SUT Journal of Mathematics
Vol. 53, No. 1 (2017), 39–58

Inequalities for quadratic operator perspective of convex functions and bounded linear operators on Hilbert spaces

Silvestru Sever Dragomir

(Received October 14, 2016)

Abstract. In this paper we introduce the concept of *quadratic operator perspective* for a continuous function Φ defined on the positive semi-axis of real numbers, the invertible operator T and operator V on a Hilbert space by

$$\bigcirc_{\Phi}(V,T) := T^{*}\Phi\left(\left|VT^{-1}\right|^{2}\right)T.$$

This generalize the *quadratic weighted operator geometric mean* of (T,V) defined by

$$T \odot_{\nu} V := \left|\left|VT^{-1}\right|^{\nu} T\right|^{2}$$

for $\nu \in [0,1]$ and the *quadratic relative operator entropy defined* by

$$\odot(T|V) := T^{*}\ln\left(\left|VT^{-1}\right|^{2}\right)T.$$

Some inequalities for this perspective of convex functions are established. Applications for quadratic weighted operator geometric mean and quadratic relative operator entropy are also provided.

AMS 2010 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

Key words and phrases. Operator perspective, Convex functions, Operator inequalities, Arithmetic mean-Geometric mean operator inequality, Relative operator entropy.

§1. Introduction

If $\Phi : I \to \mathbb{R}$ is a convex function on the real interval I and T is a selfadjoint operator on the complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$ with the spectrum $Sp(T) \subset \mathring{I}$, the interior of I, then we have the following Jensen's type inequality

$$\langle\Phi(T)x,x\rangle \geq \Phi(\langle Tx,x\rangle) \tag{1.1}$$

for any $x \in H$ with $\|x\| = 1$.

For various Jensen's type inequalities for functions of selfadjoint operators, see the recent monograph [1] and the references therein.

In the recent paper [4] we showed amongst others that if A is a positive invertible operator and B is a selfadjoint operator such that

$$Sp\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I},$$

then

$$\frac{\left\langle A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}x, x\right\rangle}{\langle Ax, x\rangle} \geq \Phi\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right), \tag{1.2}$$

for any $x \in H$, $x \neq 0$. This result can be reformulated in terms of perspective as follows.

Let Φ be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $Sp\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_\Phi(B, A)$ by setting

$$\mathcal{P}_\Phi(B, A) := A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_\Phi(B, A) = A\Phi\left(BA^{-1}\right)$$

provided $Sp\left(BA^{-1}\right) \subset \mathring{I}$.

By using the perspective notation, we have by (1.2) that

$$\frac{\langle \mathcal{P}_\Phi(B, A)x, x\rangle}{\langle Ax, x\rangle} \geq \Phi\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right), \tag{1.3}$$

for any $x \in H$ with $\|x\| = 1$.

It is well known that (see [9] and [8] or [10]), if Φ is an *operator convex function* defined in the positive half-line, then the mapping

$$(B, A) \to \mathcal{P}_\Phi(B, A)$$

defined in pairs of positive definite operators, is convex.

Assume that A, B are positive operators on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The *weighted operator arithmetic mean* for the pair (A, B) is defined by

$$A\nabla_\nu B := (1-\nu)A + \nu B.$$

In 1980, Kubo & Ando, [20] introduced the *weighted operator geometric mean* for the pair (A, B) with A positive and invertible and B positive by

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}.$$

If A, B are positive invertible operators then we can also consider the *weighted operator harmonic mean* defined by (see for instance [20])

$$A!_\nu B := \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1}.$$

We have the following fundamental operator means inequalities, or Young's inequalities

$$A!_\nu B \le A\sharp_\nu B \le A\nabla_\nu B,\ \nu \in [0,1] \tag{1.4}$$

for any A, B positive invertible operators. For $\nu = \frac{1}{2}$, we denote the above means by $A\nabla B$, $A\sharp B$ and $A!B$.

For recent results on operator Young inequality see [13]-[16], [17] and [24]-[25].

We denote by $\mathcal{B}^{-1}(H)$ the class of all bounded linear invertible operators on H. For $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$ we define the *quadratic weighted operator geometric mean* of (T, V) by [5]

$$T \bigcirc_\nu V := \left|\left|VT^{-1}\right|^\nu T\right|^2 \tag{1.5}$$

for $\nu \ge 0$. For $V \in \mathcal{B}^{-1}(H)$ we can also extend the definition (1.5) for $\nu < 0$.

By the definition of operator modulus, i.e., we recall that $|U| := \sqrt{U^*U}$, $U \in \mathcal{B}(H)$, we also have

$$T \bigcirc_\nu V = T^*\left|VT^{-1}\right|^{2\nu} T = T^*\left((T^*)^{-1}V^*VT^{-1}\right)^\nu T \tag{1.6}$$

for any $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$. For $\nu = \frac{1}{2}$ we denote

$$T \bigcirc V := \left|\left|VT^{-1}\right|^{1/2} T\right|^2 = T^*\left|VT^{-1}\right| T = T^*\left((T^*)^{-1}V^*VT^{-1}\right)^{1/2} T.$$

It has been shown in [5] that the following representation holds

$$T \bigcirc_\nu V = |T|^2 \sharp_\nu |V|^2 \tag{1.7}$$

for $T, V \in \mathcal{B}^{-1}(H)$ and any real ν.

We have the following fundamental inequalities extending (1.4):

$$|T|^2 \nabla_\nu |V|^2 \ge T \bigcirc_\nu V \ge |T|^2 !_\nu |V|^2 \tag{1.8}$$

for $T, V \in \mathcal{B}^{-1}(H)$ and for $\nu \in [0,1]$. In particular, we have

$$|T|^2 \nabla |V|^2 \geq T \bigcirc V \geq |T|^2 ! \, |V|^2 \tag{1.9}$$

for T, $V \in \mathcal{B}^{-1}(H)$.

