A note on special values of the gamma function

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The gamma function

Definition (Euler's Γ-function)

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \quad (\Re(z) > 0).$$

$$z\,\Gamma(z) = \int_0^\infty (t^z)' e^{-t} dt = \left[t^z e^{-t}\right]_0^\infty - \int_0^\infty t^z (e^{-t})' dt = \Gamma(z+1).$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 1 \quad \rightsquigarrow \quad \Gamma(n) = (n-1)!.$$

- $\Gamma(1/2) = 1.772453850\ldots.$
- $\Gamma(1/3) = 2.678938534\ldots$, $\Gamma(2/3) = 1.354117939\ldots.$
- $\Gamma(1/4) = 3.625609908\ldots$, $\Gamma(3/4) = 1.225416702\ldots.$
- $\Gamma(1/2)^2 = 3.141592653\cdots = \pi.$
- $\Gamma(1/4)\Gamma(3/4)/\pi = 1.414213562\cdots = \sqrt{2}.$
- $\Gamma(1/3)\Gamma(2/3)/\pi = 1.154700538\cdots = 2\sqrt{3}/3.$

The gamma function

Theorem (Euler's reflection formula)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \left(= \frac{2\pi i}{e^{i\pi z} - e^{-i\pi z}} \right).$$

Theorem (Multiplication formula)

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{\frac{1}{2}-2z}\sqrt{2\pi}\,\Gamma(2z).$$

$$\prod_{k=0}^{d-1} \Gamma(z+\tfrac{k}{d}) = d^{\frac{1}{2}-dz}\sqrt{2\pi}^{\,d-1}\Gamma(dz).$$

The gamma function

Theorem (Euler's reflection formula)

$$\frac{\Gamma(z)}{\sqrt{2\pi}}\frac{\Gamma(1-z)}{\sqrt{2\pi}} = \frac{1}{2\sin\pi z} = \frac{i}{e^{i\pi z}-e^{-i\pi z}} \overset{z\in\mathbb{Q}}{\doteqdot} \sqrt{\text{cyclotomic unit}}.$$

Theorem (Multiplication formula)

$$\frac{\Gamma(z)}{\sqrt{2\pi}}\frac{\Gamma(z+\frac{1}{2})}{\sqrt{2\pi}} = 2^{\frac{1}{2}-2z}\frac{\Gamma(2z)}{\sqrt{2\pi}}.$$

$$\prod_{k=0}^{d-1}\frac{\Gamma(z+\frac{k}{d})}{\sqrt{2\pi}} = d^{\frac{1}{2}-dz}\frac{\Gamma(dz)}{\sqrt{2\pi}}.$$

$\Rightarrow \Gamma_1(z) := \dfrac{\Gamma(z)}{\sqrt{2\pi}}$ seems to be more important.

In fact, $\Gamma_1(z) \overset{\text{Hurwitz-Lerch}}{=} \exp\left(\frac{d}{ds}\sum_{k=0}^{\infty}(z+k)^{-s}|_{s=0}\right)$ $(z>0)$.

A Toy Problem

Problem

"How many" monomial relations of $\Gamma_1(\frac{a}{N}) := \Gamma(\frac{a}{N})/\sqrt{2\pi}$ are there?

e.g.,

$$\prod_{(a,N)=1} \Gamma_1(\tfrac{a}{N}) = \sqrt{N(\text{“cyclotomic unit”})} = 1 \quad (N \neq p^r).$$

$$\frac{\Gamma_1(\frac{1}{18})^{30}\Gamma_1(\frac{5}{18})^{9}\Gamma_1(\frac{6}{18})^{31}\Gamma_1(\frac{7}{18})^{21}\Gamma_1(\frac{8}{18})^{12}}{\Gamma_1(\frac{2}{18})^{18}\Gamma_1(\frac{3}{18})^{32}\Gamma_1(\frac{4}{18})^{21}} = 1.$$

$\rightsquigarrow$ seems to be irregular.

A Toy Problem

Problem

"How many" relations of the form $\prod_{a=1}^{N-1} \Gamma_1(\frac{a}{N})^{k_a} \in \overline{\mathbb{Q}}^{\times}$ are there?

In other words,

- Consider the group (under the multiplication)

$$\mathcal{G}_N := \langle \Gamma_1(\tfrac{a}{N}) \mid 1 \le a \le N \rangle_{\mathbb{Q}}$$
$$= \left\{ \prod_{a=1}^{N-1} \Gamma_1(\tfrac{a}{N})^{r_a} \mid r_a \in \mathbb{Q} \right\} \subset \mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}.$$

- Then

$$N - 1 - (\#\text{ of relations}) = \dim_{\mathbb{Q}} \mathcal{G}_N = \, ? \, = \phi(N)/2.$$

A Toy Problem

Theorem

$\mathcal{G}_N := \langle \Gamma_1(\frac{a}{N}) \mid 1 \le a \le N \rangle_{\mathbb{Q}} \subset \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$. Then $\dim_{\mathbb{Q}} \mathcal{G}_N \le \phi(N)/2$.

