On a relation between CM periods, Stark units, and multiple gamma functions

Tomokazu Kashio
Tokyo University of Science

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Workshop “Special values of automorphic L-functions, periods of automorphic forms and related topics,”
Celebrating the 60th Birthday of Professor Masaaki Furusawa
This talk is based on the following two papers.

[K1] K., Fermat curves and a refinement of the reciprocity law on cyclotomic units, *J. Reine Angew. Math.*


Besides I used many of the ideas written in

First recall “CM periods” and “Stark units” shortly.

Let $K$ be a CM field (that is, an imaginary quadratic extension of a totally real field). For two complex embeddings $\sigma, \tau \in \text{Hom}(K, \mathbb{C})$, **Shimura’s period symbol** (or, CM period symbol)

$$p_K(\sigma, \tau) \in \mathbb{C}^\times / \mathbb{Q}^\times,$$

is defined in terms of periods of abelian varieties with CM by $K$.

Note that their values are well-defined only up to multiplication by an algebraic number.
Let $K$ be an imaginary quadratic field. We consider an elliptic curve $E : y^2 = x^3 + ax + b$ with $a, b \in \overline{Q}, \ K \cong \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\rho$ be the complex conjugation. Then there are 4 CM periods, which are defined as the integral of suitable differential forms on $E$:

$$p_K(\text{id}, \text{id}) = p_K(\rho, \rho) := \pi^{-1} \int_{\gamma} \frac{dx}{y} \mod \overline{Q}^\times,$$

$$p_K(\rho, \text{id}) = p_K(\text{id}, \rho) := \int_{\gamma} \frac{xdx}{y} \mod \overline{Q}^\times,$$

where $\gamma \subset E(\mathbb{C})$ is an arbitrary non-trivial closed path. Note that $H^1_{dR}(E) = \langle \frac{dx}{y}, \frac{xdx}{y} \rangle$.

When $[K : \mathbb{Q}] > 2$, we consider an abelian variety $A/\overline{Q}$ with CM by $K$, take a suitable $\omega \in H_{dR}(A, \overline{Q})$, and need to “decompose” $\int_{\gamma} \omega$. 
CM periods are closely related to the theme of this workshop: By using the symbol $p_K$, we can express the transcendental parts of

- critical values of $L$-functions associated with algebraic Hecke characters of $K$.
- special values of Hilbert modular forms at CM-points $\in K$.

**Example**

Let $f$ be an elliptic modular form of weight $k$ whose Fourier coefficients $\in \overline{\mathbb{Q}}$. For any imaginary quadratic field $K = \mathbb{Q}(\tau)$ ($\text{Im}(\tau) > 0$), we have

$$f(\tau)/p_K(\text{id}, \text{id})^k \in \overline{\mathbb{Q}}.$$
Stark units (w.r.t. real places)

Let $H/F$ be an abelian extension of number fields. We consider the partial zeta function:

$$
\zeta(s, \sigma) := \sum_{a \in \mathcal{O}_F, \left(\frac{H/F}{a}\right) = \sigma} Na^{-s} \quad (\sigma \in \text{Gal}(H/F)).
$$

We assume that $F$ is totally real and that $H$ has a real place $\nu : H \hookrightarrow \mathbb{R}$. (In particular, a real place of $F$ splits completely in $H/F$.) Then Stark’s conjecture states that

$$
\exp(2\zeta'(0, \sigma)) \in H^\times \quad \text{(in fact, } \in \nu(H^\times)),
$$

$$
\tau(\exp(2\zeta'(0, \sigma))) = \exp(2\zeta'(0, \tau\sigma)) \quad (\sigma, \tau \in \text{Gal}(H/F)),
$$

“$\exp(2\zeta'(0, \sigma))$ is a unit in many cases”, “$H(\exp(\zeta'(0, \sigma))) / F$ is abelian”.

$\exp(2\zeta'(0, \sigma))$ is called a Stark unit.
Let \( n \geq 3, \ \zeta_n := e^{\frac{2\pi i}{n}} \). We consider a CM field \( \mathbb{Q}(\zeta_n) \) and its maximal real subfield \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \). These are abelian over \( \mathbb{Q} \). Write

\[
[\sigma_a : \zeta_n \rightarrow \zeta_n^a] \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \text{Hom}(\mathbb{Q}(\zeta_n), \mathbb{C}) \quad (a \in \mathbb{Z}/n\mathbb{Z})^\times.
\]