We have the following identities [6] as well

$$(T \bigcirc_\nu V)^{-1} = (T^*)^{-1} \bigcirc_\nu (V^*)^{-1} \text{ and } T \bigcirc_{1-t} V = V \bigcirc_t T \tag{1.10}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0,1]$.

Kamei and Fujii [11], [12] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}}, \tag{1.11}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [23].

For some recent results on relative operator entropy see [2]-[3], [18]-[19] and [21]-[22].

Consider the scalar function $T_t : (0, \infty) \to \mathbb{R}$ defined for $t \neq 0$ by

$$T_t(x) := \frac{x^t - 1}{t}. \tag{1.12}$$

We have

$$T_{-t}(x) = \frac{1 - x^{-t}}{t} = \frac{x^t - 1}{t x^t} = T_t(x) \, x^{-t}. \tag{1.13}$$

For T, $V \in \mathcal{B}^{-1}(H)$ and $t > 0$ we define the *quadratic Tsallis relative operator entropy* by [7]

$$\begin{aligned} \odot_t (T|V) &:= T^* T_t \left(\left| V T^{-1} \right|^2 \right) T = T^* \frac{\left(\left| V T^{-1} \right|^2 \right)^t - 1}{t} T \\ &= \frac{T \bigcirc_t V - |T|^2}{t} = \frac{\left| \left| V T^{-1} \right|^t T \right|^2 - |T|^2}{t} \end{aligned} \tag{1.14}$$

and the *quadratic relative operator entropy* by [7]

$$\odot (T|V) := T^* \ln \left(\left| V T^{-1} \right|^2 \right) T. \tag{1.15}$$

We observe that for $T = A^{1/2} \in \mathcal{B}^{-1}(H)$ and $V = B^{1/2} \in \mathcal{B}^{-1}(H)$ we get the equalities

$$\odot_t \left(A^{1/2} | B^{1/2} \right) = T_t(A|B) := \frac{A \sharp_\nu B - A}{t}$$

and

$$\odot\left(A^{1/2}|B^{1/2}\right)=S\left(A|B\right),$$

that show the connection between the extended Tsallis and relative entropies with the classical concepts defined for positive operators.

We have for $t>0$ and $T,\,V\in\mathcal{B}^{-1}\left(H\right)$ that

$$\odot_{-t}\left(T|V\right)=T^{*}T_{-t}\left(\left|VT^{-1}\right|^{2}\right)T=\odot_{t}\left(T|V\right)\left(T\bigcirc_{t}V\right)^{-1}\left|T\right|^{2} \tag{1.16}$$

and, by (1.7), we also have the representation

$$\odot_{t}\left(T|V\right)=\frac{\left|T\right|^{2}\sharp_{\nu}\left|V\right|^{2}-\left|T\right|^{2}}{t}=T_{t}\left(\left|T\right|^{2}|\left|V\right|^{2}\right)$$

or $t>0$ and $T,\,V\in\mathcal{B}^{-1}\left(H\right)$.

The following fundamental inequalities may be stated [7]:

$$\odot_{-t}\left(T|V\right)\leq\odot\left(T|V\right)\leq\odot_{t}\left(T|V\right) \tag{1.17}$$

for any $T,\,V\in\mathcal{B}^{-1}\left(H\right)$ and $t>0$.

Let $T\in\mathcal{B}^{-1}\left(H\right)$, $V\in\mathcal{B}\left(H\right)$ and I an interval of nonnegative numbers. Assume that $Sp\left(\left|VT^{-1}\right|^{2}\right)\subset\mathring{I}$ and Φ is a continuous function defined on the interval I. Then by using the continuous functional calculus for selfadjoint operators, we can define the *quadratic operator perspective* of T, V and Φ by

$$\odot_{\Phi}\left(V,T\right):=T^{*}\Phi\left(\left|VT^{-1}\right|^{2}\right)T. \tag{1.18}$$

If $\Phi\left(U^{*}AU\right)=U^{*}\Phi\left(A\right)U$ holds for all unitary U and $A\in\mathcal{B}\left(H\right)$, then

$$\odot_{\Phi}\left(V,T\right):=\left|T\right|\Phi\left(\left|T\right|^{-1}\left|V\right|^{2}\left|T\right|^{-1}\right)\left|T\right|.$$

If we take in (1.18) $\Phi\left(x\right)=x^{\nu}$, $x>0$, $\nu\neq0$, then we recapture the definition of quadratic weighted operator geometric mean, for $\Phi\left(x\right)=\frac{x^{t}-1}{t}$, $t\neq0$, $x>0$, the definition of quadratic Tsallis relative operator entropy and for $\Phi\left(x\right)=\ln x$, $x>0$ the definition of quadratic relative operator entropy.

Motivated by the above facts, we establish in this paper some upper and lower bounds for the quadratic operator perspective and apply them for the quadratic operator entropy and geometric mean defined above.

