# On a relation between CM periods, Stark units, and multiple gamma functions

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# References

This talk is based on the following two papers.

[K1] K., Fermat curves and a refinement of the reciprocity law on cyclotomic units, *J. Reine Angew. Math.*

[K2] K., On a common refinement of Stark units and Gross-Stark units (preprint, arXiv:1706.03198).

Besides I used many of the ideas written in

[Y] H. Yoshida, *Absolute CM-Periods*, Math. Surveys Monogr. **106**, Amer. Math. Soc., 2003.

# CM periods

First recall “CM periods” and “Stark units” shortly.

Let $K$ be a CM field (that is, an imaginary quadratic extension of a totally real field). For two complex embeddings $\sigma, \tau \in \mathrm{Hom}(K, \mathbb{C})$, **Shimura’s period symbol** (or, CM period symbol)

$$p_K(\sigma, \tau) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times,$$

is defined in terms of periods of abelian varieties with CM by $K$.

Note that their values are well-defined only up to multiplication by an algebraic number.

# CM periods

## Example

Let $K$ be an imaginary quadratic field. We consider an elliptic curve $E : y^2 = x^3 + ax + b$ with $a, b \in \overline{\mathbb{Q}}$, $K \cong \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\rho$ be the complex conjugation. Then there are 4 CM periods, which are defined as the integral of suitable differential forms on $E$:

$$p_K(\mathrm{id}, \mathrm{id}) = p_K(\rho, \rho) := \pi^{-1} \int_\gamma \frac{dx}{y} \mod \overline{\mathbb{Q}}^\times,$$

$$p_K(\rho, \mathrm{id}) = p_K(\mathrm{id}, \rho) := \int_\gamma \frac{xdx}{y} \mod \overline{\mathbb{Q}}^\times,$$

where $\gamma \subset E(\mathbb{C})$ is an arbitrary non-trivial closed path. Note that $H^1_{dR}(E) = \langle \frac{dx}{y}, \frac{xdx}{y} \rangle$.

When $[K : \mathbb{Q}] > 2$, we consider an abelian variety $A/\overline{\mathbb{Q}}$ with CM by $K$, take a suitable $\omega \in H_{dR}(A, \overline{\mathbb{Q}})$, and need to "decompose" $\int_\gamma \omega$.

# CM periods

CM periods are closely related to the theme of this workshop:
By using the symbol $p_K$, we can express the transcendental parts of

- critical values of L-functions associated with algebraic Hecke characters of $K$.
- special values of Hilbert modular forms at CM-points $\in K$.

**Example**

Let $f$ be an elliptic modular form of weight $k$ whose Fourier coefficients $\in \overline{\mathbb{Q}}$. For any imaginary quadratic field $K = \mathbb{Q}(\tau)$ $(\mathrm{Im}(\tau) > 0)$, we have

$$f(\tau)/p_K(\mathrm{id}, \mathrm{id})^k \in \overline{\mathbb{Q}}.$$

# Stark units (w.r.t. real places)

Let $H/F$ be an abelian extension of number fields. We consider the partial zeta function:

$$\zeta(s,\sigma) := \sum_{\mathfrak{a}\subset\mathcal{O}_F,\ \left(\frac{H/F}{\mathfrak{a}}\right)=\sigma} N\mathfrak{a}^{-s} \qquad (\sigma\in\mathrm{Gal}(H/F)).$$

We assume that $F$ is totally real and that $H$ has a real place $\iota\colon H\hookrightarrow\mathbb{R}$. (In particular, **a real place of $F$ splits completely in $H/F$**.) Then Stark's conjecture states that

$$\exp(2\zeta'(0,\sigma))\in H^\times \qquad (\text{in fact, } \in\iota(H^\times)),$$
$$\tau(\exp(2\zeta'(0,\sigma))) = \exp(2\zeta'(0,\tau\sigma)) \qquad (\sigma,\tau\in\mathrm{Gal}(H/F)),$$

"$\exp(2\zeta'(0,\sigma))$ is a unit in many cases", "$H(\exp(\zeta'(0,\sigma)))/F$ is abelian".

$\exp(2\zeta'(0,\sigma))$ is called a **Stark unit**.

Let $n \geq 3$, $\zeta_n := e^{\frac{2\pi i}{n}}$. We consider a CM field $\mathbb{Q}(\zeta_n)$ and its maximal real subfield $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$. These are abelian over $\mathbb{Q}$. Write

$$[\sigma_a\colon \zeta_n \to \zeta_n^a] \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \mathrm{Hom}(\mathbb{Q}(\zeta_n), \mathbb{C}) \quad (a \in (\mathbb{Z}/n\mathbb{Z})^\times).$$