By **Rohrlich’s formula** (reformulated by Yoshida) and **Lerch’s formula**, we can write them explicitly:

\[
\begin{align*}
K := \mathbb{Q}(\zeta_n) \quad &p_K(\sigma_a, \sigma_b) \equiv \pi^{-\frac{\delta_{ab}}{2}} \prod_{c \in (\mathbb{Z}/n\mathbb{Z})^\times} \Gamma\left(\frac{c}{n}\right)^{\sum_{\eta \in (\mathbb{Z}/n\mathbb{Z})^\times} \frac{\eta(a^{-1}bc)}{L(0, \eta)\varphi(n)}} \\
\exp(2\zeta'(0, \sigma_a|_H)) = \left(\frac{\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right)}{2\pi}\right)^2 \quad &\text{Euler’s formulas} \\
&\frac{1}{2-\zeta_n^a-\zeta_n^{-a}}
\end{align*}
\]

Here \( \delta_{ab} := 1, -1, 0 \) if \( a = b, a = -b \), otherwise, respectively.
When abelian over $\mathbb{Q}$. Relation via Gamma function

### Problem

- **Rohrlich’s formula**: $p_K(\sigma_a, \sigma_b)$ in terms of $\Gamma(\frac{a}{n})$’s.
- **Lerch’s formula**: $\exp(2\zeta'(0, \sigma_a|_H))$ in terms of $\Gamma(\frac{a}{n})$’s.

By using these, without using Euler’s formulas, can we derive Stark’s conjecture with $F = \mathbb{Q}$?

Since CM periods are defined only up to $\mathbb{Q}^\times$, we can only derive the algebraicity of Stark units with $F = \mathbb{Q}$: roughly speaking,

\[
\text{monomial relations on CM periods} \xrightarrow{\text{Rohrlich’s formula}} \text{monomial relations on } \Gamma(\frac{a}{n})'s \xrightarrow{\text{Lerch’s formula}} \text{the algebraicity of } \exp(2\zeta'(0, \sigma_a|_H)).
\]
When abelian over $\mathbb{Q}$. Relation via Gamma function

Let $F_n : x^n + y^n = 1$ be the $n$th Fermat curve. $J(F_n)$ has CM by $\mathbb{Q}(\zeta_n)$. $H^1_{dR}(J(F_n)) \cong H^1_{dR}(F_n) = \langle \eta_{r,s} := x^r y^{n-s} \frac{dx}{x} \mid 0 < r, s < n, r + s \neq n \rangle$.

Rohrlich's formula. \[
\int_{\gamma} \eta_{r,s} \equiv B\left(\frac{r}{n}, \frac{s}{n}\right) := \frac{\Gamma\left(\frac{r}{n}\right)\Gamma\left(\frac{s}{n}\right)}{\Gamma\left(\frac{r+s}{n}\right)} \mod \mathbb{Q}(\zeta_n)^\times.
\]

The cup product induces a correspondence

$H^1(F_n) \times H^1(F_n) \rightarrow H^2(F_n) = \mathbb{Q}(-1)$ (the Lefschetz motive).

Since the period of $\mathbb{Q}(-1)$ is $2\pi i$, we obtain “monomial relations”

$B\left(\frac{r}{n}, \frac{s}{n}\right)B\left(\frac{n-r}{n}, \frac{n-s}{n}\right) \equiv \int_{\gamma} \eta_{r,s} \int_{\gamma'} \eta_{n-r,n-s} \equiv 2\pi i \mod \overline{\mathbb{Q}}^\times.$

Noting that $\Gamma\left(\frac{r}{n}\right)^n = \Gamma(r) \prod_{k=1}^{n-1} B\left(\frac{r}{n}, \frac{kr}{n}\right)$, we obtain

$\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right) \in 2\pi i \cdot \overline{\mathbb{Q}}^\times$, that is, exp($\zeta'(0, \sigma_a|_H)$) = $\frac{\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right)}{2\pi} \in \overline{\mathbb{Q}}^\times$. 

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When $F = \mathbb{Q}$, the algebraicity $\exp(2\zeta'(0, \sigma_a|_H)) \in \overline{\mathbb{Q}}^\times$ follows from

$$K := \mathbb{Q}(\zeta_n) \quad \int_{\gamma} \eta_{r,s} \quad \text{Rohrlich's formula} \quad \frac{\Gamma\left(\frac{r}{n}\right)\Gamma\left(\frac{s}{n}\right)}{\Gamma\left(\frac{r+s}{n}\right)} \quad \text{mod} \quad \mathbb{Q}(\zeta_n)^\times$$

$$H := \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \quad \exp(2\zeta'(0, \sigma_a|_H)) \quad \text{Lerch's formula} \quad \left(\frac{\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right)}{2\pi}\right)^2$$

and monomial relations on CM periods $\int_{\gamma} \eta_{r,s} \int_{\gamma} \eta_{n-r,n-s} \equiv 2\pi i \text{ mod } \overline{\mathbb{Q}}^\times$.

