A period-ring-valued gamma function and a refinement of the reciprocity law on Stark units

Tomokazu Kashio (Tokyo University of Science)

Dec, 6th (Wed), 09:20 – 10:10

References

- K., On a common refinement of Stark units and Gross-Stark units, arXiv:1706.03198.
- H. Yoshida, *Absolute CM-Periods*, Math. Surveys Monogr. **106**, AMS, 2003.

1 Introduction

The theme is **CM-periods** and **Stark units**:

CM-periods. Let K a CM-field.

- Take an abelian variety $A/\overline{\mathbb{Q}}$ with CM by K, i.e., $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K$, and its differential form ω_σ of the second kind where K acts as $\sigma(K)$ ($\sigma \in \mathrm{Hom}(K, \mathbb{C})$).
- $\int_\gamma \omega_\sigma \in \mathbb{C}$ is called a **CM-period** for an arbitrary closed path γ with $\int_\gamma \omega_\sigma \neq 0$.
- Shimura provided "generators" $p_K(\sigma, \sigma')$ of the group of all monomials of $\int_\gamma \omega_\sigma$'s $\mathrm{mod}\overline{\mathbb{Q}}^\times$ which is called **Shimura's period symbol**.

Stark units (w.r.t. real places). Let F be a totally real number field, K its abelian extension where only one real place of F splits, excepting the case $K/F = \mathbb{Q}/\mathbb{Q}$.

- Consider the partial zeta function $\zeta(s, \tau) := \sum_{(\frac{K/F}{\mathfrak{a}})=\tau} N\mathfrak{a}^{-s}$ ($\tau \in \mathrm{Gal}(K/F)$).
- The assumptions imply $\mathrm{ord}_{s=0}\zeta(s, \tau) = 1$.
- "The rank one abelian Stark conjecture" implies $\exp(2\zeta'(0, \tau)) \in K^\times$ satisfying

  $$\textbf{the reciprocity law}: \quad \tau'(\exp(2\zeta'(0, \tau))) = \exp(2\zeta'(0, \tau'\tau)).$$

- Stark's conjecture also implies $\exp(2\zeta'(0, \tau)) \in \mathcal{O}_K^\times$ when $|\mathrm{Ram}(K/F) \cup \mathrm{Arch}(F)| > 2$. $\exp(2\zeta'(0, \tau))$ is called a **Stark unit**.

There seems to be a "relation", in terms of **multiple gamma functions**.

The case $F=\mathbb{Q}$. Let $n \geq 3$, $\zeta_n := e^{\frac{2\pi i}{n}}$. Then

- (Each simple factor of) Jacobian variety $J(F_n)$ of Fermat curve $F_n\colon x^n+y^n=1$ has CM by the CM field $\mathbb{Q}(\zeta_n)$. $\eta_{r,s} := x^r y^{n-s}\frac{dx}{x}$ $(0<r,s<n,\ r+s\neq 0)$ are differential forms of the second kind.
- $\mathbb{Q}(\zeta_n+\zeta_n^{-1})$ has a real place (hence, the unique real place of $\mathbb{Q}$ splits). $\zeta(s,\sigma_{\pm a}) = \zeta(s,n,a)+\zeta(s,n,n-a)$ where $\sigma_{\pm a}\colon \zeta_n+\zeta_n^{-1}\mapsto \zeta_n^a+\zeta_n^{-a}$ $(0<a<n,\ (a,n)=1)$, $\zeta(s,n,a) := \sum_{k=0}^{\infty}(a+nk)^{-s}$ denotes the Hurwitz zeta function.