By **Rohrlich's formula** (reformulated by Yoshida) and **Lerch's formula**, we can write them explicitly:

$$K := \mathbb{Q}(\zeta_n) \qquad p_K(\sigma_a, \sigma_b) \equiv \pi^{-\frac{\delta_{ab}}{2}} \prod_{c \in (\mathbb{Z}/n\mathbb{Z})^\times} \Gamma\left(\tfrac{c}{n}\right)^{\sum_{\eta \in \widehat{(\mathbb{Z}/n\mathbb{Z})^\times_-}} \frac{\eta(a^{-1}bc)}{L(0,\eta)\varphi(n)}}$$

$$|$$

$$H := \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \quad \exp(2\zeta'(0, \sigma_a|_H)) = \left(\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}\right)^2 \overset{\text{Euler's formulas}}{=} \frac{1}{2 - \zeta_n^a - \zeta_n^{-a}}$$

$$|$$

$$\mathbb{Q}$$

Here $\delta_{ab} := 1, -1, 0$ if $a = b$, $a = -b$, otherwise, respectively.

# When abelian over $\mathbb{Q}$. Relation via Gamma function

## Problem

- **Rohrlich's formula**: $p_K(\sigma_a, \sigma_b)$ in terms of $\Gamma(\frac{a}{n})$'s.
- **Lerch's formula**: $\exp(2\zeta'(0, \sigma_a|_H))$ in terms of $\Gamma(\frac{a}{n})$'s.

By using these, without using Euler's formulas, can we derive Stark's conjecture with $F = \mathbb{Q}$?

Since CM periods are defined only up to $\overline{\mathbb{Q}}^\times$, we can only derive the algebraicity of Stark units with $F = \mathbb{Q}$: roughly speaking,

$$\begin{aligned} &\text{monomial relations on CM periods} \\ &\overset{\text{Rohrlich's formula}}{\rightsquigarrow} \text{monomial relations on } \Gamma(\tfrac{a}{n})\text{'s} \\ &\overset{\text{Lerch's formula}}{\rightsquigarrow} \text{the algebraicity of } \exp(2\zeta'(0, \sigma_a|_H)). \end{aligned}$$

# When abelian over $\mathbb{Q}$. Relation via Gamma function

Let $F_n : x^n + y^n = 1$ be the $n$th Fermat curve. $J(F_n)$ has CM by $\mathbb{Q}(\zeta_n)$.
$H^1_{dR}(J(F_n)) \cong H^1_{dR}(F_n) = \langle \eta_{r,s} := x^r y^{n-s} \frac{dx}{x} \mid 0 < r, s < n,\ r+s \neq n \rangle$.

**Rohrlich's formula**. $$\int_\gamma \eta_{r,s} \equiv B(\tfrac{r}{n}, \tfrac{s}{n}) := \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} \mod \mathbb{Q}(\zeta_n)^\times.$$

The cup product induces a correspondence

$$H^1(F_n) \times H^1(F_n) \to H^2(F_n) = \mathbb{Q}(-1) \text{ (the Lefschetz motive)}.$$

Since the period of $\mathbb{Q}(-1)$ is $2\pi i$, we obtain "monomial relations"

$$B(\tfrac{r}{n}, \tfrac{s}{n}) B(\tfrac{n-r}{n}, \tfrac{n-s}{n}) \equiv \int_\gamma \eta_{r,s} \int_{\gamma'} \eta_{n-r,n-s} \equiv 2\pi i \mod \overline{\mathbb{Q}}^\times.$$

Noting that $\Gamma(\frac{r}{n})^n = \Gamma(r) \prod_{k=1}^{n-1} B(\frac{r}{n}, \frac{kr}{n})$, we obtain

$$\Gamma(\tfrac{a}{n})\Gamma(\tfrac{n-a}{n}) \in 2\pi i \cdot \overline{\mathbb{Q}}^\times, \text{ that is, } \exp(\zeta'(0, \sigma_a|_H)) = \frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi} \in \overline{\mathbb{Q}}^\times.$$

# When abelian over $\mathbb{Q}$. Relation via Gamma function

When $F = \mathbb{Q}$, the algebraicity $\exp(2\zeta'(0, \sigma_a|_H)) \in \overline{\mathbb{Q}}^\times$ follows from

$$K := \mathbb{Q}(\zeta_n) \qquad \int_\gamma \eta_{r,s} \overset{\text{Rohrlich's formula}}{\equiv} \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} \mod \mathbb{Q}(\zeta_n)^\times$$

$$H := \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \quad \exp(2\zeta'(0, \sigma_a|_H)) \overset{\text{Lerch's formula}}{=} \left(\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}\right)^2$$

and monomial relations on CM periods $\int_\gamma \eta_{r,s} \int_\gamma \eta_{n-r,n-s} \equiv 2\pi i \mod \overline{\mathbb{Q}}^\times$. Moreover, in [K1], we show that Coleman's formula on the absolute Frobenius action on Fermat curves implies the reciprocity law

$$\tau\left(\exp(2\zeta'(0, \sigma_a|_H))\right) \equiv \exp(2\zeta'(0, \tau \circ \sigma_a|_H)) \mod \mu_\infty \quad (\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})).$$

Here $\mu_\infty$ denotes the group of all roots of unity.