Moreover, in [K1], we show that Coleman’s formula on the absolute Frobenius action on Fermat curves implies the reciprocity law

$$\tau \left(\exp(2\zeta'(0, \sigma_a|_H))\right) \equiv \exp(2\zeta'(0, \tau \circ \sigma_a|_H)) \mod \mu_\infty \quad (\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})).$$

Here $\mu_\infty$ denotes the group of all roots of unity.
When abelian over $\mathbb{Q}$. Relation via Gamma function

$p$-adic periods are defined by comparison isomorphisms of $p$-adic Hodge theory (instead of the de Rham isomorphism):

$$H_1^B(F_n) \times H^1_{dR}(F_n) \to B_{dR}, \quad (\gamma, \eta) \mapsto \int_{p, \gamma} \eta.$$ 

Here $B_{dR}$ denotes Fontaine’s $p$-adic period ring. Since abelian varieties with CM have potentially good reduction, we have $\int_{p, \gamma} \eta \in B_{cris, \overline{\mathbb{Q}}_p}$. The Weil group $W_p \subset \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts this subring as:

$$\Phi_\tau := \Phi^\text{deg \tau}_{\text{abs.Frob.}} \otimes \tau \curvearrowright B_{cris} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = B_{cris, \overline{\mathbb{Q}}_p} \quad (\tau \in W_p).$$

Assume $p \neq 2, p \nmid n, p \nmid rs(r + s)$. (A similar argument works when $p \nmid n$.) Then Coleman’s formula on $\Phi^\text{abs.Frob.} \curvearrowright H^1_{dR}(F_n/\overline{\mathbb{Q}}_p)$ implies that

$$\Phi_\tau \left( \frac{\int_{p, \gamma} \eta_{r,s}}{B_p\left(\frac{r}{n}, \frac{s}{n}\right)} \right) \equiv p^{\text{deg \tau}} \frac{\int_{p, \gamma} \eta_{\tau(r), \tau(s)}}{B_p\left(\frac{\tau(r)}{n}, \frac{\tau(s)}{n}\right)} \mod \mu_\infty,$$

where we define $\tau(r)$ by $\tau(\zeta_n^r) = \zeta_n^{\tau(r)}, \ B_p(\alpha, \beta) := \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}$. 

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When abelian over $\mathbb{Q}$. Relation via Gamma function

Now by using Rohrlich’s formula again, we can define the period-ring-valued beta function

$$B\left(\frac{r}{n}, \frac{s}{n}\right) := \frac{B\left(\frac{r}{n}, \frac{s}{n}\right) \int_{\gamma} \eta_{r,s}}{\int_{\gamma} \eta_{r,s} B_p\left(\frac{r}{n}, \frac{s}{n}\right)} \in B_{\text{cris}} \mathbb{Q}_p.$$ 

Then Coleman’s formula implies the reciprocity law on $B$:

$$\Phi_\tau(B\left(\frac{r}{n}, \frac{s}{n}\right)) \equiv p^{\deg \tau} \frac{1}{2} B\left(\frac{\tau(r)}{n}, \frac{\tau(s)}{n}\right) \mod \mu_\infty \quad (\tau \in W_p).$$

Then, by using these formulas, we obtain the reciprocity law on Stark’s units

$$\exp(2\zeta'(0, \sigma_a|_H)) = \left(\frac{\Gamma\left(\frac{n-a}{n}\right)\Gamma\left(\frac{n-a}{n}\right)}{2\pi}\right)^2 \text{ up to } \mu_\infty$$ as follows:
When abelian over $\mathbb{Q}$. Relation via Gamma function

\[ \mathcal{B}(\frac{r}{n}, \frac{s}{n}) := \frac{B(\frac{r}{n}, \frac{s}{n})}{\int_{\gamma} \eta_{r,s}} \int_{p,\gamma} \eta_{r,s} B_p(\frac{r}{n}, \frac{s}{n}), \quad \Phi_\tau(\mathcal{B}(\frac{r}{n}, \frac{s}{n})) \equiv p^{\deg \tau} \mathcal{B}(\frac{\tau(r)}{n}, \frac{\tau(s)}{n}). \]

By the correspondence $H^1(F_n) \times H^1(F_n) \to \mathbb{Q}(-1)$

\[ \sim \mathcal{B}(\frac{r}{n}, \frac{s}{n}) \mathcal{B}(\frac{n-r}{n}, \frac{n-s}{n}) = \frac{(2\pi i)_p}{2\pi i} \frac{B(\frac{r}{n}, \frac{s}{n}) B(\frac{n-r}{n}, \frac{n-s}{n})}{B_p(\frac{r}{n}, \frac{s}{n}) B_p(\frac{n-r}{n}, \frac{n-s}{n})} \in (2\pi i)_p \cdot \overline{\mathbb{Q}}_p. \]