Then we have

$$\begin{array}{llcl}
\mathbb{Q}(\zeta_n) & \int_\gamma \eta_{r,s} & \overset{\text{Rohrlich's formula}}{\equiv} & B\left(\frac{r}{n},\frac{s}{n}\right) := \dfrac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} \mod \mathbb{Q}(\zeta_n)^\times \\
\mid & & & \\
\mathbb{Q}(\zeta_n+\zeta_n^{-1}) & \exp(2\zeta'(0,\sigma_{\pm a})) & \overset{\text{Lerch's formula}}{\equiv} & \left(\dfrac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}\right)^2 \\
\mid & & \Big(\overset{\text{Euler's formulas}}{=} & \dfrac{1}{2-\zeta_n^a-\zeta_n^{-a}} \approx \text{cyclotomic unit}\Big) \\
\mathbb{Q} & & &
\end{array}$$

We formulate conjectures which state, roughly speaking,

§2. Monomial relations of CM-periods imply the algebraicity of Stark units.

§3. The absolute Frobenius actions on p-adic CM-periods imply the reciprocity law on Stark units.

Indeed, when $F=\mathbb{Q}$, we showed that (K., Fermat curves and a refinement of the reciprocity law on cyclotomic units, *Crelle's Journal* (online version))

- The cup product $H^1(F_n)\times H^1(F_n)\to H^2(F_n)=\mathbb{Q}(-1)$ (the Lefschetz motive) induces "monomial relations"

$$B(\tfrac{r}{n},\tfrac{s}{n})B(\tfrac{n-r}{n},\tfrac{n-s}{n}) \overset{\text{Rohrlich's formula}}{\equiv} \int_\gamma \eta_{r,s}\int_{\gamma'}\eta_{n-r,n-s} \equiv 2\pi i \mod \overline{\mathbb{Q}}^\times$$

  since the period of $\mathbb{Q}(-1)$ is $2\pi i$.

- Noting that $\Gamma(\frac{r}{n})^n = \Gamma(r)\prod_{k=1}^{n-1}B(\frac{r}{n},\frac{kr}{n})$ (thank to an anonymous referee), we obtain $\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n}) \in 2\pi i\cdot\overline{\mathbb{Q}}^\times$. It follows that $\exp(2\zeta'(0,\sigma_{\pm a}))\in\overline{\mathbb{Q}}^\times$ by Lerch's formula, without Euler's formulas.
- Furthermore, we see that Coleman's formula on Frobenius action on F_m implies $\sigma_{\pm b}(\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}) \equiv \frac{\Gamma(\frac{ab}{n})\Gamma(\frac{n-ab}{n})}{2\pi}$, at least modulo the group μ_∞ of all roots of unity.

2 Yoshida's conjecture and its slight refinement

2.1 Multiple gamma functions

Recall Lerch's formula:

$$\frac{\Gamma(x)}{\sqrt{2\pi}} = \exp\left(\frac{d}{ds}\left[\sum_{m=0}^{\infty}(x+m)^{-s}\right]_{s=0}\right).$$

For a "good" subset $Z \subset \mathbb{R}$, we put

$$\Gamma(Z) := \exp\left(\frac{d}{ds}\left[\sum_{z\in Z} z^{-s}\right]_{s=0}\right)$$

Here we say Z is "good" if $\sum_{z\in Z} z^{-s}$ converges for $\mathrm{Re}(s) >> 0$, has a meromorphic continuation, is analytic at $s=0$. In particular, for $x>0$, $\boldsymbol{\omega} := (\omega_1,\ldots,\omega_r)$ with $\omega_1,\ldots,\omega_r > 0$, a "lattice in cone"

$$L_{x,\boldsymbol{\omega}} := \{x + m_1\omega_1 + \cdots + m_r\omega_r \mid 0 \le m_1,\ldots,m_r \in \mathbb{Z}\}$$

is "good" and $\Gamma(L_{x,\boldsymbol{\omega}})$ is called **Barnes' multiple gamma function**.