# When abelian over $\mathbb{Q}$. Relation via Gamma function

$p$-adic periods are defined by comparison isomorphisms of $p$-adic Hodge theory (instead of the de Rham isomorphism):

$$H_1^B(F_n) \times H^1_{dR}(F_n) \to B_{dR}, \quad (\gamma, \eta) \mapsto \int_{p,\gamma} \eta.$$

Here $B_{dR}$ denotes Fontaine's $p$-adic period ring. Since abelian varieties with CM have potentially good reduction, we have $\int_{p,\gamma} \eta \in B_{cris}\overline{\mathbb{Q}_p}$. The Weil group $W_p \subset \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts this subring as:

$$\Phi_\tau := \Phi_{abs.Frob.}^{\deg \tau} \otimes \tau \curvearrowright B_{cris} \otimes_{\mathbb{Q}_p^{ur}} \overline{\mathbb{Q}_p} = B_{cris}\overline{\mathbb{Q}_p} \qquad (\tau \in W_p).$$

Assume $p \neq 2$, $p \mid n$, $p \nmid rs(r+s)$. (A similar argument works when $p \nmid n$.) Then **Coleman's formula** on $\Phi_{abs.Frob.} \curvearrowright H^1_{dR}(F_n/\overline{\mathbb{Q}_p})$ implies that

$$\Phi_\tau \left( \frac{\int_{p,\gamma} \eta_{r,s}}{B_p(\frac{r}{n}, \frac{s}{n})} \right) \equiv p^{\frac{\deg \tau}{2}} \frac{\int_{p,\gamma} \eta_{\tau(r),\tau(s)}}{B_p(\frac{\tau(r)}{n}, \frac{\tau(s)}{n})} \bmod \mu_\infty,$$

where we define $\tau(r)$ by $\tau(\zeta_n^r) = \zeta_n^{\tau(r)}$, $B_p(\alpha, \beta) := \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}$.

# When abelian over $\mathbb{Q}$. Relation via Gamma function

Now by using Rohrlich's formula again, we can define the **period-ring-valued beta function**

$$\mathfrak{B}(\tfrac{r}{n}, \tfrac{s}{n}) := \frac{B(\frac{r}{n}, \frac{s}{n})}{\int_\gamma \eta_{r,s}} \frac{\int_{p,\gamma} \eta_{r,s}}{B_p(\frac{r}{n}, \frac{s}{n})} \in B_{cris}\overline{\mathbb{Q}_p}.$$

Then Coleman's formula implies the **reciprocity law on $\mathfrak{B}$**:

$$\Phi_\tau(\mathfrak{B}(\tfrac{r}{n}, \tfrac{s}{n})) \equiv p^{\frac{\deg \tau}{2}} \mathfrak{B}(\tfrac{\tau(r)}{n}, \tfrac{\tau(s)}{n}) \bmod \mu_\infty \quad (\tau \in W_p).$$

Then, by using these formulas, we obtain the reciprocity law on Stark's units $\exp(2\zeta'(0, \sigma_a|_H)) = \left(\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}\right)^2$ up to $\mu_\infty$ as follows:

# When abelian over $\mathbb{Q}$. Relation via Gamma function

$$\mathfrak{B}(\tfrac{r}{n},\tfrac{s}{n}) := \frac{B(\frac{r}{n},\frac{s}{n})}{\int_{\gamma}\eta_{r,s}}\frac{\int_{p,\gamma}\eta_{r,s}}{B_p(\frac{r}{n},\frac{s}{n})}, \qquad \Phi_\tau(\mathfrak{B}(\tfrac{r}{n},\tfrac{s}{n})) \equiv p^{\frac{\deg\tau}{2}}\mathfrak{B}(\tfrac{\tau(r)}{n},\tfrac{\tau(s)}{n}).$$

By the correspondence $H^1(F_n)\times H^1(F_n)\to\mathbb{Q}(-1)$

$$\rightsquigarrow \mathfrak{B}(\tfrac{r}{n},\tfrac{s}{n})\mathfrak{B}(\tfrac{n-r}{n},\tfrac{n-s}{n}) = \frac{(2\pi i)_p}{2\pi i}\frac{B(\frac{r}{n},\frac{s}{n})B(\frac{n-r}{n},\frac{n-s}{n})}{B_p(\frac{r}{n},\frac{s}{n})B_p(\frac{n-r}{n},\frac{n-s}{n})} \in (2\pi i)_p\cdot\overline{\mathbb{Q}_p}.$$