§2. Operator Inequalities for Quadratic Perspectives

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi:I\to\mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and

right derivatives at each point of $\mathring{I}$. Moreover, if $t,\ s \in \mathring{I}$ and $t < s$, then $\Phi'_-(t) \le \Phi'_+(t) \le \Phi'_-(s) \le \Phi'_+(s)$ which shows that both Φ'_- and Φ'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi(t) \ge \Phi(a) + (t-a)\varphi(a) \text{ for any } t,\ a \in I. \tag{2.1}$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_-, $\Phi'_+ \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_-(t) \le \varphi(t) \le \Phi'_+(t) \text{ for any } t \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

We need the following simple fact, see also [5]:

Lemma 2.1. *Let $T,\ V \in \mathcal{B}^{-1}(H)$ and $0 < m < M < \infty$. Then the following statements are equivalent:*

(i) The inequality

$$m\|Tx\| \le \|Vx\| \le M\|Tx\| \tag{2.2}$$

holds for any $x \in H$;

(ii) We have the operator inequality

$$m1_H \le \left|VT^{-1}\right| \le M1_H. \tag{2.3}$$

Proof. The inequality (2.2) is equivalent to

$$m^2\|Tx\|^2 \le \|Vx\|^2 \le M^2\|Tx\|^2$$

for any $x \in H$, namely

$$m^2\langle T^*Tx, x\rangle \le \langle V^*Vx, x\rangle \le M^2\langle T^*Tx, x\rangle$$

for any $x \in H$, which can be written in the operator order as

$$m^2T^*T \le V^*V \le M^2T^*T.$$

Since $T \in \mathcal{B}^{-1}(H)$, then this inequality is equivalent to

$$m^2 1_H \le \left(T^{-1}\right)^* V^*VT^{-1} \le M^2 1_H,$$

namely

$$m^2 1_H \le \left|VT^{-1}\right|^2 \le M^2 1_H,$$

which in its turn is equivalent to (2.3). □

We have:

Theorem 2.2. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of positive numbers I, $T, V \in \mathcal{B}^{-1}(H)$ such that there exists the positive numbers $m < M$ with $\left[m^2, M^2\right] \subset \mathring{I}$ satisfying either the condition (2.2), or, equivalently, the condition (2.3). Then for any $\varphi \in \partial\Phi$ and any $t \in \mathring{I}$*

$$\odot_{\Phi}(V,T) \geq \Phi(t)|T|^2 + \varphi(t)\left(|V|^2 - t|T|^2\right). \tag{2.4}$$

In particular,

$$\begin{aligned}\odot_{\Phi}(V,T) \geq{}& \Phi\left(\frac{m^2+M^2}{2}\right)|T|^2 \\ &+ \varphi\left(\frac{m^2+M^2}{2}\right)\left(|V|^2 - \frac{m^2+M^2}{2}|T|^2\right).\end{aligned} \tag{2.5}$$

Proof. From (2.1)

$$\Phi(s) \geq \Phi(t) + (s-t)\varphi(t) \tag{2.6}$$

for any $s \in \left[m^2, M^2\right]$ and $t \in \mathring{I}$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp(X) \subseteq \left[m^2, M^2\right] \subset \mathring{I}$ we have from (2.6) in the operator order that

$$\Phi(X) \geq \Phi(t)1_H + \varphi(t)(X - t1_H) \tag{2.7}$$

for any $t \in \mathring{I}$.

Now, if we take $X = \left|VT^{-1}\right|^2$ in (2.7), then we get

$$\Phi\left(\left|VT^{-1}\right|^2\right) \geq \Phi(t)1_H + \varphi(t)\left(\left|VT^{-1}\right|^2 - t1_H\right) \tag{2.8}$$

for any $t \in \mathring{I}$.

It is well know that, if $P \geq 0$ then by multiplying at left with T^* and at right with T, where $T \in \mathcal{B}(H)$ we have that $T^*PT \geq 0$. If A, B are selfadjoint operators with $A \geq B$ then for any $T \in \mathcal{B}(H)$ we have $T^*AT \geq T^*BT$.

So, if we multiply (2.8) at left with T^* and at right with T, then we get

$$\begin{aligned}T^*\Phi\left(\left|VT^{-1}\right|^2\right)T &\geq \Phi(t)|T|^2 + \varphi(t)T^*\left(\left|VT^{-1}\right|^2 - t1_H\right)T \\ &= \Phi(t)|T|^2 + \varphi(t)T^*\left((T^*)^{-1}V^*VT^{-1} - t1_H\right)T \\ &= \Phi(t)|T|^2 + \varphi(t)\left(|V|^2 - t|T|^2\right)\end{aligned}$$

for any $t \in \mathring{I}$, which proves the desired inequality (2.4). □

Corollary 2.3. *Under the same assumptions of Theorem 2.2, for any $x \in H \setminus \{0\}$,*

$$\odot_{\Phi}(V,T) \geq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)|T|^2 + \varphi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\left(|V|^2 - \frac{\|Vx\|^2}{\|Tx\|^2}|T|^2\right) \tag{2.9}$$

in the operator order of $\mathcal{B}(H)$.

In particular, we have the Jensen's type inequality

$$\frac{\langle \odot_{\Phi}(V,T)x, x\rangle}{\|Tx\|^2} \geq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right) \tag{2.10}$$

$x \in H \setminus \{0\}$.

Proof. For $x \in H \setminus \{0\}$ we have

$$\begin{aligned} t_{A,B} &= \frac{\|Vx\|^2}{\|Tx\|^2} = \frac{\left\langle |V|^2 x, x\right\rangle}{\left\langle |T|^2 x, x\right\rangle} = \frac{\left\langle T^*(T^*)^{-1}V^*VT^{-1}Tx, x\right\rangle}{\langle Tx, Tx\rangle} \\ &= \frac{\left\langle \left((T^*)^{-1}V^*VT^{-1}\right)Tx, Tx\right\rangle}{\langle Tx, Tx\rangle} = \frac{\left\langle \left((T^*)^{-1}V^*VT^{-1}\right)Tx, Tx\right\rangle}{\|Tx\|^2}. \end{aligned}$$

If we put

$$u = \frac{Tx}{\|Tx\|} \neq 0,$$

then $\|u\| = 1$ and

$$t_{A,B} = \left\langle \left((T^*)^{-1}V^*VT^{-1}\right)u, u\right\rangle \in \left[m^2, M^2\right] \subset \mathring{I}.$$

By taking $t = t_{A,B}$ in (2.4) we get (2.9).