2.2 Shintani's formula and Yoshida's class invariants

Let F be a totally real field, $\mathfrak{f}$ an integral ideal of F, $C_\mathfrak{f}$ the ideal class group modulo $\mathfrak{f}$, in the narrow sense. Let D be Shintani's fundamental domain of $F_+/\mathcal{O}_{F,+}^\times$. Here $+$ denotes their totally positive parts. For $c \in C_\mathfrak{f}$, we take an ideal $\mathfrak{a} \in c$ and consider a subset:

$$Z_c := \{z \in D \cap \mathfrak{a}^{-1} \mid z\mathfrak{a} \in c\} \subset F_+.$$

Remark 1. *Shintani expressed D as a finite disjoint union of cones and provided an expression $Z_c = \coprod_{i=1}^k L_{x_i,\boldsymbol{\omega}_i}$. In particular $\iota(Z_c)$ is "good" for any $\iota \in \mathrm{Hom}(F,\mathbb{R})$.*

Yoshida defined a **class invariant**

$$\Gamma(c,\iota) := \Gamma(\iota(Z_c)) \times \prod_i \iota(a_i)^{\iota(b_i)} \quad (c \in C_\mathfrak{f},\ \iota \in \mathrm{Hom}(F,\mathbb{R}))$$

for suitable $a_i, b_i \in F$. Although Z_c, a_i, b_i depend on the choices of $D, \mathfrak{a}$, we have

- **Shintani's formula** states that $\exp(\zeta'(0,c)) = \prod_{\iota\in\mathrm{Hom}(F,\mathbb{R})} \Gamma(c,\iota)$.
- $\Gamma(c,\iota) \bmod \iota(\mathcal{O}_{F,+}^\times)^\mathbb{Q}$ does not depend on $D, \mathfrak{a}$: there exist $\epsilon \in \mathcal{O}_{F,+}^\times$, $N \in \mathbb{N}$ s.t.

$$\Gamma(c,\iota;D,\mathfrak{a})/\Gamma(c,\iota;D',\mathfrak{a}') = \iota(\epsilon)^{\frac{1}{N}}.$$

We fix $\mathrm{id}\colon F \hookrightarrow \mathbb{R}$ and put $\Gamma(c) := \Gamma(c,\mathrm{id})$.

2.3 Shimura's period symbol (well-known restatement)

Let K be a CM-field, $\sigma, \tau \in \mathrm{Hom}(K, \mathbb{C})$. We take an algebraic Hecke character χ of K^τ, which takes values in K, whose infinite type is $l \cdot (\tau^{-1} - \rho \circ \tau^{-1})$ with l large enough. We consider the associated motive $M(\chi)/K^\tau$ with coefficients in K. By the de Rham isomorphism, we define

$$H_B(M(\chi)) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{dR}(M(\chi)) \otimes_{K^\tau} \mathbb{C}, \quad c_B \otimes P(\chi) \to c_{dR} \otimes 1,$$
$$K \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} \mathbb{C}, \quad P(\chi) \to (P(\sigma, \chi))_{\sigma \in \mathrm{Hom}(K,\mathbb{C})}$$

with c_* a K or $K \otimes_{\mathbb{Q}} K^\tau$-basis of $H_*(M(\chi))$. Then we have

$$p_K(\sigma, \tau) \equiv (2\pi i)^{-\frac{\delta_{\sigma\tau}}{2}} P(\sigma, \chi)^{\frac{1}{2l}} \bmod \overline{\mathbb{Q}}^\times,$$

where we put $\delta_{\sigma\tau} := 1, -1, 0$ if $\sigma = \tau, \rho \circ \tau$, otherwise, respectively.

2.4 Yoshida's conjecture

Yoshida formulated a conjecture which expresses Shimura's period symbol p_K as a finite product of rational powers of $\Gamma(c)$'s. Here we introduce its slight generalization:

Conjecture 2. *Assume that the narrow ray class field $H_{\mathfrak{f}}$ modulo $\mathfrak{f}$ contains a CM-field. Let K be the maximal CM subfield of $H_{\mathfrak{f}}$. Then we have*

$$\Gamma(c) \equiv \pi^{\zeta(0,c)} \prod_{c' \in C_{\mathfrak{f}}} p_K(c, c')^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}} \bmod \overline{\mathbb{Q}}^\times.$$

Here c, c' in $p_K(\ )$ denotes the images of them under the Artin map $\mathrm{Art}\colon C_{\mathfrak{f}} \to \mathrm{Gal}(K/F)$.