Example ($N = 6$)

Write $\alpha \sim \beta$ if $\alpha \equiv \beta \bmod \overline{\mathbb{Q}}^\times$.
duplication: $\Gamma_1(a/6)\Gamma_1(a/6+1/2) \sim \Gamma_1(2a/6)$ $(a = 1, 2, 3)$
triplication: $\Gamma_1(a/6)\Gamma_1(a/6+1/3)\Gamma_1(a/6+2/3) \sim \Gamma_1(3a/6)$ $(a = 1, 2)$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}}_{\text{rank}=4} \begin{bmatrix} \Gamma_1(1/6) \\ \Gamma_1(2/6) \\ \Gamma_1(3/6) \\ \Gamma_1(4/6) \\ \Gamma_1(5/6) \end{bmatrix} = \begin{bmatrix} \Gamma_1(1/6)\Gamma_1(2/6)^{-1}\Gamma_1(4/6) \\ \Gamma_1(2/6)\Gamma_1(4/6)^{-1}\Gamma_1(5/6) \\ \Gamma_1(3/6) \\ \Gamma_1(1/6)\Gamma_1(5/6) \\ \Gamma_1(2/6)\Gamma_1(4/6) \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \dim \mathcal{G}_6 \le 5 - 4 = 1.$

CM-periods (Shimura's period symbol)

CM-field K, complex embeddings $\sigma, \tau \in \mathrm{Hom}(K, \mathbb{C})$
$\rightsquigarrow$ Shimura's period symbol $p_K(\sigma, \tau) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ is characterized by

$$p_K(\sigma, \Xi) := \prod_{\tau \in \Xi} p_K(\sigma, \tau) \equiv \begin{cases} \pi^{-1} \int_\gamma \omega_\sigma & (\sigma \in \Xi) \\ \int_\gamma \omega_\sigma & (\sigma \notin \Xi) \end{cases} \mod \overline{\mathbb{Q}}^\times$$

for abelian variety A of CM-type (K, Ξ), "K-eigen" differential form ω_σ:

- $A/\overline{\mathbb{Q}}$: abelian variety with $K = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $\Xi = \Xi_A$ is a half of $\mathrm{Hom}(K, \mathbb{C})$ defined by

$$\begin{array}{ccccc} K & \curvearrowright & H^1_{dR}(A, \mathbb{C}) \cong \mathbb{C}^{[K:\mathbb{C}]} & \text{via} & \bigoplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} \sigma \\ & & \cup & & \cup \\ K & \curvearrowright & H^0(A, \Omega^1_A) \cong \mathbb{C}^{[K:\mathbb{C}]/2} & \text{via} & \bigoplus_{\sigma \in \Xi} \sigma. \end{array}$$

- $K \curvearrowright \mathbb{C} \cdot \omega_\sigma \subset H^1_{dR}(A, \mathbb{C})$ via $\sigma \in \mathrm{Hom}(K, \mathbb{C})$: $k^*(\omega_\sigma) = \sigma(k)\omega_\sigma$.
- γ: arbitrary closed path with $\int_\gamma \omega_\sigma \neq 0$.

CM-periods (Shimura's period symbol)

$$p_K(\sigma,\Xi) \equiv \begin{cases} \pi^{-1}\int_\gamma \omega_\sigma & (\sigma\in\Xi) \\ \int_\gamma \omega_\sigma & (\sigma\notin\Xi) \end{cases} \mod \overline{\mathbb{Q}}^\times$$

Example

$K=\mathbb{Q}(\sqrt{-1})$, $\mathrm{Hom}(K,\mathbb{C})=\{\mathrm{id},\rho\}$, $E\colon y^2=x^3-x$,

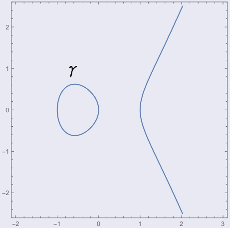
$\mathrm{End}(E)=\mathbb{Z}[\text{“}\sqrt{-1}\text{”}]$, “$\sqrt{-1}$” : $(x,y)\mapsto(-x,iy)$.

$\rightsquigarrow \omega_{\mathrm{id}}=\frac{dx}{y}$, $\omega_\rho=\frac{xdx}{y}$, $\Xi=\{\mathrm{id}\}$:

“$\sqrt{-1}$”$^*\frac{dx}{y}=\frac{-dx}{iy}=i\frac{dx}{y}$, “$\sqrt{-1}$”$^*\frac{xdx}{y}=\frac{xdx}{iy}=-i\frac{xdx}{y}$.

$$p_{\mathbb{Q}(\sqrt{-1})}(\mathrm{id},\mathrm{id})=\pi^{-1}\int_\gamma\frac{dx}{y}=2\pi^{-1}\int_{-1}^0\frac{dx}{\sqrt{x^3-x}}$$

$$\overset{x=-\sqrt{t}}{=}\pi^{-1}\int_0^1\frac{dt}{t^{\frac34}(1-t)^{\frac12}}=\pi^{-1}B(\tfrac14,\tfrac12)=\pi^{\frac{-1}{2}}\frac{\Gamma(\frac14)}{\Gamma(\frac34)}.$$



CM-periods (Shimura's period symbol)

Shimura's period symbol $p_K(\sigma,\tau)$ ($\fallingdotseq$ CM-periods) provides the transcendental pars of

- critical values of L-functions of algebraic Hecke characters.
- special values of Hilbert modular forms at CM-points.
- the exponentials of the derivative values of partial zeta functions of totally real fields at $s=0$
(Yoshida's conjecture on "absolute CM-periods").