$\Phi_\tau = \Phi_{\text{deg } \tau}^{\text{abs.Frob.}} \otimes \tau$ acts on $\overline{\mathbb{Q}}_p$ as $\tau$, on $(2\pi i)_p$ as $p^{\text{deg } \tau}$

\[ \sim \tau \left( \frac{1}{2\pi i} \frac{B(\frac{r}{n}, \frac{s}{n}) B(\frac{n-r}{n}, \frac{n-s}{n})}{B_p(\frac{r}{n}, \frac{s}{n}) B_p(\frac{n-r}{n}, \frac{n-s}{n})} \right) \equiv \frac{1}{2\pi i} \frac{B(\frac{\tau(r)}{n}, \frac{\tau(s)}{n}) B(\frac{\tau(n-r)}{n}, \frac{\tau(n-s)}{n})}{B_p(\frac{\tau(r)}{n}, \frac{\tau(s)}{n}) B_p(\frac{\tau(n-r)}{n}, \frac{\tau(n-s)}{n})}. \]

$\Gamma_*(\frac{r}{n})^n = \Gamma(r) \prod_{k=1}^{n-1} B_*(\frac{r}{n}, \frac{kr}{n}) (* =, p), \Gamma_p(\frac{r}{n}) \Gamma_p(\frac{n-r}{n}) \in \mu_\infty$

\[ \sim \tau \left( \frac{1}{2\pi i} \frac{\Gamma(\frac{r}{n}) \Gamma(\frac{n-r}{n})}{\Gamma_p(\frac{r}{n}) \Gamma_p(\frac{n-r}{n})} \right) \equiv \frac{1}{2\pi i} \frac{\Gamma(\frac{\tau(r)}{n}) \Gamma(\frac{\tau(n-r)}{n})}{\Gamma_p(\frac{\tau(r)}{n}) \Gamma_p(\frac{\tau(n-r)}{n})} \mod \mu_\infty \]

only for $\tau \in \mathcal{W}_p \subset \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subset \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).$ Then we vary $p$. Q.E.D.
Why mod $\mu_\infty$?

We need $B_p(\frac{r}{n}, \frac{s}{n}) := \frac{\Gamma_p(\frac{r}{n})\Gamma_p(\frac{s}{n})}{\Gamma_p(\frac{r+s}{n})}$ for $p \mid n$, $p \nmid rs(r+s)$. So we can not use Morita's $p$-adic gamma function:

$$\Gamma_p(z): \mathbb{Z}_p \to \mathbb{Z}_p^\times, \quad z \mapsto \lim_{N \to z} (-1)^n \prod_{1 \leq k \leq n-1, \ p \nmid k} k.$$  

Recall Lerch’s formula:

$$\frac{\Gamma(x)}{\sqrt{2\pi}} = \exp\left(\frac{d}{ds}\left[\sum_{m=0}^{\infty} (x + m)^{-s}\right]_{s=0}\right).$$  

We need to put

$$\Gamma_p(\frac{r}{n}) := \exp_p\left(\frac{p^N}{p} \frac{d}{ds}\left[\sum_{m=0}^{\infty} (\frac{r}{n} + m)^{-s}(p^{\text{ord}_p n})^{-s}\right]_{p\text{-adic interpolation}}\right)^{\frac{1}{p^N}}$$  

for some $N \in \mathbb{N}$ since $\exp_p(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$ converges only on a neighborhood of 0. To be honest there is one more reason: the “$\mu_\infty$-part” of Coleman’s formula is very complicated and I gave up calculating.
When abelian over an arbitrary totally real field $F$

**Summary.** In [K1], studying classical or $p$-adic CM periods for CM fields **abelian over** $\mathbb{Q}$, we prove the reciprocity law on $\mathcal{B}(\frac{r}{n}, \frac{s}{n})$ and provide an alternative proof of a part of Stark’s conjecture with $F = \mathbb{Q}$.

Note that not only an alternative proof, but also a refinement since

$$\exp(2\zeta'(0, \sigma_a|_H)) = \left(\frac{\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right)}{2\pi}\right)^2$$

is a finite product of $\mathcal{B}(\frac{r}{n}, \frac{s}{n})$'s.

**Motivation.** Generalize this to totally real fields $F$.

We have some good **Guidelines**.

- We have **Shimura’s period symbol** $\rho_K(\sigma, \tau)$, which is a generalization of periods $\int_\gamma \eta_{r,s}$ of Fermat curves.
- We have **Shintani’s formula** on partial zeta functions of totally real fields, which is a generalization of Lerch’s formula

  $$\exp(\zeta'(0, \sigma_a|_H)) = \frac{\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right)}{2\pi}.$$  

- We have **Yoshida’s conjecture** on “absolute CM periods”, which is a conjectural generalization of Rohrlich’s formula $\int_\gamma \eta_{r,s} \equiv \frac{\Gamma\left(\frac{r}{n}\right)\Gamma\left(\frac{s}{n}\right)}{\Gamma\left(\frac{r+s}{n}\right)}$. 
Now we explain multiple gamma functions. Recall Lerch’s formula:

\[
\frac{\Gamma(x)}{\sqrt{2\pi}} = \exp \left( \frac{d}{ds} \left[ \sum_{m=0}^{\infty} (x + m)^{-s} \right]_{s=0} \right).
\]