The inequality (2.9) is equivalent to

$$\begin{aligned} \langle \odot_{\Phi}(V,T)y, y\rangle &\geq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\left\langle |T|^2 y, y\right\rangle \\ &\quad + \varphi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\left(\left\langle |V|^2 y, y\right\rangle - \frac{\|Vx\|^2}{\|Tx\|^2}\left\langle |T|^2 y, y\right\rangle\right) \end{aligned}$$

for any $y \in H$.

It can be written as

$$\begin{aligned} \langle \odot_{\Phi}(V,T)y, y\rangle &\geq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\|Ty\|^2 \\ &\quad + \varphi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\left(\|Vy\|^2 - \frac{\|Vx\|^2}{\|Tx\|^2}\|Ty\|^2\right). \end{aligned} \tag{2.11}$$

This is an inequality of interest in itself.

In particular, if we take in (2.11) $y = x$, then we get the desired result (2.10). □

Corollary 2.4. *Under the same assumptions of Theorem 2.2, we have*

$$\begin{aligned}(2.12)\quad & \odot_{\Phi}(V,T) \\ & \geq 2\left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\right)|T|^2 \\ & -\frac{1}{M^2-m^2}\left[\Phi\left(M^2\right)\left(M^2|T|^2-|V|^2\right)+\Phi\left(m^2\right)\left(|V|^2-m^2|T|^2\right)\right].\end{aligned}$$

Proof. If we take the integral mean in the interval $\left[m^2, M^2\right]$ of the inequality (2.4), then we get

$$\begin{aligned}(2.13)\qquad \odot_{\Phi}(V,T) \geq & \left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\right)|T|^2 \\ & +\left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\varphi(t)\,dt\right)|V|^2 \\ & -\left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}t\varphi(t)\,dt\right)|T|^2.\end{aligned}$$

Observe that, since $\varphi \in \partial\Phi$, hence

$$\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\varphi(t)\,dt = \frac{\Phi\left(M^2\right)-\Phi\left(m^2\right)}{M^2-m^2}$$

and

$$\begin{aligned}\frac{1}{M^2-m^2}\int_{m^2}^{M^2}t\varphi(t)\,dt &= \frac{1}{M^2-m^2}\left[t\Phi(t)\big|_{m^2}^{M^2}-\int_{m^2}^{M^2}\Phi(t)\,dt\right] \\ &= \frac{M^2\Phi\left(M^2\right)-m^2\Phi\left(m^2\right)}{M^2-m^2}-\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\end{aligned}$$

and by (2.13) we get

$$\begin{aligned}\odot_{\Phi}(V,T) &\geq \left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\right)|T|^2+\frac{\Phi(M^2)-\Phi(m^2)}{M^2-m^2}|V|^2\\ &-\left(\frac{M^2\Phi(M^2)-m^2\Phi(m^2)}{M^2-m^2}-\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\right)|T|^2\\ &=2\left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\right)|T|^2\\ &-\frac{1}{M^2-m^2}\left[\Phi(M^2)\left(M^2|T|^2-|V|^2\right)+\Phi(m^2)\left(|V|^2-m^2|T|^2\right)\right]\end{aligned}$$

that proves the desired result (2.12). □

The following result also provides upper bounds for the quadratic perspective.

Theorem 2.5. *Let $\Phi : I \to \mathbb{R}$ be a continuously differentiable convex function on $\mathring{I}$, T, $V \in \mathcal{B}^{-1}(H)$ such that there exists the positive numbers $m < M$ with $\left[m^2, M^2\right] \subset \mathring{I}$ satisfying either the condition (2.2), or, equivalently, the condition (2.3). Then for any $t \in \mathring{I}$*

$$\begin{aligned}\odot_{\Phi}(V,T) &\leq \Phi(t)|T|^2+\odot_{\Phi'\ell}(V,T)-t\odot_{\Phi'}(V,T)\\ &\leq \Phi(t)|T|^2+\Phi'(t)\left(|V|^2-t|T|^2\right)\\ &+\left[\Phi'(M^2)-\Phi'(m^2)\right]\odot_{|\cdot|,t}(V,T),\end{aligned}\tag{2.14}$$

where ℓ is the identity function, i.e. $\ell(t)=t$ and

$$\odot_{|\cdot|,t}(V,T) := T^*\left|(T^*)^{-1}\left(|V|^2-t|T|^2\right)T^{-1}\right|T.$$

In particular, we have

$$\begin{aligned}&\odot_{\Phi}(V,T)\\ &\leq \Phi\left(\frac{m^2+M^2}{2}\right)|T|^2+\odot_{\Phi'\ell}(V,T)-\frac{m^2+M^2}{2}\odot_{\Phi'}(V,T)\\ &\leq \Phi\left(\frac{m^2+M^2}{2}\right)|T|^2+\Phi'\left(\frac{m^2+M^2}{2}\right)\left(|V|^2-\frac{m^2+M^2}{2}|T|^2\right)\\ &+\left[\Phi'(M^2)-\Phi'(m^2)\right]\odot_{|\cdot|,\frac{m^2+M^2}{2}}(V,T)\\ &\leq \Phi\left(\frac{m^2+M^2}{2}\right)|T|^2+\Phi'\left(\frac{m^2+M^2}{2}\right)\left(|V|^2-\frac{m^2+M^2}{2}|T|^2\right)\\ &+\frac{1}{2}\left(M^2-m^2\right)\left[\Phi'(M^2)-\Phi'(m^2)\right]|T|^2.\end{aligned}\tag{2.15}$$

Proof. By the gradient inequality we have

$$\Phi'(s)(s-t)+\Phi(t)\geq\Phi(s) \tag{2.16}$$

for any $s\in\left[m^2,M^2\right]$ and $t\in\mathring{I}$.