Enjoys some "nice" properties:

- For $\iota\colon K' \cong K$, $p_K(\sigma,\tau) = p_{K'}(\sigma\circ\iota,\tau\circ\iota)$.
- For $K \subset L$, $\tilde{\sigma} \in \mathrm{Hom}(L,\mathbb{C})$, $\tau \in \mathrm{Hom}(K,\mathbb{C})$,
$p_K(\tilde{\sigma}|_K,\tau) = p_L(\tilde{\sigma},\mathrm{Inf}(\tau)) := \prod_{\tilde{\tau}\in\mathrm{Hom}(L,\mathbb{C}),\tilde{\tau}|K=\tau} p_L(\tilde{\sigma},\tilde{\tau})$.
- For the complex conjugation ρ, $p_K(\sigma,\tau)p_K(\sigma,\rho\circ\tau) = 1$.

CM-periods (Shimura's period symbol)

Theorem (Rohrlich, Bannai-Otsubo, Yoshida)

$$\Gamma_1(\tfrac{a}{N}) \sim \pi^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle},$$

where $[\sigma_b \colon \zeta_N \mapsto \zeta_N^b] \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, $\langle \alpha \rangle \in (0,1]$ denotes the fraction part of $\alpha \in \mathbb{Q}$.

sketch of proof.

Explicit computation of $\int_\gamma x^{r-1} y^{s-N} dx$ on the Nth Fermat curve $F_N \colon x^N + y^N = 1$:

$$\int_\gamma x^{r-1} y^{s-N} dx \doteqdot \int_0^1 t^{\frac{r}{N}} (1-t)^{\frac{s}{N}} dt = B(\tfrac{r}{N}, \tfrac{s}{N}) = \frac{\Gamma(\frac{r}{N})\Gamma(\frac{s}{N})}{\Gamma(\frac{r+s}{N})}.$$

& linear algebra using "nice" properties. □

CM-periods (Shimura's period symbol)

$$\Gamma_1(\tfrac{a}{N}) \sim \pi^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N}\rangle}.$$

proof of $\dim_{\mathbb{Q}} \mathcal{G}_N \leq \phi(N)/2$.

$\mathcal{G}_N := \langle \Gamma_1(\frac{a}{N}) \mid 1 \leq a \leq N-1 \rangle$

$= \langle \pi^{\frac{\delta_b}{2}} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b) \mid (b,N)=1 \rangle$ ($\delta_b := 1, -1, 0$ if $b \equiv 1, -1$,otherwise)

$\overset{p_K(\sigma,\tau)p_K(\sigma,\rho\circ\tau)=1}{=} \langle \pi^{\frac{\delta_b}{2}} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b) \mid (b,N)=1,\ 1 \leq b \leq N/2 \rangle.$ □

CM-periods (Shimura's period symbol)

$$\Gamma_1(\tfrac{a}{N}) \sim \pi^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N}\rangle}.$$

"Multiplication formula"

$$\prod_{k=0}^{d-1} \Gamma_1(\tfrac{a}{N}+\tfrac{k}{d}) \sim \Gamma_1(\tfrac{da}{N}).$$

also follows: may assume that $d \mid N$ (due to $p_K(\tilde{\sigma}|_K,\tau) = p_L(\tilde{\sigma},\mathrm{Inf}(\tau))$).

$$\prod_{k=0}^{d-1} \Gamma_1(\tfrac{a}{N}+\tfrac{k}{d}) \sim \pi^{\sum_{k=0}^{d-1}\frac{1}{2}-\langle\frac{a}{N}+\frac{k}{d}\rangle} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\sum_{(b,N)=1}\sum_{k=0}^{d-1}\frac{1}{2}-\langle\frac{ab}{N}+\frac{kb}{d}\rangle}$$
$$= \pi^{\frac{1}{2}-\langle\frac{ad}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{abd}{N}\rangle} \sim \Gamma_1(\tfrac{ad}{N})$$

by multiplication formula $\sum_{k=0}^{d-1} B_1(x+\frac{k}{d}) = B_1(dx)$ for $B_1(x) = x - \frac{1}{2}$.

CM-periods (Shimura's period symbol)

$$\Gamma_1(\tfrac{a}{N}) \sim \pi^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle}.$$

"Reflection formula"

$$\Gamma_1(\tfrac{a}{N})\Gamma_1(1-\tfrac{a}{N}) \in \overline{\mathbb{Q}}^{\times}$$

follows from $p_K(\sigma,\tau)p_K(\sigma,\rho\circ\tau)=1$.

Moreover the "reciprocity law"

$$\tilde{\sigma}_b\left(\Gamma_1(\langle \tfrac{a}{N} \rangle)\Gamma_1(1-\langle \tfrac{a}{N} \rangle)\right) \equiv \Gamma_1(\langle \tfrac{ab}{N} \rangle)\Gamma_1(1-\langle \tfrac{ab}{N} \rangle) \mod \mu_\infty$$

for any lift $\tilde{\sigma}_b \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\sigma_b \colon \zeta_N \mapsto \zeta_N^b$, follows from Coleman's formula on the absolute Frobenius action on $F_N \colon x^N + y^N = 1$.