For a “good” subset \( Z \subset \mathbb{R} \), we put

\[
\Gamma(Z) := \exp \left( \frac{d}{ds} \left[ \sum_{z \in Z} z^{-s} \right]_{s=0} \right).
\]

Here we say \( Z \) is “good” if \( \sum_{z \in Z} z^{-s} \) converges for \( \text{Re}(s) >> 0 \), has a meromorphic continuation, is analytic at \( s = 0 \). In particular, for \( x > 0 \), \( \omega := (\omega_1, \ldots, \omega_r) \) with \( \omega_1, \ldots, \omega_r > 0 \), the lattice-like set

\[
L_{x,\omega} := \{ x + m_1\omega_1 + \cdots + m_r\omega_r \mid 0 \leq m_1, \ldots, m_r \in \mathbb{Z} \}
\]

is “good” and \( \Gamma(L_{x,\omega}) \) is called \textbf{Barnes’ multiple gamma function}. 
Yoshida’s class invariant

Let $F$ be a totally real field, $\mathfrak{f}$ an integral ideal of $\mathfrak{f}$, $C_{\mathfrak{f}}$ the ideal class group modulo $\mathfrak{f}$, in the narrow sense. Let $D$ be Shintani’s fundamental domain of $F_+/\mathcal{O}_F^\times$. For $c \in C_{\mathfrak{f}}$, we take an ideal $\alpha \in c$ and consider a subset:

$$Z_c := \{z \in D \cap \alpha^{-1} \mid z\alpha \in c\} \subset F_+.$$

Shintani provided an expression $Z_c = \coprod_{i=1}^k L_{x_i, \omega_i}$. In particular $\nu(Z_c)$ is “good” for any $\nu \in \text{Hom}(F, \mathbb{R})$. Yoshida defined a class invariant

$$\Gamma(c, \nu) := \Gamma(\nu(Z_c)) \times \prod_i \nu(a_i)^{\nu(b_i)} \quad (c \in C_{\mathfrak{f}}, \ \nu \in \text{Hom}(F, \mathbb{R}))$$

for certain $a_i, b_i \in F$. Although $Z_c, a_i, b_i$ depend on $D, \alpha$, we have

- $\prod_{\nu \in \text{Hom}(F, \mathbb{R})} \Gamma(c, \nu) = \exp(\zeta'(0, c))$: Shintani’s formula, which is a generalization of Lerch’s formula.
- $\Gamma(c, \nu) \mod \nu(\mathcal{O}_F^\times, +)^\mathbb{Q}$ does not depend on the choices of $D, \alpha$,

  that is, $\Gamma(c, \nu; D, \alpha)/\Gamma(c, \nu; D', \alpha') = \nu(\epsilon)^{1/N}$ with $\epsilon \in \mathcal{O}_F^\times, \ N \in \mathbb{N}$.

We fix $\text{id} : F \hookrightarrow \mathbb{R}$ and put $\Gamma(c) := \Gamma(c, \text{id})$. 

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Example \((F = \mathbb{Q})\)

- \(F_+/\mathcal{O}_F^\times_+ = \mathbb{Q}_+/\{1\}, \ D = \mathbb{Q}_+\).
- \(C_{(f)} = \{[(a)] \mid 1 \leq a \leq f, \ (a, f) = 1\} \cong (\mathbb{Z}/f\mathbb{Z})^\times \ (f \in \mathbb{N})\).
- \(Z_{[(a)]} := \{z \in \mathbb{Q}_+ \cap (a)^{-1} \mid (za) \in [(a)]\} = \left\{\frac{a+km}{a} \mid 0 \leq k \in \mathbb{Z}\right\}\).

\[
\Gamma([(a)]) := \Gamma(Z_{[(a)]}) \times \text{a correction term}
\]

\[
= \exp\left(\frac{d}{ds}\left[\sum_{k \geq 0} \left(\frac{a+km}{a}\right)^{-s}\right]_{s=0}\right) \times a^{\frac{a}{f} - \frac{1}{2}}
\]

\[
= \Gamma\left(\frac{a}{f}\right)f^{\frac{a}{f} - \frac{1}{2}}(2\pi)^{-\frac{1}{2}}
\]

\[
= \exp(\zeta'(0, [(a)]))
\]

Note that \(\text{Hom}(\mathbb{Q}, \mathbb{R}) = \{\text{id}\}\).
Yoshida’s conjecture

Example ([F : Q] = 2)

- \( F_+/\mathcal{O}_{F,+}^\times = F_+/<\epsilon> \), \( D = \{ a + b\epsilon \mid a, b \in \mathbb{Q}, \ a \geq 0, \ b > 0 \} \).