Remark 3.

- *When $F = \mathbb{Q}$, this conjecture holds true by Rohrlich's formula.*
- *The original one is equivalent to* $\prod_{c \in \mathrm{Art}^{-1}(\sigma)} LHS \equiv \prod_{c \in \mathrm{Art}^{-1}(\sigma)} RHS$ *for* $\sigma \in \mathrm{Gal}(K/F)$.
- *Conjecture 1 implies the algebraicity of Stark units:*

  "$\exp(\zeta'(0, \sigma)) \in \overline{\mathbb{Q}}^\times$ *for* $\sigma \in \mathrm{Gal}(H/F)$
  if F is a totally real field, H has a real place, H/F is abelian"

  follows from the monomial relation $p_K(\sigma, \sigma')p_K(\sigma, \rho \circ \sigma') \equiv 1 \bmod \overline{\mathbb{Q}}^\times$.
- *More precisely, we can show that*
  - *Generalized one implies this algebraicity.*
  - *"Original one + this algebraicity" implies generalized one.*

  (K., On the algebraicity of some products of special values of Barnes' multiple gamma function, Amer. J. Math. (online version))

2.5 A numerical example

Let $K := \mathbb{Q}(\sqrt{2\sqrt{5}-26})$, $\sigma\colon \sqrt{2\sqrt{5}-26} \mapsto -\sqrt{-2\sqrt{5}-26} \in \mathrm{Hom}(K,\mathbb{C})$, ρ the complex conjugation. Then $\mathrm{Hom}(K,\mathbb{C}) = \{\mathrm{id},\rho,\sigma,\rho\circ\sigma\}$. Let

$$C\colon y^2 = \tfrac{7+\sqrt{41}}{2}x^6 + (-10-2\sqrt{41})x^5 + 10x^4 + \tfrac{41+\sqrt{41}}{2}x^3 + (3-2\sqrt{41})x^2 + \tfrac{7-\sqrt{41}}{2}x + 1.$$

Then $J(C)$ has CM by $(K,\{\mathrm{id},\sigma\})$. (Bouyer and Streng, Examples of CM curves of genus two defined over the reflex field, *LMS J. Comput. Math.*) In fact, $\omega_{\mathrm{id}} = \frac{2dx}{y} + \frac{(\sqrt{5}-1)xdx}{y}$, $\omega_\sigma = \frac{(-\sqrt{5}+\sqrt{41})xdx}{y}$ are differential forms of the first kind where K acts by id,σ respectively. Numerically we have (Maple's command "periodmatrix")

$$\begin{aligned} \pi p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\sigma) &:= \int \omega_{\mathrm{id}} = -0.4929421793\ldots - 0.8116152991\ldots i,\\ \pi p_K(\sigma,\mathrm{id})p_K(\sigma,\sigma) &:= \int \omega_{\sigma} = -0.1395619319\ldots + 0.1323795194\ldots i. \end{aligned}$$

Similarly, we define C' where $J(C')$ has CM by $(K,\{\mathrm{id},\rho\circ\sigma\})$ by replacing $\sqrt{41}$ with $-\sqrt{41}$. Then we have

$$\begin{aligned} \pi p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\rho\circ\sigma) &= -0.4443866005\ldots - 0.3099403507\ldots i,\\ \pi p_K(\rho\circ\sigma,\mathrm{id})p_K(\rho\circ\sigma,\rho\circ\sigma) &= -2.0247186165\ldots + 0.4533729269\ldots i. \end{aligned}$$