Coleman's formula in terms of p-adic periods

For simplicity, assume that $p \nmid 2N$. Let $B_{cris} \subset B_{dR}$ be Fontaine's period rings, $\Phi_{cris} \curvearrowright B_{cris}$ the absolute Frobenius action. Similarly to

$$\int : H_1^{sing}(F_N(\mathbb{C})) \times H_{dR}^1(F_N, \mathbb{Q}) \to \mathbb{C} \rightsquigarrow p_{\mathbb{Q}(\zeta_N)}(\sigma, \tau) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times,$$

we can define the p-adic period symbol

$$\int_p : H_1^{sing}(F_N(\mathbb{C})) \times H_{dR}^1(F_N, \mathbb{Q}) \to B_{cris}$$
$$\rightsquigarrow p_{\mathbb{Q}(\zeta_N),p}(\sigma, \tau) \in (B_{cris} - \{0\})^{\mathbb{Q}} / (\overline{\mathbb{Q}} \cap \mathbb{Q}_p^{ur})^{\times \mathbb{Q}}.$$

(Note that F_N has good reduction at $p \nmid N$.) By Complex Multiplication, the ratio $[\int_\gamma \omega_\sigma : \int_{p,\gamma} \omega_\sigma]$ depends only on Ξ, σ, i.e., $\frac{\int_\gamma \omega_\sigma}{\int_{\gamma'} \omega'_\sigma} = \frac{\int_{p,\gamma} \omega_\sigma}{\int_{p,\gamma'} \omega'_\sigma} \in \overline{\mathbb{Q}}$.

$$\rightsquigarrow [p_{\mathbb{Q}(\zeta_N)}(\sigma, \tau) : p_{\mathbb{Q}(\zeta_N),p}(\sigma, \tau)] \in (\mathbb{C}^\times \times B_{cris}^{\mathbb{Q}}) / (\mu_\infty \times \mu_\infty)(\overline{\mathbb{Q}} \cap \mathbb{Q}_p^{ur})^{\times \mathbb{Q}}.$$

Coleman's formula in terms of p-adic periods

Theorem (Coleman, $p \nmid 2N$ for simplicity)

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N}) \cdot (2\pi i)_p^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle}}{(2\pi i)^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle}} \in B_{cris}^{\mathbb{Q}}/\mu_\infty.$$

$$\rightsquigarrow \Gamma_p(\langle \tfrac{pa}{N} \rangle) \equiv p^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \frac{\mathfrak{G}(\langle \frac{pa}{N} \rangle)}{\Phi_{cris}(\mathfrak{G}(\langle \frac{a}{N} \rangle))} \mod \mu_\infty$$

with Morita's $\Gamma_p \colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$, $\Gamma_p(n) := (-1)^n \prod_{1 \le k \le n-1 \ p \nmid k} k.$

sketch of proof.

Explicit computation of Φ_{cris} on $H^1_{cris}(F_N)$ by the following lemma & linear algebra using "nice" properties. □

Coleman's formula in terms of p-adic periods

Lemma

Let $C/\mathbb{Q}_p$ be a projective smooth connected algebraic curve having good (or arboreal) reduction,

$$\omega = \sum_{n=1}^{\infty} a_n t^n \frac{dt}{t}, \ \eta = \sum_{n=1}^{\infty} b_n t^n \frac{dt}{t} \text{ s.t. } \Phi_{cris}(\omega) = \alpha\eta.$$

Then

$$\alpha = \lim_{k\to\infty} \frac{p\sigma_p(a_{n_k})}{b_{pn_k}} \quad \text{when} \quad \lim_{k\to\infty} \frac{n_k}{a_{n_k}} = 0.$$

Note that

$$x^{r-1}y^{s-N}dx = x^r(1-x^N)^{\frac{s-N}{N}}\frac{dx}{x} = \sum_{n=0}^{\infty}(-1)^n\binom{\frac{s-N}{N}}{n}x^{r+nN}\frac{dx}{x}.$$

Coleman's formula in terms of p-adic periods

Coleman's formula

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N}) \cdot (2\pi i)_p^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N}\rangle}}{(2\pi i)^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N}\rangle}},$$

$$\Gamma_p(\langle \tfrac{pa}{N}\rangle) \equiv p^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \frac{\mathfrak{G}(\langle \frac{pa}{N}\rangle)}{\Phi_{cris}(\mathfrak{G}(\langle \frac{a}{N}\rangle))} \mod \mu_\infty$$

implies the "reciprocity law on cyclotomic units"

$$\tilde{\sigma}_p \left(\Gamma_1(\langle \tfrac{a}{N}\rangle)\Gamma_1(1-\langle \tfrac{a}{N}\rangle)\right) \equiv \Gamma_1(\langle \tfrac{ab}{N}\rangle)\Gamma_1(1-\langle \tfrac{ab}{N}\rangle) \mod \mu_\infty$$

since

$$\mathfrak{G}(\langle \tfrac{a}{N}\rangle)\mathfrak{G}(1-\langle \tfrac{a}{N}\rangle) \equiv \Gamma_1(\langle \tfrac{a}{N}\rangle)\Gamma_1(1-\langle \tfrac{a}{N}\rangle) \mod \mu_\infty,$$

$$\Gamma_p(z)\Gamma_p(1-z) \in \mu_\infty,$$

$\Phi_{cris} = \tilde{\sigma}_p$ on $\mathbb{Q}_p^{ur} \cap \overline{\mathbb{Q}}$.