$\Phi_\tau = \Phi_{abs.Frob.}^{\deg\tau}\otimes\tau$ acts on $\overline{\mathbb{Q}_p}$ as $\tau$, on $(2\pi i)_p$ as $p^{\deg\tau}$

$$\rightsquigarrow \tau\left(\frac{1}{2\pi i}\frac{B(\frac{r}{n},\frac{s}{n})B(\frac{n-r}{n},\frac{n-s}{n})}{B_p(\frac{r}{n},\frac{s}{n})B_p(\frac{n-r}{n},\frac{n-s}{n})}\right) \equiv \frac{1}{2\pi i}\frac{B(\frac{\tau(r)}{n},\frac{\tau(s)}{n})B(\frac{\tau(n-r)}{n},\frac{\tau(n-s)}{n})}{B_p(\frac{\tau(r)}{n},\frac{\tau(s)}{n})B_p(\frac{\tau(n-r)}{n},\frac{\tau(n-s)}{n})}.$$

$\Gamma_*(\frac{r}{n})^n = \Gamma(r)\prod_{k=1}^{n-1}B_*(\frac{r}{n},\frac{kr}{n})$ $(* = , p)$, $\Gamma_p(\frac{r}{n})\Gamma_p(\frac{n-r}{n})\in\mu_\infty$

$$\rightsquigarrow \tau\left(\frac{1}{2\pi i}\frac{\Gamma(\frac{r}{n})\Gamma(\frac{n-r}{n})}{\cancel{\Gamma_p(\frac{r}{n})\Gamma_p(\frac{n-r}{n})}}\right) \equiv \frac{1}{2\pi i}\frac{\Gamma(\frac{\tau(r)}{n})\Gamma(\frac{\tau(n-r)}{n})}{\cancel{\Gamma_p(\frac{\tau(r)}{n})\Gamma_p(\frac{\tau(n-r)}{n})}} \bmod \mu_\infty$$

only for $\tau\in W_p\subset\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)\subset\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then we vary $p$. Q.E.D.

# Why mod $\mu_\infty$?

We need $B_p(\frac{r}{n},\frac{s}{n}) := \frac{\Gamma_p(\frac{r}{n})\Gamma_p(\frac{s}{n})}{\Gamma_p(\frac{r+s}{n})}$ for $p \mid n$, $p \nmid rs(r+s)$. So we can not use Morita's $p$-adic gamma function:

$$\Gamma_p(z)\colon \mathbb{Z}_p \to \mathbb{Z}_p^\times, \quad z \mapsto \lim_{\mathbb{N}\ni n \to z} (-1)^n \prod_{1\le k\le n-1,\ p\nmid k} k.$$

Recall Lerch's formula:

$$\frac{\Gamma(x)}{\sqrt{2\pi}} = \exp\left(\frac{d}{ds}\left[\sum_{m=0}^{\infty}(x+m)^{-s}\right]_{s=0}\right).$$

We need to put

$$\Gamma_p(\tfrac{r}{n}) := \exp_p\left(p^N \frac{d}{ds}\left[\sum_{m=0}^{\infty}(\tfrac{r}{n}+m)^{-s}(p^{\mathrm{ord}_p n})^{-s}\Big|_{p\text{-adic interpolation}}\right]_{s=0}\right)^{\frac{1}{p^N}}$$

for some $N \in \mathbb{N}$ since $\exp_p(z) := \sum_{k=0}^{\infty}\frac{z^k}{k!}$ converges only on a neighborhood of 0. To be honest there is one more reason: the "$\mu_\infty$-part" of Coleman's formula is very complicated and I gave up calculating.

# When abelian over an arbitrary totally real field $F$

**Summary**. In [K1], studying classical or $p$-adic CM periods for CM fields **abelian over** $\mathbb{Q}$, we prove the reciprocity law on $\mathfrak{B}(\frac{r}{n}, \frac{s}{n})$ and provide an alternative proof of a part of Stark's conjecture with $F = \mathbb{Q}$.
Note that not only an alternative proof, but also a refinement since $\exp(2\zeta'(0, \sigma_a|_H)) = (\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi})^2$ is a finite product of $\mathfrak{B}(\frac{r}{n}, \frac{s}{n})$'s.

**Motivation**. Generalize this to totally real fields $F$.

We have some good **Guidelines**.

- We have **Shimura's period symbol** $p_K(\sigma, \tau)$, which is a generalization of periods $\int_\gamma \eta_{r,s}$ of Fermat curves.
- We have **Shintani's formula** on partial zeta functions of totally real fields, which is a generalization of Lerch's formula $\exp(\zeta'(0, \sigma_a|_H)) = \frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}$.
- We have **Yoshida's conjecture** on "absolute CM periods", which is a conjectural generalization of Rohrlich's formula $\int_\gamma \eta_{r,s} \equiv \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})}$.