Using the continuous functional calculus for a selfadjoint operator X with $Sp(X)\subseteq\left[m^2,M^2\right]\subset\mathring{I}$ we have from (2.16) in the operator order that

$$\Phi'(X)(X-t1_H)+\Phi(t)1_H\geq\Phi(X) \tag{2.17}$$

for any $t\in\mathring{I}$.

Now, if we take $X=\left|VT^{-1}\right|^2$ in (2.17) and since

$$Sp\left(\left|VT^{-1}\right|^2\right)\subseteq\left[m^2,M^2\right],$$

then we get

$$\Phi'\left(\left|VT^{-1}\right|^2\right)\left(\left|VT^{-1}\right|^2-t1_H\right)+\Phi(t)1_H\geq\Phi\left(\left|VT^{-1}\right|^2\right) \tag{2.18}$$

for any $t\in\mathring{I}$.

So, if we multiply (2.18) at left with T^* and at right with T, then we get

$$T^*\Phi'\left(\left|VT^{-1}\right|^2\right)\left(\left|VT^{-1}\right|^2-t1_H\right)T+\Phi(t)|T|^2\geq T^*\Phi\left(\left|VT^{-1}\right|^2\right)T \tag{2.19}$$

for any $t\in\mathring{I}$.

Since

$$T^*\Phi'\left(\left|VT^{-1}\right|^2\right)\left(\left|VT^{-1}\right|^2-t1_H\right)T=\odot_{\Phi'\ell}(V,T)-t\odot_{\Phi'}(V,T),$$

hence by (2.19) we get the first inequality in (2.14).

Now, observe that

$$\begin{aligned}
&T^*\Phi'\left(\left|VT^{-1}\right|^2\right)\left(\left|VT^{-1}\right|^2-t1_H\right)T+\Phi(t)|T|^2\\
&=T^*\left(\Phi'\left(\left|VT^{-1}\right|^2\right)-\Phi'(t)1_H\right)\left(\left|VT^{-1}\right|^2-t1_H\right)T+\Phi(t)|T|^2\\
&+\Phi'(t)T^*\left(\left|VT^{-1}\right|^2-t1_H\right)T\\
&=T^*\left(\Phi'\left(\left|VT^{-1}\right|^2\right)-\Phi'(t)1_H\right)\left(\left|VT^{-1}\right|^2-t1_H\right)T+\Phi(t)|T|^2\\
&+\Phi'(t)\left(|V|^2-t|T|^2\right)
\end{aligned}$$

for any $t\in\mathring{I}$.

Since Φ' is nondecreasing on $\mathring{I}$ we have for any $s \in \left[m^2, M^2\right]$ and $t \in \mathring{I}$ that

$$\begin{aligned} 0 &\leq \left(\Phi'(s) - \Phi'(t)\right)(s-t) = \left|\left(\Phi'(s) - \Phi'(t)\right)(s-t)\right| \\ &= \left|\Phi'(s) - \Phi'(t)\right| |s-t| \leq \left[\Phi'\left(M^2\right) - \Phi'\left(m^2\right)\right] |s-t|, \end{aligned}$$

which, as above, implies in the operator order that

$$\begin{aligned} & T^* \left(\Phi'\left(\left|VT^{-1}\right|^2\right) - \Phi'(t) 1_H\right)\left(\left|VT^{-1}\right|^2 - t1_H\right) T \\ & \leq \left[\Phi'\left(M^2\right) - \Phi'\left(m^2\right)\right] \left|T^* \left|VT^{-1}\right|^2 - t1_H\right| T \\ & = \left[\Phi'\left(M^2\right) - \Phi'\left(m^2\right)\right] \odot_{|\cdot|,t}(V,T). \end{aligned}$$

This proves the second inequality in (2.14).

We need to prove only the last part of (2.15).

Since $s \in \left[m^2, M^2\right]$, then $\left|s - \frac{m^2+M^2}{2}\right| \leq \frac{1}{2}\left(M^2 - m^2\right)$ that implies in the operator order

$$\left|\left|VT^{-1}\right|^2 - \frac{m^2+M^2}{2} 1_H\right| \leq \frac{1}{2}\left(M^2 - m^2\right) 1_H,$$

which by multiplying at left with T^* and at right with T gives that

$$\odot_{|\cdot|,\frac{m^2+M^2}{2}}(V,T) \leq \frac{1}{2}\left(M^2 - m^2\right)|T|^2.$$

□

Corollary 2.6. *With the assumptions of Theorem 2.5, we have for any $x \in H \setminus \{0\}$ that*

$$\begin{aligned} \odot_\Phi(V,T) &\leq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)|T|^2 + \odot_{\Phi'\ell}(V,T) - \frac{\|Vx\|^2}{\|Tx\|^2} \odot_{\Phi'}(V,T) \\ &\leq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)|T|^2 + \Phi'\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\left(|V|^2 - \frac{\|Vx\|^2}{\|Tx\|^2}|T|^2\right) \\ &\quad + \left[\Phi'\left(M^2\right) - \Phi'\left(m^2\right)\right] \odot_{|\cdot|,\frac{\|Vx\|^2}{\|Tx\|^2}}(V,T). \end{aligned} \tag{2.20}$$

In particular

$$\begin{aligned} & \langle \odot_\Phi(V,T)x, x\rangle \\ & \leq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\|Tx\|^2 + \langle \odot_{\Phi'\ell}(V,T)x, x\rangle - \frac{\|Vx\|^2}{\|Tx\|^2}\langle \odot_{\Phi'}(V,T)x, x\rangle \\ & \leq \Phi\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)\|Tx\|^2 + \left[\Phi'\left(M^2\right) - \Phi'\left(m^2\right)\right]\left\langle \odot_{|\cdot|,\frac{\|Vx\|^2}{\|Tx\|^2}}(V,T)x, x\right\rangle \end{aligned} \tag{2.21}$$

for any $x \in H \setminus \{0\}$.