Coleman's formula in terms of p-adic periods

Coleman's formula

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N}) \cdot (2\pi i)_p^{\frac{1}{2}-\langle\frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}}{(2\pi i)^{\frac{1}{2}-\langle\frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}},$$

$$\Gamma_p(\langle\tfrac{pa}{N}\rangle) \equiv p^{\frac{1}{2}-\langle\frac{a}{N}\rangle} \frac{\mathfrak{G}(\langle\frac{pa}{N}\rangle)}{\Phi_{cris}(\mathfrak{G}(\langle\frac{a}{N}\rangle))} \mod \mu_\infty$$

also implies the "Anderson-Gross-Koblitz formula"

$$\prod_{k=0}^{d-1} \Gamma_p(\langle\tfrac{p^k a}{N}\rangle) \equiv \text{“Gauss sum”} := \prod_{(b,N)=1} \pi_{\mathfrak{P}}^{\frac{1}{h}(\langle\frac{ab}{N}\rangle-\frac{1}{2})\cdot\sigma_b^{-1}} \mod \mu_\infty$$

for $d :=$ order of $\overline{p} \in (\mathbb{Z}/N\mathbb{Z})^\times$, the maximal subfield $H \subset \mathbb{Q}(\zeta_N)$ where p splits completely, the prime ideal $\mathfrak{P}$ corresponding to $\mathrm{id}\colon H \hookrightarrow \mathbb{Q}_p$, $\mathfrak{P}^h = (\pi_{\mathfrak{P}})$ with $h := h_H$.

Coleman's formula in terms of p-adic periods

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N}) \cdot (2\pi i)_p^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}}{(2\pi i)^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}},$$

$$\Gamma_p(\langle \tfrac{pa}{N}\rangle) \equiv p^{\frac{1}{2}-\langle\frac{a}{N}\rangle} \frac{\mathfrak{G}(\langle\frac{pa}{N}\rangle)}{\Phi_{cris}(\mathfrak{G}(\langle\frac{a}{N}\rangle))} \mod \mu_\infty$$

sketch of proof of $\prod_{k=0}^{d-1} \Gamma_p(\langle \frac{p^k a}{N}\rangle) \equiv \prod_{(b,N)=1} \pi_{\mathfrak{P}}^{\frac{1}{h}(\langle\frac{ab}{N}\rangle - \frac{1}{2})\cdot\sigma_b^{-1}}$.

Note that $d = \deg \mathfrak{P}_{\mathbb{Q}(\zeta_N)}$. Hence, for an algebraic Hecke character χ of $\mathbb{Q}(\zeta_N)$ whose infinity type is the "reflex of the CM-type",

$$\prod_{k=0}^{d-1} \Gamma_p(\langle \tfrac{p^k a}{N}\rangle) \doteqdot \text{“}\Phi_{cris}^d \curvearrowright H_{cris}(M(\chi))\text{”} \doteqdot \chi(\mathfrak{P}_{\mathbb{Q}(\zeta_N)}).$$

□

A characterization of Γ_p (on-going)

Morita's $\Gamma_p \colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is the unique continuous function satisfying

$$\Gamma_p(0) = 1,\ \Gamma_p(z+1) = z^*\Gamma_p(z),\ z^* := \begin{cases} -z & (z \in \mathbb{Z}_p^\times), \\ -1 & (z \in p\mathbb{Z}_p). \end{cases}$$

Proposition (for the proof, see Koblitz, Robert, Schikhof, etc.)

$\Gamma_p(z) \bmod \mu_\infty$ satisfies "multiplication formula":

$$\prod_{k=0}^{d-1} \Gamma_p(z + \tfrac{k}{d}) \equiv d^{1-dz+(dz)_1}\Gamma_p(dz) \mod \mu_\infty \quad (p \nmid d \in \mathbb{N}),$$

where $z_1 \in \mathbb{Z}_p$ is defined by $z = z_0 + pz_1$ with $z_0 \in \{1, 2, \dots, p\}$.

Note that $z + \frac{k}{p} \notin \mathbb{Z}_p$ when $z \in \mathbb{Z}_p$.