# multiple gamma functions

Now we explain multiple gamma functions. Recall Lerch's formula:

$$\frac{\Gamma(x)}{\sqrt{2\pi}} = \exp\left(\frac{d}{ds}\left[\sum_{m=0}^{\infty}(x+m)^{-s}\right]_{s=0}\right).$$

For a "good" subset $Z \subset \mathbb{R}$, we put

$$\Gamma(Z) := \exp\left(\frac{d}{ds}\left[\sum_{z\in Z} z^{-s}\right]_{s=0}\right)$$

Here we say $Z$ is "good" if $\sum_{z\in Z} z^{-s}$ converges for $\mathrm{Re}(s) >> 0$, has a meromorphic continuation, is analytic at $s=0$. In particular, for $x>0$, $\boldsymbol{\omega} := (\omega_1,\dots,\omega_r)$ with $\omega_1,\dots,\omega_r > 0$, the lattice-like set

$$L_{x,\boldsymbol{\omega}} := \{x + m_1\omega_1 + \cdots + m_r\omega_r \mid 0 \le m_1,\dots,m_r \in \mathbb{Z}\}$$

is "good" and $\Gamma(L_{x,\boldsymbol{\omega}})$ is called **Barnes' multiple gamma function**.

# Yoshida's class invariant

Let $F$ be a totally real field, $\mathfrak{f}$ an integral ideal of $\mathfrak{f}$, $C_\mathfrak{f}$ the ideal class group modulo $\mathfrak{f}$, in the narrow sense. Let $D$ be Shintani's fundamental domain of $F_+/\mathcal{O}_{F,+}^\times$. For $c \in C_\mathfrak{f}$, we take an ideal $\mathfrak{a} \in c$ and consider a subset:

$$Z_c := \{z \in D \cap \mathfrak{a}^{-1} \mid z\mathfrak{a} \in c\} \subset F_+.$$

Shintani provided an expression $Z_c = \coprod_{i=1}^k L_{x_i,\omega_i}$. In particular $\iota(Z_c)$ is "good" for any $\iota \in \mathrm{Hom}(F,\mathbb{R})$. Yoshida defined a **class invariant**

$$\Gamma(c,\iota) := \Gamma(\iota(Z_c)) \times \prod_i \iota(a_i)^{\iota(b_i)} \quad (c \in C_\mathfrak{f},\ \iota \in \mathrm{Hom}(F,\mathbb{R}))$$

for certain $a_i, b_i \in F$. Although $Z_c, a_i, b_i$ depend on $D, \mathfrak{a}$, we have

- $\prod_{\iota \in \mathrm{Hom}(F,\mathbb{R})} \Gamma(c,\iota) = \exp(\zeta'(0,c))$: **Shintani's formula**, which is a generalization of Lerch's formula.
- $\Gamma(c,\iota) \mod \iota(\mathcal{O}_{F,+}^\times)^\mathbb{Q}$ does not depend on the choices of $D, \mathfrak{a}$, that is, $\Gamma(c,\iota; D,\mathfrak{a})/\Gamma(c,\iota; D',\mathfrak{a}') = \iota(\epsilon)^{\frac{1}{N}}$ with $\epsilon \in \mathcal{O}_{F,+}^\times$, $N \in \mathbb{N}$.

We fix $\mathrm{id}\colon F \hookrightarrow \mathbb{R}$ and put $\Gamma(c) := \Gamma(c,\mathrm{id})$.

# Yoshida's class invariant

## Example ($F = \mathbb{Q}$)

- $F_+/\mathcal{O}_{F,+}^\times = \mathbb{Q}_+/\{1\}$, $D = \mathbb{Q}_+$.
- $C_{(f)} = \{[(a)] \mid 1 \leq a \leq f,\ (a,f)=1\} \cong (\mathbb{Z}/f\mathbb{Z})^\times$ $(f \in \mathbb{N})$.
- $Z_{[(a)]} := \{z \in \mathbb{Q}_+ \cap (a)^{-1} \mid (za) \in [(a)]\} = \{\frac{a+kf}{a} \mid 0 \leq k \in \mathbb{Z}\}$.

$$\begin{aligned}
\Gamma([(a)]) &:= \Gamma(Z_{[(a)]}) \times \text{ a correction term} \\
&= \exp\left(\frac{d}{ds}\left[\sum_{k\geq 0}\left(\tfrac{a+kf}{a}\right)^{-s}\right]_{s=0}\right) \times a^{\frac{a}{f}-\frac{1}{2}} \\
&= \Gamma(\tfrac{a}{f}) f^{\frac{a}{f}-\frac{1}{2}} (2\pi)^{-\frac{1}{2}} \\
&= \exp(\zeta'(0,[(a)])).
\end{aligned}$$

Note that $\mathrm{Hom}(\mathbb{Q},\mathbb{R}) = \{\mathrm{id}\}$.