If we take the integral mean in the interval $\left[m^2, M^2\right]$ of the inequality (2.14) we can also state the following result.

Corollary 2.7. *With the assumptions of Theorem 2.5, we have*

(2.22)

$$\begin{aligned}
&\odot_{\Phi}(V,T) \\
&\leq \left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\right)|T|^2 + \odot_{\Phi'\ell}(V,T) - \frac{m^2+M^2}{2}\odot_{\Phi'}(V,T) \\
&\leq 2\left(\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\Phi(t)\,dt\right)|T|^2 \\
&- \frac{1}{M^2-m^2}\left[\Phi\left(M^2\right)\left(M^2|T|^2-|V|^2\right)+\Phi\left(m^2\right)\left(|V|^2-m^2|T|^2\right)\right] \\
&+ \left[\Phi'\left(M^2\right)-\Phi'\left(m^2\right)\right]\frac{1}{M^2-m^2}\int_{m^2}^{M^2}\odot_{|\cdot|,t}(V,T)\,dt.
\end{aligned}$$

§3. Applications for Quadratic Weighted Geometric Mean

For $x \neq y$ and $p \in \mathbb{R}\setminus\{-1,0\}$, we define the *p-logarithmic mean* (*generalized logarithmic mean*) $L_p(x,y)$ by

$$L_p(x,y) := \left[\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

In fact the singularities at $p=-1$, 0 are removable and L_p can be defined for $p=-1$, 0 so as to make $L_p(x,y)$ a continuous function of p. In the limit as $p\to 0$ we obtain the *identric mean* $I(x,y)$, given by

$$I(x,y) := \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}, \tag{3.1}$$

and in the case $p\to -1$ the *logarithmic mean* $L(x,y)$, given by

$$L(x,y) := \frac{y-x}{\ln y - \ln x}.$$

In each case we define the mean as x when $y=x$, which occurs as the limiting value of $L_p(x,y)$ for $y\to x$.

If we consider the continuous function $f_\nu : [0,\infty)\to[0,\infty)$, $f_\nu(t)=t^\nu$ then the *quadratic weighted operator geometric mean* can be interpreted as

the quadratic perspective $\odot_{f_\nu}(B,A)$ of T, $V \in \mathcal{B}^{-1}(H)$ and f_ν, namely, see for instance [5],

$$\odot_{f_\nu}(V,T) = T \bigcirc_\nu V = |T|^2 \sharp_\nu |V|^2,$$

Consider the convex function $f = -f_\nu$. Then by applying the inequalities (2.4) and (2.5) we have

$$|T|^2 \sharp_\nu |V|^2 \le (1-\nu)\, t^\nu |T|^2 + \nu t^{\nu-1} |V|^2 = \left(t^\nu |T|^2\right) \nabla_\nu \left(t^{\nu-1} |V|^2\right), \tag{3.2}$$

for any $t > 0$ and $\nu \in [0,1]$, and

$$|T|^2 \sharp_\nu |V|^2 \le (1-\nu) \left(\frac{m^2+M^2}{2}\right)^\nu |T|^2 + \nu \left(\frac{m^2+M^2}{2}\right)^{\nu-1} |V|^2 \tag{3.3}$$

for any $\nu \in [0,1]$, provided either the condition (2.2), or, equivalently, the condition (2.3) is valid.

From (2.9) and (2.10) we have for any $x \in H \setminus \{0\}$ and $\nu \in [0,1]$ that

$$|T|^2 \sharp_\nu |V|^2 \le (1-\nu) \left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)^\nu |T|^2 + \nu \left(\frac{\|Tx\|^2}{\|Vx\|^2}\right)^{1-\nu} |V|^2 \tag{3.4}$$

and

$$\left\langle |T|^2 \sharp_\nu |V|^2 x, x \right\rangle \le \|Tx\|^{2(1-\nu)} \|Vx\|^{2\nu}, \tag{3.5}$$

for any $\nu \in [0,1]$.

The inequality (1.8) can be written as

$$\left\langle |T|^2 \sharp_\nu |V|^2 x, x \right\rangle \le (1-\nu) \|Tx\|^2 + \nu \|Vx\|^2 \tag{3.6}$$

for any $x \in H$.

By utilizing the scalar arithmetic mean-geometric mean inequality we also have

$$\|Tx\|^{2(1-\nu)} \|Vx\|^{2\nu} \le (1-\nu) \|Tx\|^2 + \nu \|Vx\|^2 \tag{3.7}$$

for any $x \in H$.

Therefore by (3.5) and (3.7) we have the following vector inequality improving (3.6)

$$\left\langle |T|^2 \sharp_\nu |V|^2 x, x \right\rangle \le \|Tx\|^{2(1-\nu)} \|Vx\|^{2\nu} \le (1-\nu) \|Tx\|^2 + \nu \|Vx\|^2 \tag{3.8}$$

for any $x \in H$.

From (2.12) we have

(3.9)
$$|T|^2 \sharp_\nu |V|^2 \le 2L_\nu^\nu(m^2, M^2)\,|T|^2 - \frac{1}{M^2-m^2}\left[M^{2\nu}\left(M^2|T|^2 - |V|^2\right) + m^{2\nu}\left(|V|^2 - m^2|T|^2\right)\right]$$

for any $\nu \in (0,1)$, provided either the condition (2.2), or, equivalently, the condition (2.3) is valid.