A characterization of Γ_p (on-going)

Definition

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N}) \cdot (2\pi i)_p^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle}}{(2\pi i)^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle}},$$

$$\Gamma_{\mathrm{period}}(\langle \tfrac{pa}{N} \rangle) := p^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \frac{\mathfrak{G}(\langle \frac{pa}{N} \rangle)}{\Phi_{cris}(\mathfrak{G}(\langle \frac{a}{N} \rangle))} \quad \text{on} \quad \mathbb{Z}_{(p)} \cap (0,1).$$

We can derive "multiplication formula":

$$\prod_{k=0}^{d-1} \Gamma_{\mathrm{period}}(z + \tfrac{k}{d}) \equiv d^{1-dz+(dz)_1} \Gamma_{\mathrm{period}}(dz) \mod \mu_\infty \quad (p \nmid d \in \mathbb{N}),$$

from properties of classical Γ-function or period symbols, independently of Coleman's formula $\Gamma_p(z) \equiv \Gamma_{\mathrm{period}}(z)$.

A characterization of Γ_p (on-going)

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N})\cdot(2\pi i)_p^{\frac{1}{2}-\langle\frac{a}{N}\rangle}\prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}}{(2\pi i)^{\frac{1}{2}-\langle\frac{a}{N}\rangle}\prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}},$$

$$\Gamma_{\mathrm{period}}(\langle\tfrac{pa}{N}\rangle) := p^{\frac{1}{2}-\langle\frac{a}{N}\rangle}\frac{\mathfrak{G}(\langle\frac{pa}{N}\rangle)}{\Phi_{cris}(\mathfrak{G}(\langle\frac{a}{N}\rangle))} \quad \text{on} \quad \mathbb{Z}_{(p)}\cap(0,1).$$

Proof of "multiplication formula" for period symbols.

$z := \langle px\rangle$ $(x\in\mathbb{Z}_{(p)}\cap(0,1)) \Rightarrow z = px - n$ $(0\le n\le p-1)$

$\Rightarrow \mathbb{Z}_{(p)} \ni x = \frac{z+n}{p} \Rightarrow n = p - z_0$ $(z = z_0 + pz_1,\ 1\le z_0\le p) \Rightarrow x = z_1 + 1$

i.e., $\Gamma_{\mathrm{period}}(z) = p^{\frac{1}{2}-(z_1+1)}\frac{\mathfrak{G}(z)}{\Phi_{cris}(\mathfrak{G}(z_1+1))}$ $(z\in\mathbb{Z}_{(p)}\cap(0,1))$

$z\in(0,\frac{1}{d}) \Rightarrow \frac{\prod_{k=0}^{d-1}\Gamma_{\mathrm{period}}(z+\frac{k}{d})}{\Gamma_{\mathrm{period}}(dz)} = \frac{\prod_{k=0}^{d-1}\Gamma_1(z+\frac{k}{d})}{\Gamma_1(dz)}\cdot\frac{\Gamma_1((dz)_1+1)}{\prod_{k=0}^{d-1}\Gamma_1((z+\frac{k}{d})_1+1)}.$

"p-power" $\cdot$ "period symbols" $\equiv d^{\frac{1}{2}-dz}\cdot d^{(dz)_1+1-\frac{1}{2}} \equiv d^{1-dz+(dz)_1}.$

Note $\{(z+\frac{k}{d})_1+1 \mid k=0,\cdots,d-1\} = \{\frac{(dz)_1+1}{d}+\frac{k}{d} \mid k=0,\cdots,d-1\}$
since both sets are contained in $d^{-1}\mathbb{Z}\cap[\frac{z}{p},\frac{z}{p}+1-\frac{1}{pd}]$. □

A characterization of Γ_p (on-going)

Definition

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N}) \cdot (2\pi i)_p^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle}}{(2\pi i)^{\frac{1}{2}-\langle \frac{a}{N} \rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N} \rangle}},$$

$$\Gamma_{\mathrm{period}}(z) := p^{\frac{1}{2}-(z_1+1)} \frac{\mathfrak{G}(z)}{\Phi_{cris}(\mathfrak{G}(z_1+1))} \quad \text{on} \quad z \in \mathbb{Z}_{(p)} \cap (0,1).$$

Remark

p-adic continuity of $\Gamma_{\mathrm{period}}(\langle \frac{pa}{N} \rangle)$ follows from the facts that

- We can take γ so that $\int_\gamma x^{r-1} y^{s-N} dx = B(\frac{r}{N}, \frac{s}{N})$ (Bannai-Otsubo).
- Coefficients of $x^{r-1}y^{s-N}dx = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{s-N}{N}}{n} x^{r+nN} \frac{dx}{x}$ are "continuous" on $\frac{r}{N}, \frac{s}{N} \in \mathbb{Z}_p$.

Then we can define $\Gamma_{\mathrm{period}}(z)$ on $\mathbb{Z}_p$.

A characterization of Γ_p (on-going)

Problem

Is $\Gamma_p(z) \bmod \mu_\infty$ characterized by "multiplication formula"

$$\prod_{k=0}^{d-1} \Gamma_p(z + \tfrac{k}{d}) \equiv d^{1-dz+(dz)_1} \Gamma_p(dz) \mod \mu_\infty \quad (p \nmid d \in \mathbb{N})?$$

$\rightsquigarrow$? $\rightsquigarrow$ an alternative proof for Coleman's formula without explicit calculation.