# Yoshida's conjecture

## Example ($[F:\mathbb{Q}]=2$)

- $F_+/\mathcal{O}_{F,+}^\times = F_+/\langle\epsilon\rangle$, $D=\{a+b\epsilon \mid a,b\in\mathbb{Q},\ a\geq 0,\ b>0\}$.
- Let $c\in C_\mathfrak{f}$, $\mathfrak{a}\in c$, $Z_c := \{z\in D\cap\mathfrak{a}^{-1} \mid z\mathfrak{a}\in c\}$. We defined $\Gamma(c):=\Gamma(Z_c)\times$ correction terms. In this case, it can be expressed in terms of Barnes' double gamma function:

$$\Gamma(x,(\omega_1,\omega_2)) := \exp\Big(\frac{d}{ds}\Big[\sum_{k_1,k_2\geq 0}(x+k_1\omega_1+k_2\omega_2)^{-s}\Big]_{s=0}\Big).$$

$\exists$ a finite set $R\subset D$, an element $\alpha\in F$, a generator $\pi_{\mathfrak{af}}$ of $(\mathfrak{af})^{h_F^+}$ s.t.

$$\Gamma(c,\iota)=\prod_{x\in R}\Gamma(\iota(x),(1,\iota(\epsilon)))\times\iota(\epsilon)^{\iota(\alpha)}\times\iota(\pi_{\mathfrak{af}})^{-\frac{\zeta(0,c)}{h_F^+}},$$

$$\exp(\zeta'(0,c))=\prod_{\iota\in\mathrm{Hom}(F,\mathbb{R})}\Gamma(c,\iota).$$

# Yoshida's conjecture

Yoshida formulated a conjecture in [Y] which expresses Shimura's period symbol $p_K$ as a finite product of rational powers of $\Gamma(c)$'s. Here we introduce its slight generalization: The original conjecture in [Y] is equivalent to

## Conjecture

Assume that the narrow ray class field $H_{\mathfrak{f}}$ modulo $\mathfrak{f}$ contains a CM field. Let $K$ be the maximal CM subfield of $H_{\mathfrak{f}}$. Then we have for $\sigma \in \mathrm{Gal}(K/F)$

$$\prod_{c \in \mathrm{Art}^{-1}(\sigma)} \Gamma(c) \equiv \prod_{c \in \mathrm{Art}^{-1}(\sigma)} \pi^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}} \bmod \overline{\mathbb{Q}}^{\times}.$$

Strictly speaking, $c, c'$ in $p_K(c,c')$ are the images of $c, c'$ under the Artin map $\mathrm{Art} \colon C_{\mathfrak{f}} \to \mathrm{Gal}(K/F)$.

When $F = \mathbb{Q}$, this conjecture holds true by Rohrlich's formula.

# A $p$-adic analogue of Yoshida's class invariant

Let $\mathfrak{p}$ be the prime ideal corresponding to the $p$-adic topology on $F$.
Let $D$ be Shintani's fundamental domain of $F_+^\times/\mathcal{O}_+^\times$. For $c \in C_\mathfrak{f}$ with $\mathfrak{p} \mid \mathfrak{f}$, we take an ideal $\mathfrak{a} \in c$ and put

$$Z_c := \{z \in D \cap \mathfrak{a}^{-1} \mid z\mathfrak{a} \in c\} \subset F \overset{\mathrm{id}}{\hookrightarrow} \mathbb{R}.$$

For $c \in C_\mathfrak{f}$, we define

$$\Gamma_p(c) := \Gamma_p(Z_c) \times \prod_i \exp_p(b_i \log_p a_i) \quad (a_i, b_i \in F),$$

$$\Gamma_p(Z_c) := \exp_p\left(\frac{d}{ds}\left[\sum_{z \in Z_c} z^{-s}\Big|_{\text{analytic continuation}}\Big|_{p\text{-adic interpolation}}\right]_{s=0}\right)$$

for some $a_i, b_i \in F$ satisfying that

$$\Gamma_p(c) \mod (\mathcal{O}_{F,+}^\times)^\mathbb{Q} \text{ does not depend on } \mathfrak{a}, D.$$

Moreover **the "ratio"** $[\Gamma(c) : \Gamma_p(c)] \bmod \mu_\infty$ **does not depend on** $\mathfrak{a}, D$,
i.e., $\Gamma(c; D, \mathfrak{a})/\Gamma(c; D', \mathfrak{a}') \equiv \Gamma_p(c; D, \mathfrak{a}))/\Gamma_p(c; D', \mathfrak{a}') \bmod \mu_\infty$.

# A $p$-adic analogue of Shimura's period symbol

Let $K$ be a CM field, $\sigma, \tau \in \mathrm{Hom}(K, \mathbb{C})$. We take a suitable algebraic Hecke character $\chi$ of $K^\tau$:

- the infinite type of $\chi = \ell \cdot (\tau^{-1} - \rho \circ \tau^{-1})$ with $\ell$ large enough.
- $\chi(\mathfrak{a}) \in K$ for all $\mathfrak{a} \subset \mathcal{O}_{K^\tau}$ relatively prime to the conductor of $\chi$.