If T, $V \in \mathcal{B}^{-1}(H)$ satisfy the condition (2.2), then by (2.15) we have

(3.10)
$$\begin{aligned}
|T|^2 \sharp_\nu |V|^2 &\ge \left(\frac{m^2+M^2}{2}\right)^\nu |T|^2 + \nu |T|^2 \sharp_\nu |V|^2 - \nu\frac{m^2+M^2}{2}|T|^2 \sharp_{\nu-1} |V|^2 \\
&\ge (1-\nu)\left(\frac{m^2+M^2}{2}\right)^\nu |T|^2 + \nu\left(\frac{m^2+M^2}{2}\right)^{\nu-1}|V|^2 \\
&\quad + \nu\left(M^{2(\nu-1)} - m^{2(\nu-1)}\right) \odot_{|\cdot|,\frac{m^2+M^2}{2}} (V,T) \\
&\ge (1-\nu)\left(\frac{m^2+M^2}{2}\right)^\nu |T|^2 + \nu\left(\frac{m^2+M^2}{2}\right)^{\nu-1}|V|^2 \\
&\quad + \frac{1}{2}\nu\left(M^2-m^2\right)\left(M^{2(\nu-1)} - m^{2(\nu-1)}\right).
\end{aligned}$$

From the last inequality in (3.10) we get

(3.11)
$$\begin{aligned}
&\frac{1}{2}\nu\left(M^2-m^2\right)\left(\frac{M^{2(1-\nu)} - m^{2(1-\nu)}}{m^{2(1-\nu)}M^{2(1-\nu)}}\right) \\
&\ge (1-\nu)\left(\frac{m^2+M^2}{2}\right)^\nu |T|^2 + \nu\left(\frac{m^2+M^2}{2}\right)^{\nu-1}|V|^2 - |T|^2 \sharp_\nu |V|^2 \ge 0,
\end{aligned}$$

for any $\nu \in [0,1]$, which provides a simple reverse for (3.3).

§4. Applications for Quadratic Relative Operator Entropy

Consider the logarithmic function $\ln$. Then the quadratic relative operator entropy can be interpreted as the perspective of ln, namely, see for instance [7],

$$\odot_{\ln}(V,T) = \odot(T|V) = T^* \ln\left(\left|VT^{-1}\right|^2\right) T = S\left(|T|^2 \,|\, |V|^2\right),$$

provided T, $V \in \mathcal{B}^{-1}(H)$.

If we use the inequalities (2.4) and (2.5) for the convex function $f = -\ln$ we have

$$S\left(|T|^2 \,|\, |V|^2\right) \leq (\ln t)\,|T|^2 - |T|^2 + t^{-1}\,|V|^2\,, \tag{4.1}$$

for any $t > 0$ and T, $V \in \mathcal{B}^{-1}(H)$.

In particular, if T, V satisfy the condition (2.2), then

$$\begin{aligned} S\left(|T|^2 \,|\, |V|^2\right) \leq & \left[\ln\left(\frac{m^2+M^2}{2}\right)\right]|T|^2 \\ & + \left(\frac{m^2+M^2}{2}\right)^{-1}\left(|V|^2 - \frac{m^2+M^2}{2}\,|T|^2\right). \end{aligned} \tag{4.2}$$

From the inequalities (2.9) and (2.10) we have

$$S\left(|T|^2 \,|\, |V|^2\right) \leq \ln\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right)|T|^2 + \frac{\|Tx\|^2}{\|Vx\|^2}\,|V|^2 - |T|^2 \tag{4.3}$$

and

$$\left\langle S\left(|T|^2 \,|\, |V|^2\right)x, x\right\rangle \leq \|Tx\|^2 \ln\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right), \tag{4.4}$$

for any $x \in H$, $x \neq 0$.

The following inequality for the relative operator entropy is known

$$S\left(|T|^2 \,|\, |V|^2\right) \leq |V|^2 - |T|^2 \tag{4.5}$$

for any T, $V \in \mathcal{B}^{-1}(H)$.

This inequality is equivalent to

$$\left\langle S\left(|T|^2 \,|\, |V|^2\right)x, x\right\rangle \leq \|Vx\|^2 - \|Tx\|^2 \tag{4.6}$$

for any $x \in H$.

We know the following elementary inequality that holds for the logarithm

$$\ln t \leq t - 1 \text{ for any } t > 0.$$

If we take in this inequality $t = \frac{\|Vx\|^2}{\|Tx\|^2} > 0$, $x \in H$, $x \neq 0$ and multiply with $\|Tx\|^2 > 0$, then we get

$$\|Tx\|^2 \ln\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right) \leq \|Vx\|^2 - \|Tx\|^2 \tag{4.7}$$

for any $x \in H$, $x \neq 0$.

Therefore, by (4.4) and (4.7) we have

$$\left\langle S\left(|T|^2 \,|\, |V|^2\right) x, x\right\rangle \leq \|Tx\|^2 \ln\left(\frac{\|Vx\|^2}{\|Tx\|^2}\right) \leq \|Vx\|^2 - \|Tx\|^2$$

for any $x \in H$, $x \neq 0$ that is an improvement of (4.6).

From (2.12) we also have

$$\begin{aligned} S\left(|T|^2 \,|\, |V|^2\right) &\leq 2\left[\ln I\left(m^2, M^2\right)\right] |T|^2 \\ &\quad - \frac{1}{M^2 - m^2} \\ &\quad \times \left[\ln M^2\left(M^2 |T|^2 - |V|^2\right) + \ln m^2\left(|V|^2 - m^2 |T|^2\right)\right], \end{aligned} \tag{4.8}$$

where $I(\cdot,\cdot)$ is the identric mean defined in (3.1) and

$$\frac{1}{M^2 - m^2}\int_{m^2}^{M^2} \ln t dt = \ln I\left(m^2, M^2\right).$$

From (2.15) we also have

$$\begin{aligned} & S\left(|T|^2 \,|\, |V|^2\right) \\ &\geq \left[\ln\left(\frac{m^2 + M^2}{2}\right)\right] |T|^2 + |T|^2 - \frac{m^2 + M^2}{2} |T|^2 |V^*|^{-2} |T|^2 \\ &\geq \left[\ln\left(\frac{m^2 + M^2}{2}\right)\right] |T|^2 + \left(\frac{m^2 + M^2}{2}\right)^{-1}\left(|V|^2 - \frac{m^2 + M^2}{2}|T|^2\right) \\ &- \frac{M^2 - m^2}{m^2 M^2} \odot_{|\cdot|, \frac{m^2+M^2}{2}} (T|V) \\ &\geq \left[\ln\left(\frac{m^2 + M^2}{2}\right)\right] |T|^2 + \left(\frac{m^2 + M^2}{2}\right)^{-1}\left(|V|^2 - \frac{m^2 + M^2}{2}|T|^2\right) \\ &- \frac{1}{2}\frac{\left(M^2 - m^2\right)^2}{m^2 M^2}, \end{aligned} \tag{4.9}$$

provided T, $V \in \mathcal{B}^{-1}(H)$ satisfying the condition (2.2).