A characterization of Γ_p (on-going)

Problem

$$\prod_{k=0}^{d-1} f(z+\tfrac{k}{d}) \equiv f(dz)\ (p \nmid d) \quad \Rightarrow ? \Rightarrow \quad f(z) \equiv 1$$

$\prod_{k=0}^{d-1} f(z+\frac{k}{d}) \equiv f(dz)$

$\Rightarrow \prod_{k=1}^{d} f(z+\frac{k}{d}) \equiv f(dz+1)$ (replace z with $z+\frac{1}{d}$)

$\Rightarrow \frac{f(z+1)}{f(z)} \equiv \frac{f(dz+1)}{f(dz)}$

$\Rightarrow g(z) := \frac{f(z+1)}{f(z)} \equiv g(dz)\ (p \nmid d \in \mathbb{N})$

$\Rightarrow g(z) \equiv c_n$ ($\exists c_n$ depends only on $n := \operatorname{ord}_p z$)

A characterization of Γ_p (on-going)

Problem

$$\prod_{k=0}^{d-1} f(z+\tfrac{k}{d}) \equiv f(dz)\ (p \nmid d)$$

$$\Rightarrow g(z) := \frac{f(z+1)}{f(z)} \equiv c_n\ (n := \operatorname{ord}_p z)$$

$$\Rightarrow ? \Rightarrow f(z) \equiv 1$$

Write $z-1 = x_0 + x_1 p + \cdots + x_n p^n + \cdots \in \mathbb{Z}_p\ (0 \le x_n \le p-1)$

$\Rightarrow$ $f(z) \equiv \lim_{n\to\infty} f(1)g(1)g(2)\cdots g(x_0 + x_1 p + \cdots + x_n p^n)$

$\equiv \lim_{n\to\infty} f(1)\alpha_0^{x_0}\alpha_1^{x_1}\cdots\alpha_n^{x_n}\ (\alpha_k := c_0^{p^{k-1}(p-1)} c_1^{p^{k-2}(p-1)} \cdots c_{k-1}^{p-1} c_k)$

A characterization of Γ_p (on-going)

Problem

$$\prod_{k=0}^{d-1} f(z+\tfrac{k}{d}) \equiv f(dz)\ (p \nmid d)$$

$$\Rightarrow f(1+\sum_{k=0}^{\infty} x_k p^k) \equiv f(1)\prod_{k=0}^{\infty} \alpha_k^{x_k}$$

$$(c_k := \frac{f(p^k+1)}{f(p^k)},\ \alpha_k := c_0^{p^{k-1}(p-1)} \cdots c_{k-1}^{p-1} c_k \to 1)$$

$$\Rightarrow ? \Rightarrow f(z) \equiv 1$$

Consider the case $d=2, z=\frac{1}{2}$: $f(\frac{1}{2})f(1) \equiv f(1)$

$$\Rightarrow f(\tfrac{1}{2}) = f(1+\tfrac{-1}{2}) = f(1+\textstyle\sum_{k=0}^{\infty} \frac{p-1}{2} p^k) \equiv f(1)\prod_{k=0}^{\infty} \alpha_k^{\frac{p-1}{2}} \equiv 1$$

$$\Rightarrow f(1) \equiv \textstyle\prod_{k=0}^{\infty} \alpha_k^{-\frac{p-1}{2}}$$

$$\Rightarrow f(1+\textstyle\sum_{k=0}^{\infty} x_k p^k) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}}$$

A characterization of Γ_p (on-going)

Proposition

A continuous function $f(z)$ on $\mathbb{Z}_p$ satisfies $\prod_{k=0}^{d-1} f(z+\frac{k}{d}) \equiv f(dz)$ $(p \nmid d)$

$\Leftrightarrow$ $\exists \alpha_k$ satisfying $f(1+\sum_{k=0}^{\infty} x_k p^k) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}}$ $(0 \le x_k \le p-1)$.

In particular, there exist infinitely many parameters.

sketch of proof.

$z+z'=1 \Rightarrow x_k + x'_k = p-1 \Rightarrow f(z)f(z') \equiv \prod_{k=0}^{\infty} \alpha_k^0 \equiv 1 \Rightarrow$ the case $z=0$: $\prod_{k=1}^{d-1} f(\frac{k}{d}) \equiv 1$.

Then "mathematical induction" by $\prod_{k=0}^{d-1} f(z+1+\frac{k}{d}) \equiv \prod_{k=0}^{d-1} f(z+\frac{k}{d}) g(z+\frac{k}{d})$,

$f(dz+d) \equiv f(dz)g(dz)\cdots g(dz+d-1)$ and $g(dz+k) \equiv g(z+\frac{k}{d})$ $(k=0,\ldots,d-1)$. □

A characterization of Γ_p (on-going)

Proposition

Assume a continuous function $f(z)$ on $\mathbb{Z}_p$ satisfies

$\prod_{k=0}^{d-1} f(z+\frac{k}{d}) \equiv f(dz)$ $(p \nmid d)$ and put $c_n := \frac{f(p^n+1)}{f(p^n)}$. Then

1. $c_0 \equiv c_1 \equiv \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}}$.
2. $c_1 \equiv c_2 \equiv \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}}(c_1/c_0)^{z_1+\frac{1}{2}}$ $(z = z_0 + pz_1,\ 1 \leq z_0 \leq p)$.

Proof.