Let $F := \mathbb{Q}(\sqrt{5})$, $\mathfrak{f} := (\frac{13-\sqrt{5}}{2})$. We easily see that $C_{\mathfrak{f}} = \{c_1 := [(1)], c_2 := [(3)]\} \cong \mathrm{Gal}(K/F)$, $c_1 \leftrightarrow \mathrm{id}$, $c_2 \leftrightarrow \rho$, $\zeta(0,c_1) = 1$, $\zeta(0,c_2) = -1$. Then we obtain numerically

$$\begin{aligned} \pi p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\rho)^{-1} &\equiv \pi p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\sigma)p_K(\mathrm{id},\mathrm{id})p_K(\mathrm{id},\rho\circ\sigma)\\ &\stackrel{?}{=} \Gamma(c_1)(\tfrac{\sqrt{5}-1}{2})^{\frac{14}{41}}\frac{\sqrt{-8\sqrt{5}+20+(\sqrt{5}+15)\sqrt{2\sqrt{5}-26}}}{80}. \end{aligned}$$

3 p-adic analogues

3.1 p-adic analogue of Yoshida's class invariant

Assume that

$$\text{the prime ideal } \mathfrak{p} \text{ corresponding to } F \hookrightarrow \mathbb{C}_p \text{ divides } \mathfrak{f}.$$

We define

$$\Gamma_p(c) := \Gamma_p(Z_c)) \times \prod_i \exp_p(b_i \log_p a_i)$$

for the same $Z_c \subset F$, $a_i, b_i \in F$ as those in the definition of $\Gamma(c)$. Here we put

$$\Gamma_p(Z) := \exp_p\left(\frac{d}{ds}\left[p\text{-adic interpolation of } \sum_{z\in Z} z^{-s}\right]_{s=0}\right).$$

- We have a p-adic analogue of Shintani's formula.
- $\Gamma_p(c) \bmod (\mathcal{O}_{F,+}^{\times})^{\mathbb{Q}}$ does not depend on the choices of $D, \mathfrak{a}$.
- The "ratio" $[\Gamma(c) : \Gamma_p(c)] \bmod \mu_\infty$ does not depend on $\mathfrak{a}, D$, that is,

$$\Gamma(c; D, \mathfrak{a})/\Gamma(c; D', \mathfrak{a}') \equiv \Gamma_p(c; D, \mathfrak{a}))/\Gamma_p(c; D', \mathfrak{a}') \mod \mu_\infty.$$

3.2 p-adic analogue of Shimura's period symbol

We define

$$p_{K,p}(\sigma, \tau) \in B_{dR}^{\times}$$

by replacing the de Rham isomorphism with comparison isomorphisms of p-adic Hodge theory, $\mathbb{C}$ with Fontaine's p-adic period ring B_{dR}, and $2\pi i$ with the p-adic period $(2\pi i)_p$ of the Lefschetz motive. Since abelian varieties with CM have potentially good reductions, we see that

$$p_{K,p}(\sigma, \tau) \in (B_{cris}\overline{\mathbb{Q}_p})^{\mathbb{Q}}.$$

Moreover

$$[p_K(\sigma, \tau) : p_{K,p}(\sigma, \tau)] \bmod \mu_\infty$$

is well-defined when we take the same basis c_B, c_{dR} of cohomology groups for $p_K, p_{K,p}$.

3.3 Reciprocity laws

Let $W_\mathfrak{p} \subset \mathrm{Gal}(\overline{F_\mathfrak{p}}/F_\mathfrak{p})$ be the Weil group, that is, $\tau \in W_\mathfrak{p} \Leftrightarrow \tau|_{F_\mathfrak{p}^{ur}} = \mathrm{Fr}_\mathfrak{p}^{\deg_\mathfrak{p} \tau}$ where $\deg_\mathfrak{p} \tau \in \mathbb{Z}$, $\mathrm{Fr}_\mathfrak{p}$ denotes the Frobenius automorphism at $\mathfrak{p}$. We consider a natural action $W_\mathfrak{p} \curvearrowright B_{cris}\overline{\mathbb{Q}_p}$ defined by $\Phi_\tau := (\mathrm{ab.Fr.})^{\deg \mathfrak{p} \deg_\mathfrak{p} \tau} \otimes \tau$.