From the last part of (4.9) we get

$$
\begin{aligned}
(4.10) \qquad & \frac{1}{2}\frac{\left(M^2-m^2\right)^2}{m^2M^2} \\
& \geq \left[\ln\left(\frac{m^2+M^2}{2}\right)\right]|T|^2 \\
& + \left(\frac{m^2+M^2}{2}\right)^{-1}\left(|V|^2-\frac{m^2+M^2}{2}|T|^2\right) - S\left(|T|^2\,|\,|V|^2\right) \\
& \geq 0
\end{aligned}
$$

that provides a simple reverse of (4.2).

If one considers the convex function $f(t) = t\ln t$ for $t > 0$, that one can get other logarithmic inequalities as above. The details are left to the interested reader.

Acknowledgement. The author would like to thank the anonymous referee for valuable suggestions that have been implemented in the final version of the manuscript.

References

[1] S. S. Dragomir, Operator Inequalities of the Jensen, Čebyšev and Grüss Type. Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[2] S. S. Dragomir, Some inequalities for relative operator entropy, RGMIA Res. Rep. Coll. **18** (2015), Art. 145. [Online http://rgmia.org/papers/v18/v18a145.pdf].

[3] S. S. Dragomir, Further inequalities for relative operator entropy, RGMIA Res. Rep. Coll. **18** (2015), Art. 160.[Online http://rgmia.org/papers/v18/v18a160.pdf].

[4] S. S. Dragomir, Inequalities for operator noncommutative perspectives of convex functions, RGMIA Res. Rep. Coll. **19** (2016), Art. 21.[http://rgmia.org/papers/v19/v19a21.pdf].

[5] S. S. Dragomir, The quadratic weighted geometric mean for bounded linear operators in Hilbert spaces, Preprint, RGMIA Res. Rep. Coll. **19** (2016), Art. 145. [http://rgmia.org/papers/v19/v19a145.pdf].

[6] S. S. Dragomir, Some properties of quadratic weighted geometric mean of bounded linear operators in Hilbert spaces via Kato's inequality, Preprint RGMIA Res. Rep. Coll. **19** (2016), Art. 151. [http://rgmia.org/papers/v19/v19a151.pdf].

[7] S. S. Dragomir, The quadratic relative entropy for bounded linear operators in Hilbert spaces, Preprint RGMIA Res. Rep. Coll. **19** (2016), Art. 155. [http://rgmia.org/papers/v19/v19a155.pdf].

[8] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, Proc. Natl. Acad. Sci. USA, **108** (2011), no. 18, 7313–7314.

[9] E. G. Effros, A matrix convexity approach to some celebrated quantum inequalities, Proc. Natl. Acad. Sci. USA **106** (2009), 1006–1008.

[10] E. G. Effros and F. Hansen, Noncomutative perspectives, Ann. Funct. Anal. **5** (2014), no. 2, 74–79.

[11] J. I. Fujii and E. Kamei, Uhlmann's interpolational method for operator means. Math. Japon. **34** (1989), no. 4, 541–547.

[12] J. I. Fujii and E. Kamei, Relative operator entropy in noncommutative information theory. Math. Japon. **34** (1989), no. 3, 341–348.

[13] S. Furuichi, Refined Young inequalities with Specht's ratio, J. Egyptian Math. Soc. **20** (2012), 46–49.

[14] S. Furuichi, On refined Young inequalities and reverse inequalities, J. Math. Inequal. **5** (2011), 21-31.

[15] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, J. Math. Anal. Appl. **361** (2010), 262-269.

[16] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, Linear Multilinear Algebra, **59** (2011), 1031-1037.

[17] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, Taiwanese J. Math. **19** (2015), No. 2, pp. 467-479.

[18] I. H. Kim, Operator extension of strong subadditivity of entropy, J. Math. Phys. 53(2012), 122204

[19] P. Kluza and M. Niezgoda, Inequalities for relative operator entropies, Elec. J. Lin. Alg. **27** (2014), Art. 1066.

[20] F. Kubo and T. Ando, Means of positive operators, Math. Ann. **264** (1980), 205–224.

[21] M. S. Moslehian, F. Mirzapour, and A. Morassaei, Operator entropy inequalities. Colloq. Math., **130** (2013), 159–168.

[22] I. Nikoufar, On operator inequalities of some relative operator entropies, Adv. Math. **259** (2014), 376-383.

[23] M. Nakamura and H. Umegaki, A note on the entropy for operator algebras. Proc. Japan Acad. **37** (1961) 149–154.

[24] M. Tominaga, Specht's ratio in the Young inequality, *S*ci. Math. Japon., **55** (2002), 583-588.

[25] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, J. Math. Inequal., **5** (2011), 551-556.

S.S. Dragomir
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia

DST-NRF Centre of Excellence
in the Mathematical and Statistical Sciences,
School of Computer Science & Applied Mathematics,
University of the Witwatersrand,
Private Bag 3, Johannesburg 2050, South Africa
E-mail: `sever.dragomir@vu.edu.au`