefore

$$\left\|\phi_{m,R}\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} + \left\|\phi'_{m,R}\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le |m| + C\left\|\phi_{m,R}\right\|_{\mathbf{Z}}^3.$$

Thus we obtain

$$\left\|\phi_{m,R}\right\|_{\mathbf{Z}} \le 2|m| + C\left\|\phi_{m,R}\right\|_{\mathbf{Z}}^3$$

from which we find that there exists m such that $\left\|\phi_{m,R}\right\|_{\mathbf{Z}} \le 3|m|$. We take the limit $R \to \infty$. Then there exists a unique solution ϕ_m of (5.1) in Z. By the definition of $\phi_m(\eta)$, we obtain

$$\left\|\partial_\xi\widehat{\varphi_m}\right\|_{\mathbf{L}^2} = t^{\frac{1}{6}}\left\|\phi'_m\right\|_{\mathbf{L}^2} \le C|m|\,t^{\frac{1}{6}}, \left\|\widehat{\varphi_m}\right\|_{\mathbf{L}^\infty} \le 3|m|.$$

In the same way as in the proof of (3.4) we have the $\mathbf{L}^p$ estimate of v_m stated in the theorem for $p > 4$.

5.2. Proof of Theorem 1.3

Now we turn to the proof of asymptotic formula (1.4) for the solutions u of the Cauchy problem (1.1). Let $v_m(t,x)$ be the self-similar solution with the total mass condition $m = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} u_0(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} v_m(t,x)\,dx \neq 0$. Note that $\|v_m\|_{\mathbf{X}_\infty} \le C\varepsilon$ by Theorem 1.4 and $\|u\|_{\mathbf{X}_\infty} \le C\varepsilon$ by Theorem 1.3. Also $m = \widehat{\varphi}(t,0) = \widehat{v_m}(t,0)$ for $t \ge 1$.Then by Lemma 3.3 we find for $x > 0$

$$u(t,x) = v_m(t,x) + O\left(\varepsilon t^{-\frac{1}{2}+\gamma}\right)$$

and for $x \le 0$

$$u(t,x) = v_m(t,x) + O\left(\varepsilon t^{-\frac{1}{2}+\gamma}\left\langle x t^{-\frac{1}{3}}\right\rangle^{-\frac{3}{4}}\right)$$

Thus asymptotics (1.4) and (1.5) follow. By Theorem 1.3 and Theorem 1.4, we have the estimates

$$\|u(t)\|_{\mathbf{L}^p} \le C\varepsilon t^{-\frac{1}{3}+\frac{1}{3p}}$$

and

$$\|v_m(t)\|_{\mathbf{L}^p} = t^{-\frac{1}{3}}\left(\int\left|f_m\left(x t^{-\frac{1}{3}}\right)\right|^p dx\right)^{\frac{1}{p}} = t^{-\frac{1}{3}+\frac{1}{3p}}\|f_m\|_{\mathbf{L}^p}$$

for $4 < p < \infty$. Hence by Lemma 3.3

$$\begin{aligned}\|u(t) - v_m(t)\|_{\mathbf{L}^q} &\le C\|u(t) - v_m(t)\|_{\mathbf{L}^\infty}^{1-\frac{p}{q}}\|u(t) - v_m(t)\|_{\mathbf{L}^p}^{\frac{p}{q}} \\ &\le C\varepsilon t^{-\left(\frac{1}{2}-\gamma\right)\left(1-\frac{p}{q}\right)-\frac{1}{3}\left(1-\frac{1}{p}\right)\frac{p}{q}} \\ &= C\varepsilon t^{-\left(\frac{1}{6}-\gamma\right)\left(1-\frac{p}{q}\right)-\frac{1}{3}\left(1-\frac{1}{q}\right)}\end{aligned}$$

for $q > p$, which implies the lower bound of time decay of solutions

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^q} &\ge \|v_m(t)\|_{\mathbf{L}^q} - \|u(t) - v_m(t)\|_{\mathbf{L}^q} \\ &\ge t^{-\frac{1}{3}\left(1-\frac{1}{q}\right)}\|f_m\|_{\mathbf{L}^q} - C\varepsilon t^{-\left(\frac{1}{6}-\gamma\right)\left(1-\frac{p}{q}\right)-\frac{1}{3}\left(1-\frac{1}{q}\right)}.\end{aligned}$$

Theorem 1.5 is proved.

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