- Let \( c \in C_f, \ a \in c, \ Z_c := \{ z \in D \cap a^{-1} \mid za \in c \} \). We defined 
  \( \Gamma(c) := \Gamma(Z_c) \times \) correction terms. In this case, it can be expressed in terms of Barnes’ double gamma function:

\[
\Gamma(x, (\omega_1, \omega_2)) := \exp\left( \frac{d}{ds} \left[ \sum_{k_1, k_2 \geq 0} (x + k_1\omega_1 + k_2\omega_2)^{-s} \right]_{s=0} \right).
\]

\( \exists \) a finite set \( R \subset D \), an element \( \alpha \in F \), a generator \( \pi_{af} \) of \( (af)^{h_F^+} \) s.t.

\[
\Gamma(c, \iota) = \prod_{x \in R} \Gamma(\iota(x), (1, \iota(\epsilon))) \times \iota(\epsilon)^{\iota(\alpha)} \times \iota(\pi_{af})^{-\zeta(0,c)_{h_F^+}},
\]

\[
\exp(\zeta'(0, c)) = \prod_{\iota \in \text{Hom}(F, \mathbb{R})} \Gamma(c, \iota).
\]
Yoshida’s conjecture

Yoshida formulated a conjecture in [Y] which expresses Shimura’s period symbol $p_K$ as a finite product of rational powers of $\Gamma(c)$’s. Here we introduce its slight generalization: The original conjecture in [Y] is equivalent to

**Conjecture**

Assume that the narrow ray class field $H_f$ modulo $f$ contains a CM field. Let $K$ be the maximal CM subfield of $H_f$. Then we have for $\sigma \in \text{Gal}(K/F)$

$$
\prod_{c \in \text{Art}^{-1}(\sigma)} \Gamma(c) \equiv \prod_{c \in \text{Art}^{-1}(\sigma)} \pi^{\zeta(0,c)} \prod_{c' \in C_f} p_K(c, c')^{\zeta(0,c')^{[H_f:K]}} \mod \mathbb{Q}^\times.
$$

Strictly speaking, $c, c'$ in $p_K(c, c')$ are the images of $c, c'$ under the Artin map $\text{Art} : C_f \rightarrow \text{Gal}(K/F)$.

When $F = \mathbb{Q}$, this conjecture holds true by Rohrlich’s formula.
A $p$-adic analogue of Yoshida’s class invariant

Let $p$ be the prime ideal corresponding to the $p$-adic topology on $F$. Let $D$ be Shintani’s fundamental domain of $F_+^\times / \mathcal{O}_+^\times$. For $c \in C_f$ with $p \mid f$, we take an ideal $a \in c$ and put

$$Z_c := \{ z \in D \cap a^{-1} \mid za \in c \} \subset F \xrightarrow{\text{id}} \mathbb{R}.$$  

For $c \in C_f$, we define

$$\Gamma_p(c) := \Gamma_p(Z_c) \times \prod_i \exp_p \left( b_i \log_p a_i \right) \quad (a_i, b_i \in F),$$  

$$\Gamma_p(Z_c) := \exp_p \left( \frac{d}{ds} \left[ \sum_{z \in Z_c} z^{-s} \right]_{\text{analytic continuation}} \right)_{s=0}$$

for some $a_i, b_i \in F$ satisfying that

$$\Gamma_p(c) \mod (\mathcal{O}_F^\times)_+^{\mathbb{Q}} \text{ does not depend on } a, D.$$  

Moreover the “ratio” $[\Gamma(c) : \Gamma_p(c)] \mod \mu_\infty \text{ does not depend on } a, D,$ i.e., $\Gamma(c; D, a)/\Gamma(c; D', a') \equiv \Gamma_p(c; D, a))/\Gamma_p(c; D', a') \mod \mu_\infty.$
A $p$-adic analogue of Shimura’s period symbol

Let $K$ be a CM field, $\sigma, \tau \in \text{Hom}(K, \mathbb{C})$. We take a suitable algebraic Hecke character $\chi$ of $K^\tau$:

- the infinite type of $\chi = \ell \cdot (\tau^{-1} - \rho \circ \tau^{-1})$ with $\ell$ large enough.
- $\chi(\alpha) \in K$ for all $\alpha \subset \mathcal{O}_{K^\tau}$ relatively prime to the conductor of $\chi$.