Conjecture 4. *Assume that $\mathfrak{p} \mid \mathfrak{f}$. Under Conjecture 1, we define*

$$\mathfrak{G}(c) := \frac{\Gamma(c)}{(2\pi i)^{\zeta(0,c)} \prod_{c' \in C_\mathfrak{f}} p_K(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}}} \frac{(2\pi i)_p^{\zeta(0,c)} \prod_{c' \in C_\mathfrak{f}} p_{K,p}(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}}}{\Gamma_p(c)} \in (B_{cris}\overline{\mathbb{Q}_p})^{\mathbb{Q}}/\mu_\infty.$$

Then we have for $\tau \in W_\mathfrak{p}$

$$\Phi_\tau\left(\mathfrak{G}(c)\right) \equiv \mathfrak{G}(c_\tau c) \bmod \mu_\infty,$$

where $c_\tau := \mathrm{Art}^{-1}(\tau|_{H_\mathfrak{f}}) \in C_\mathfrak{f}$.

Conjecture 5. *Assume that $\mathfrak{p} \nmid \mathfrak{f}$. Under Conjecture 1, we define*

$$\mathfrak{G}(c;D,\mathfrak{a}) := \frac{\Gamma(c;D,\mathfrak{a})}{(2\pi i)^{\zeta(0,c)} \prod_{c'} p_K(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}}} (2\pi i)_p^{\zeta(0,c)} \prod_{c'} p_{K,p}(c,c')^{\frac{\zeta(0,c')}{[H_\mathfrak{f}:K]}} \in (B_{cris}\overline{\mathbb{Q}_p})^{\mathbb{Q}}/\mu_\infty.$$

Then we have for $\tau \in W_\mathfrak{p}$ with $\deg_\mathfrak{p} \tau = 1$

$$\Phi_\tau(\mathfrak{G}(c;D,\mathfrak{a})) \equiv \frac{\pi_\mathfrak{p}^{\frac{\zeta(0,[\mathfrak{p}]c)}{h_F^+}} \mathfrak{G}([\mathfrak{p}]c;D,\mathfrak{p}\mathfrak{a})}{\prod_{\substack{\tilde{c} \in C_{\mathfrak{f}\mathfrak{p}} \\ \tilde{c} \mapsto [\mathfrak{p}]c \in C_\mathfrak{f}}} \Gamma_p(\tilde{c};D,\mathfrak{p}\mathfrak{a})} \mod \mu_\infty,$$

where h_F^+ is the narrow class number, $\pi_\mathfrak{p}$ is a suitable generator of $\mathfrak{p}^{h_F^+}$.

Remark 6.

- *When $H_\mathfrak{f}$ does not contain any CM-field, we see that $\zeta(0,c) = 0$. Hence we regard $p_K = p_{K,p} = 1$ in this case.*
- *"$\mathrm{mod}\mu_\infty$ ambiguity" occurs when we take rational powers of periods or consider $\exp_p$ outside of the convergence region. This may be avoidable by "S,T-modified".*

4 Main results

Proposition 7 (Norm relation?). *Let $c \in C_{\mathfrak{f}}$, $\mathfrak{q}$ a prime ideal, $\phi\colon C_{\mathfrak{f}\mathfrak{q}} \to C_{\mathfrak{f}}$ the natural projection. For simplicity assume that $\mathfrak{p} \mid \mathfrak{f}$. Then we have*

$$\prod_{\tilde{c} \in C_{\mathfrak{f}\mathfrak{q}},\ \phi(\tilde{c})=c} \mathfrak{G}(\tilde{c}) \equiv \begin{cases} \mathfrak{G}(c)\mathfrak{G}([\mathfrak{q}]c)^{-1} & (\mathfrak{q} \nmid \mathfrak{f}) \\ \mathfrak{G}(c) & (\mathfrak{q} \mid \mathfrak{f}) \end{cases} \mod \mu_\infty.$$