and consider the associated motive $M(\chi)/K^\tau$ with coefficients in $K$.
By comparison isomorphisms of $p$-adic Hodge theory, we define

$$H_B(M(\chi)) \otimes_{\mathbb{Q}} B_{dR} \cong H_{dR}(M(\chi)) \otimes_{K^\tau} B_{dR}, \quad c_B \otimes P_p(\chi) \to c_{dR} \otimes 1,$$

$$K \otimes_{\mathbb{Q}} B_{dR} \cong \bigoplus_{\sigma \in \mathrm{Hom}(K, B_{dR})} B_{dR}, \quad P_p(\chi) \to (P_p(\sigma, \chi))_{\sigma \in \mathrm{Hom}(K, B_{dR})}$$

with $c_*$ a $K$ or $K \otimes_{\mathbb{Q}} K^\tau$-basis of $H_*(M(\chi))$. Then we define

$$p_{K,p}(\sigma, \tau) \equiv (2\pi i)_p^{-\frac{\delta_{\sigma\tau}}{2}} P_p(\sigma, \chi)^{\frac{1}{2\ell}} \bmod \overline{\mathbb{Q}}^\times,$$

where we put $\delta_{\sigma\tau} := 1, -1, 0$ if $\sigma = \tau, \rho \circ \tau$, otherwise, respectively.
Moreover $[p_K(\sigma, \tau) : p_{K,p}(\sigma, \tau)] \bmod \mu_\infty$ **is well-defined**.

# Main results in [K2]

So far, I have explained the following things: For a totally real field $F$, $c \in C_{\mathfrak{f}}$ with $\mathfrak{p} \mid \mathfrak{f}$, we defined the ratio of two class invariants

$$[\Gamma(c) : \Gamma_p(c)] \in (\mathbb{C}^\times/\mu_\infty \times \mathbb{C}_p^\times/\mu_\infty)/(\mathcal{O}_{F,+}^\times)^{\mathbb{Q}}.$$

For a CM field $K$, $\sigma, \tau \in \mathrm{Hom}(K, \mathbb{C})$, we defined the ratio of periods

$$[p_K(\sigma,\tau) : p_{K,p}(\sigma,\tau)] \in (\mathbb{C}^\times/\mu_\infty \times B_{dR}^\times/\mu_\infty)/\overline{\mathbb{Q}}^\times.$$

Then (a slight generalization of) Yoshida's conjecture implies that

$$\mathfrak{G}(c) := \frac{\Gamma(c)}{(2\pi i)^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}} \frac{(2\pi i)_p^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_{K,p}(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}}{\Gamma_p(c)}$$

$\in B_{dR}^\times/\mu_\infty$ is well-defined. Furthermore, by CM, $\mathfrak{G}(c) \in (B_{cris}\overline{\mathbb{Q}_p} - \{0\})^{\mathbb{Q}}/\mu_\infty$ where $\tau \in W_p$ acts as $\Phi_\tau = \Phi_{abs.Frob.}^{\deg \tau} \otimes \tau$.

# Main results in [K2]

Now we can state the main result in [K2].

### Conjecture

Assume that $\mathfrak{p} \mid \mathfrak{f}$. For $c \in C_{\mathfrak{f}}$, $\tau \in W_{\mathfrak{p}} \subset \mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})$, we have the following reciprocity law on this period-ring-valued function:

$$\Phi_\tau\left(\mathfrak{G}(c)\right) \equiv \mathfrak{G}(c_\tau c) \bmod \mu_\infty \text{ where } c_\tau := \mathrm{Art}^{-1}(\tau|_{H_{\mathfrak{f}}}) \in C_{\mathfrak{f}}.$$

In [K2], we proved the following:

### Theorem

Conjecture holds true when $H_{\mathfrak{f}}$ is abelian over $\mathbb{Q}$ and $\mathfrak{p} \nmid 2$.

The case $F = \mathbb{Q}$ follows from Rohrlich's formula and Coleman's formula as we have seen. We reduce the problem to the case $F = \mathbb{Q}$, as follows:

# Reducing the problem to the case $F = \mathbb{Q}$

$\mathfrak{G}(c) := \frac{\Gamma(c)}{(2\pi i)^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}} \frac{(2\pi i)_p^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_{K,p}(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}}{\Gamma_p(c)}$. Recall that $\exp(\zeta'(0,c)) = \prod_{\iota \in \mathrm{Hom}(F,\mathbb{R})} \Gamma(c,\iota)$. When $H_{\mathfrak{f}}/\mathbb{Q}$ is abelian, $\iota(F)$, $\iota(\mathfrak{f})$ and $\iota(c) \in C_{\iota(\mathfrak{f})}$ do not depend on $\iota \in \mathrm{Hom}(F,\mathbb{R})$. Hence, we obtain an expression like $\exp(\zeta'(0,c)) = \Gamma(c)^{[F\,:\,\mathbb{Q}]} \times$ (correction terms) by Yoshida's technique. Since the same holds true for $\exp_p(\zeta_p'(0,c))$ we have

$$[\exp(\zeta'(0,c)) : \exp_p(\zeta_p'(0,c))] \equiv [\Gamma(c)^{[F\,:\,\mathbb{Q}]} : \Gamma_p(c)^{[F\,:\,\mathbb{Q}]}] \bmod \mu_\infty.$$