and consider the associated motive $M(\chi)/K^\tau$ with coefficients in $K$. By comparison isomorphisms of $p$-adic Hodge theory, we define

$$H_B(M(\chi)) \otimes_{\mathbb{Q}} B_{dR} \cong H_{dR}(M(\chi)) \otimes_{K^\tau} B_{dR}, \quad c_B \otimes P_p(\chi) \rightarrow c_{dR} \otimes 1,$$

$$K \otimes_{\mathbb{Q}} B_{dR} \cong \bigoplus_{\sigma \in \text{Hom}(K, B_{dR})} B_{dR}, \quad P_p(\chi) \rightarrow (P_p(\sigma, \chi))_{\sigma \in \text{Hom}(K, B_{dR})},$$

with $c_*$ a $K$ or $K \otimes_{\mathbb{Q}} K^\tau$-basis of $H_*(M(\chi))$. Then we define

$$p_{K,p}(\sigma, \tau) \equiv (2\pi i)_p^\delta_{\sigma \tau} P_p(\sigma, \chi)^{\frac{1}{2\ell}} \mod \overline{\mathbb{Q}}^\times,$$

where we put $\delta_{\sigma \tau} := 1, -1, 0$ if $\sigma = \tau, \rho \circ \tau$, otherwise, respectively.

Moreover $[p_K(\sigma, \tau) : p_{K,p}(\sigma, \tau)] \mod \mu_\infty$ is well-defined.
Main results in [K2]

So far, I have explained the following things: For a totally real field $F$, $c \in C_f$ with $p \mid f$, we defined the ratio of two class invariants

$$[\Gamma(c) : \Gamma_p(c)] \in (\mathbb{C}^\times / \mu_\infty \times \mathbb{C}_p^\times / \mu_\infty) / (\mathcal{O}_{F,+})^\mathbb{Q}.$$ 

For a CM field $K$, $\sigma, \tau \in \text{Hom}(K, \mathbb{C})$, we defined the ratio of periods

$$[p_K(\sigma, \tau) : p_K, p(\sigma, \tau)] \in (\mathbb{C}^\times / \mu_\infty \times B_{dR}^\times / \mu_\infty) / \mathbb{Q}^\times.$$ 

Then (a slight generalization of) Yoshida’s conjecture implies that

$$\mathcal{G}(c) := \frac{\Gamma(c)}{(2\pi i)^{\zeta(0,c)}} \prod_{c' \in C_f} p_K(c, c')^{\zeta(0,c') \text{ [H_f:K] } / \Gamma_p(c)} \in B_{dR}^\times / \mu_\infty$$

is well-defined. Furthermore, by CM,

$$\mathcal{G}(c) \in (B_{cris} \mathbb{Q}_p - \{0\})^\mathbb{Q} / \mu_\infty$$

where $\tau \in W_p$ acts as $\Phi_\tau = \Phi_{\text{deg} \tau}^{\text{abs.Frob}} \otimes \tau$. 

T. Kashio, Tokyo Univ. of Sci.  
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Main results in [K2]

Now we can state the main result in [K2].

**Conjecture**

Assume that $p \mid f$. For $c \in C_f$, $\tau \in W_p \subset \text{Gal}(\overline{F}_p/F_p)$, we have the following reciprocity law on this period-ring-valued function:

$$\Phi_{\tau} (\mathcal{G}(c)) \equiv \mathcal{G}(c_\tau c) \mod \mu_\infty$$

where $c_\tau := \text{Art}^{-1}(\tau|_{H_f}) \in C_f$.

In [K2], we proved the following:

**Theorem**

Conjecture holds true when $H_f$ is abelian over $\mathbb{Q}$ and $p \nmid 2$.

The case $F = \mathbb{Q}$ follows from Rohrlich’s formula and Coleman’s formula as we have seen. We reduce the problem to the case $F = \mathbb{Q}$, as follows:
Reducing the problem to the case $F = \mathbb{Q}$

$$\mathcal{G}(c) := \frac{\Gamma(c)}{(2\pi i)^{\zeta(0,c)} \prod_{c' \in C_f} p_K(c,c')^{\zeta(0,c')^{[H_f:K]}} \Gamma_p(c)} \prod_{c' \in C_f} p_K(c,c')^{\zeta(0,c')^{[H_f:K]}}$$

Recall that $\exp(\zeta'(0, c)) = \prod_{\iota \in \text{Hom}(F, \mathbb{R})} \Gamma(c, \iota)$. When $H_f/\mathbb{Q}$ is abelian, $\iota(F)$, $\iota(f)$ and $\iota(c) \in C_{\iota(f)}$ do not depend on $\iota \in \text{Hom}(F, \mathbb{R})$. Hence, we obtain an expression like $\exp(\zeta'(0, c)) = \Gamma(c)^{[F: \mathbb{Q}]} \times$ (correction terms) by Yoshida's technique. Since the same holds true for $\exp_p(\zeta'_p(0, c))$ we have

$$[\exp(\zeta'(0, c)) : \exp_p(\zeta'_p(0, c))] \equiv [\Gamma(c)^{[F: \mathbb{Q}]} : \Gamma_p(c)^{[F: \mathbb{Q}]})] \mod \mu_\infty.$$