Theorem 8. *Conjectures 1, 2, 3 imply the reciprocity law on Stark's units up to μ_∞.*

Proof. Let H be the maximal subfield of $H_{\mathfrak{f}}$ where the real place $\mathrm{id}\colon F \hookrightarrow \mathbb{R}$ splits. Then we can show that

$$\prod_{c \mapsto \sigma} \mathfrak{G}(c) \equiv \exp(\zeta'(0,\sigma)) \bmod \mu_\infty \quad (\sigma \in \mathrm{Gal}(H/F)).$$

Since Φ_τ is τ-semilinear, we obtain $\tau(\exp(\zeta'(0,\sigma)) \equiv \exp(\zeta'(0,\tau\circ\sigma)) \bmod \mu_\infty$ for $\tau \in W_{\mathfrak{p}}$. Then we vary $\mathfrak{p}$. □

By a similar argument, we can show that

Theorem 9. *Conjectures 1, 3 imply a refinement (by K.-Yoshida) of the rank one abelian Gross-Stark conjecture.*

Sketch of proof. Let H be the maximal subfield of $H_{\mathfrak{f}}$ where $\mathfrak{p}$ splits completely. Then, roughly speaking, we can show that

$$\prod_{c \mapsto \sigma} \mathfrak{G}(c) \equiv \text{“a Gross-Stark unit”} \bmod \mu_\infty \quad (\sigma \in \mathrm{Gal}(H/F)).$$

□

Theorem 10. *Conjecture 2 holds true when $H_{\mathfrak{f}}$ is abelian over $\mathbb{Q}$ and $\mathfrak{p} \nmid 2$.*

Sketch of proof. The case $F = \mathbb{Q}$ follows from Rohrlich's formula and Coleman's formula. We reduce the problem to this case, by well-known formula on L-functions

$$L(s,\chi) = \prod_{\psi \in \widehat{G}, \psi|_H = \chi} L(s,\psi) \quad (\chi \in \widehat{G},\ G := \mathrm{Gal}(H_{\mathfrak{f}}/\mathbb{Q}) \supset H := \mathrm{Gal}(H_{\mathfrak{f}}/F)).$$

By this, we can express $\exp(\zeta'(0,c))$'s of F in terms of those of $\mathbb{Q}$. Recall Shintani's formula:

$$\exp(\zeta'(0,c)) = \prod_{\iota \in \mathrm{Hom}(F,\mathbb{R})} \Gamma(c,\iota).$$

We need just $\Gamma(c, \mathrm{id})$, not their product. By definition,

$$Z_c := \{z \in D \cap \mathfrak{a}^{-1} \mid z\mathfrak{a} \in c\},$$
$$\Gamma(c,\iota) := \Gamma(\iota(Z_c)) \times \prod_i \iota(a_i)^{\iota(b_i)}.$$

Hence $\Gamma(c,\iota)$ depends on $\iota(c)$, rather than on c. When $H_{\mathfrak{f}}/\mathbb{Q}$ is abelian, $\iota(c) \in C_{\iota(\mathfrak{f})}$ (and $\iota(\mathfrak{f})$) do not depend on $\iota \in \mathrm{Hom}(F,\mathbb{R})$. Hence, we obtain an expression like

$$\exp(\zeta'(0,c)) = \Gamma(c)^{[F:\mathbb{Q}]} \times \text{ explicit correction terms}$$

by Yoshida's technique. Since the same holds true for $\exp_p(\zeta_p'(0,c))$ we have

$$[\exp(\zeta'(0,c)) : \exp_p(\zeta_p'(0,c))] \equiv [\Gamma(c)^{[F:\mathbb{Q}]} : \Gamma_p(c)^{[F:\mathbb{Q}]}] \bmod \mu_\infty.$$

□

Similarly we can show that

Theorem 11 (which has not yet written)**.** *Conjecture 3 holds true when $H_{\mathfrak{f}}$ is abelian over $\mathbb{Q}$ and p remains prime in F.*