Put $G := \mathrm{Gal}(H_{\mathfrak{f}}/\mathbb{Q})$ and consider $C_{\mathfrak{f}} = \mathrm{Gal}(H_{\mathfrak{f}}/F) \subset G$. Then we have

$$\sum_{c \in C_{\mathfrak{f}}} \chi(c)\zeta_*(s,c) = L_*(s,\chi) = \prod_{\psi \in \widehat{G}, \psi|_{C_{\mathfrak{f}}} = \chi} L_*(s,\psi) \quad (\chi \in \widehat{C_{\mathfrak{f}}},\ * = \emptyset, p).$$

Hence we obtain an explicit relation between $[\Gamma(c) : \Gamma_p(c)]$'s of $H_{\mathfrak{f}}/F$ and those of $H_{\mathfrak{f}}/\mathbb{Q}$. The part of $[p_K(\dots) : p_{K,p}(\dots)]$ is simpler. Q.E.D.

## Remark

- We also formulated a conjecture in the case $\mathfrak{p} \nmid \mathfrak{f}$, which is rather complicated since we do not have $\Gamma_p(c)$ with $\mathfrak{p} \nmid \mathfrak{f}$.
- Our conjecture is consistent with Stark's conjecture w.r.t. real places and Gross' $p$-adic analogue:
  - Slight generalization of Yoshida's conjecture "implies" the algebraicity of Stark's units:
  - Our conjectures in both cases $\mathfrak{p} \mid \mathfrak{f}$, $\mathfrak{p} \nmid \mathfrak{f}$ imply the reciprocity law on Stark's units up to $\mu_\infty$.
  - Our conjecture in the case $\mathfrak{p} \nmid \mathfrak{f}$ implies Gross' $p$-adic analogue which was proved by Dasgupta-Darmon-Pollack and Ventullo, and its refinements by K.-Yoshida under a certain assumption.

# Yoshida's conjecture vs. Stark's conjecture

## Conjecture (Slight generalization of Yoshida's conjecture)

Assume that the narrow ray class field $H_{\mathfrak{f}}$ modulo $\mathfrak{f}$ contains a CM field. Let $K$ be the maximal CM subfield of $H_{\mathfrak{f}}$. Then we have for $\sigma \in \mathrm{Gal}(K/F)$

$$\prod_{c \in \mathrm{Art}^{-1}(\sigma)} \Gamma(c) \equiv \prod_{c \in \mathrm{Art}^{-1}(\sigma)} \pi^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}} \bmod \overline{\mathbb{Q}}^{\times}.$$

This "difference" is equivalent to the algebraicity of Stark's units:

$$\exp(\zeta'(0,\sigma)) \in \overline{\mathbb{Q}}^{\times} \text{ for } \sigma \in \mathrm{Gal}(H/F)$$

if $F$ is a totally real field, $H$ has a real place, $H/F$ is abelian.

Namely, we have

- Generalized version implies this algebraicity.
- "Original version + this algebraicity" implies generalized version.

(K., On the algebraicity of some products of special values of Barnes' multiple gamma function, to appear in *Amer. J. Math.*)

# Main results in [K2]

## Remark

- We also formulated a conjecture in the case $\mathfrak{p} \nmid \mathfrak{f}$, which is rather complicated since we do not have $\Gamma_p(c)$ with $\mathfrak{p} \nmid \mathfrak{f}$.
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# Main results in [K2]

## Conjecture

Assume that $\mathfrak{p} \nmid \mathfrak{f}$. We put for $c \in C_{\mathfrak{f}}$

$$\mathfrak{G}(c; D, \mathfrak{a}) := \frac{\Gamma(c; D, \mathfrak{a})}{(2\pi i)^{\zeta(0,c)} \prod_{c'} p_K(c, c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}} (2\pi i)_p^{\zeta(0,c)} \prod_{c'} p_{K,p}(c, c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}$$

$\in (B_{cris}\overline{\mathbb{Q}_p} - \{0\})^{\mathbb{Q}}/\mu_\infty$. Then we have for $\tau \in W_{\mathfrak{p}}$ with $\deg_{\mathfrak{p}} \tau = 1$

$$\Phi_\tau(\mathfrak{G}(c; D, \mathfrak{a})) \equiv \frac{\pi_{\mathfrak{p}}^{\frac{\zeta(0,[\mathfrak{p}]c)}{h_F^+}} \mathfrak{G}([\mathfrak{p}]c; D, \mathfrak{p}\mathfrak{a})}{\prod_{\substack{\tilde{c} \in C_{\mathfrak{f}\mathfrak{p}} \\ \tilde{c} \mapsto [\mathfrak{p}]c \in C_{\mathfrak{f}}}} \Gamma_p(\tilde{c}; D, \mathfrak{p}\mathfrak{a})} \mod \mu_\infty.$$

The truth of Conjecture does not depend on the choices of $D, \mathfrak{a}$. The case $F = \mathbb{Q}$ follows from Rohrlich's formula and Coleman's (other) formula.