Put $G := \text{Gal}(H_f/\mathbb{Q})$ and consider $C_f = \text{Gal}(H_f/F) \subset G$. Then we have

$$\sum_{c \in C_f} \chi(c)\zeta_*(s, c) = L_*(s, \chi) = \prod_{\psi \in \hat{G}, \psi|_{C_f} = \chi} L_*(s, \psi) \quad (\chi \in \hat{C}_f, \ * = \emptyset, p).$$

Hence we obtain an explicit relation between $[\Gamma(c) : \Gamma_p(c)]$'s of $H_f/F$ and those of $H_f/\mathbb{Q}$. The part of $[p_K(\ldots) : p_K.p(\ldots)]$ is simpler. Q.E.D.
Remark

- We also formulated a conjecture in the case $p \nmid f$, which is rather complicated since we do not have $\Gamma_p(c)$ with $p \nmid f$.
- Our conjecture is consistent with Stark’s conjecture w.r.t. real places and Gross’ $p$-adic analogue:
  - Slight generalization of Yoshida’s conjecture “implies” the algebraicity of Stark’s units:
    - Our conjectures in both cases $p \mid f, p \nmid f$ imply the reciprocity law on Stark’s units up to $\mu_\infty$.
    - Our conjecture in the case $p \mid f$ implies Gross’ $p$-adic analogue which was proved by Dasgupta-Darmon-Pollack and Ventullo, and its refinements by K.-Yoshida under a certain assumption.
Conjecture (Slight generalization of Yoshida’s conjecture)

Assume that the narrow ray class field $H_f$ modulo $f$ contains a CM field. Let $K$ be the maximal CM subfield of $H_f$. Then we have for $\sigma \in \text{Gal}(K/F)$

$$\prod_{c \in \text{Art}^{-1}(\sigma)} \Gamma(c) \equiv \prod_{c \in \text{Art}^{-1}(\sigma)} \pi^{\zeta(0,c)} \prod_{c' \in C_f} p_K(c, c')^{\zeta(0,c')_{[H_f:K]}} \mod \mathbb{Q}^\times.$$ 

This “difference” is equivalent to the algebraicity of Stark’s units:

$$\exp(\zeta'(0, \sigma)) \in \mathbb{Q}^\times \text{ for } \sigma \in \text{Gal}(H/F)$$

if $F$ is a totally real field, $H$ has a real place, $H/F$ is abelian.

Namely, we have

- Generalized version implies this algebraicity.
- “Original version + this algebraicity” implies generalized version.

(K., On the algebraicity of some products of special values of Barnes’ multiple gamma function, to appear in Amer. J. Math.)
Remark

- We also formulated a conjecture in the case $p \nmid \mathfrak{f}$, which is rather complicated since we do not have $\Gamma_p(c)$ with $p \nmid \mathfrak{f}$.
- Our conjecture is consistent with Stark’s conjecture w.r.t. real places and Gross’ $p$-adic analogue:
  - Slight generalization of Yoshida’s conjecture “implies” the algebraicity of Stark’s units.
  - Our conjectures in both cases $p \mid \mathfrak{f}$, $p \nmid \mathfrak{f}$ imply the reciprocity law on Stark’s units up to $\mu_\infty$.
  - Our conjecture in the case $p \nmid \mathfrak{f}$ implies Gross’ $p$-adic analogue which was proved by Dasgupta-Darmon-Pollack and Ventullo, and its refinements by K.-Yoshida under a certain assumption.
Conjecture

Assume that \( p \nmid f \). We put for \( c \in C_f \)

\[
\mathcal{G}(c; D, a) := \frac{\Gamma(c; D, a)}{(2\pi i)^{\zeta(0,c)}} \prod_{c'} p_{K,c'}(c, c')^{\zeta(0,c')} \prod_{c'} p_{K,c'}(c, c')^{\zeta(0,c')} \prod_{c'} p_{K,c'}(c, c')^{\zeta(0,c')} \prod_{c'} p_{K,c'}(c, c')^{\zeta(0,c')}
\]

\[
\in (B_{cris} \overline{\mathbb{Q}}_p - \{0\})^\mathbb{Q}/\mu_\infty. \text{ Then we have for } \tau \in W_p \text{ with } \deg_p \tau = 1
\]

\[
\Phi_\tau(\mathcal{G}(c; D, a)) \equiv \frac{\zeta(0,[p]c)}{\prod_{\tilde{c} \in C_{fp}} \Gamma_p(\tilde{c}; D, p\alpha)} \text{ mod } \mu_\infty.
\]

The truth of Conjecture does not depend on the choices of \( D, a \). The case \( F = \mathbb{Q} \) follows from Rohrlich’s formula and Coleman’s (other) formula.