1. $\alpha_k = c_0^{p^{k-1}(p-1)} \cdots c_{k-1}^{p-1} c_k = c_0^{p^k}$

$\Rightarrow f(1+\sum_{k=0}^{\infty} x_k p^k) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}} = c_0^{\sum_{k=0}^{\infty} x_k p^k - \frac{p-1}{2} p^k} = c_0^{z-\frac{1}{2}}$.

2. $\alpha_0 = c_0$, $\alpha_k = c_0^{p^k}(c_1/c_0)^{p^{k-1}}$ $(k \geq 1)$

$\Rightarrow f(1+\sum_{k=0}^{\infty} x_k p^k) \equiv c_0^{\sum_{k=0}^{\infty} x_k p^k - \frac{p-1}{2} p^k}(c_1/c_0)^{\sum_{k=1}^{\infty} x_k p^{k-1} - \frac{p-1}{2} p^{k-1}}$. □

A characterization of Γ_p (on-going)

Compare $\frac{c_n}{c_{n+1}}$ of $\Gamma_{\mathrm{period}}(z)$, $\Gamma_p(z)$.

Idea:

- $\frac{\Gamma_{\mathrm{period}}(pz)\Gamma_{\mathrm{period}}(z+1)}{\Gamma_{\mathrm{period}}(pz+1)\Gamma_{\mathrm{period}}(z)} = ? \equiv \frac{\Gamma_p(pz)\Gamma_p(z+1)}{\Gamma_p(pz+1)\Gamma_p(z)} = \begin{cases} 1 & (p \mid z) \\ z & (p \nmid z) \end{cases}$

$\Uparrow$ $\frac{\Gamma_{\mathrm{period}}(z)\Gamma_{\mathrm{period}}(z+\frac{1}{p})\cdots\Gamma_{\mathrm{period}}(z+\frac{p-1}{p})}{\Gamma_{\mathrm{period}}(pz)} = ?$

$\Uparrow$ $\Gamma_{\mathrm{period}}(z) := p^{\frac{1}{2}-z_1}\frac{\mathfrak{G}(z)}{\Phi_{cris}(\mathfrak{G}(z_1+1))}$ $(z \in \mathbb{Z}_{(p)} \cap (0,1),\ z = z_0 + pz_1)$

$$\mathfrak{G}(\tfrac{a}{N}) := \frac{\Gamma_1(\frac{a}{N}) \cdot (2\pi i)_p^{\frac{1}{2}-\langle\frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}}{(2\pi i)^{\frac{1}{2}-\langle\frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}}.$$

Seems to be natural to put $\Gamma_{\mathrm{period}}(z) := p^{???}\frac{\mathfrak{G}(z)}{\Phi_{cris}(\mathfrak{G}(???))}$ even for $z \in p^{-1}\mathbb{Z}_{(p)}^{\times} \cap (0,1)$.

- When $p \mid N$, $J(F_N)$ has potentially good reduction

$$\rightsquigarrow [p_{\mathbb{Q}(\zeta_N)}(\sigma,\tau) : p_{\mathbb{Q}(\zeta_N),p}(\sigma,\tau)] \in (\mathbb{C}^\times \times (B_{cris}\overline{\mathbb{Q}_p})^{\mathbb{Q}})/(\mu_\infty \times \mu_\infty)\overline{\mathbb{Q}}^\times.$$

for the composite ring $B_{cris}\overline{\mathbb{Q}_p}$ of B_{cris} and $\overline{\mathbb{Q}_p}$ in B_{dR}.

- The action of the Weil group:

$$W_{\mathbb{Q}_p} := \{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \mid \tau|_{\mathbb{Q}_p^{ur}} = \mathrm{Frob}_p^{\deg\tau} \text{ with } \deg\tau \in \mathbb{Z}\},$$

$$\tau \in W_{\mathbb{Q}_p} \rightsquigarrow \begin{cases} \Phi_\tau := \Phi_{cris}^{\deg\tau} \otimes \tau \curvearrowright B_{cris}\overline{\mathbb{Q}_p} = B_{cris} \otimes_{\mathbb{Q}_p^{ur}} \overline{\mathbb{Q}_p}, \\ \tau \curvearrowright \mathbb{Q} \cap (0,1) \text{ by } \tau(\zeta_N^a) = \zeta_N^b \Rightarrow \tau(\frac{a}{N}) := \frac{b}{N}. \end{cases}$$

- Define p-adic gamma function on $z \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,\infty)$ by

$$\Gamma_p(z) := \exp_p\left(\frac{d}{ds}\ p\text{-adic interpolation of } \sum_{k=0}^{\infty}(z+k)^{-s}|_{s=0}\right).$$

Coleman's formula in the case $p \mid N$

- $[p_{\mathbb{Q}(\zeta_N)}(\sigma,\tau) : p_{\mathbb{Q}(\zeta_N),p}(\sigma,\tau)] \in (\mathbb{C}^\times \times (B_{cris}\overline{\mathbb{Q}_p})^{\mathbb{Q}})/(\mu_\infty \times \mu_\infty)\overline{\mathbb{Q}}^\times$.
- $\tau \in W_{\mathbb{Q}_p} \rightsquigarrow \Phi_\tau \curvearrowright (B_{cris}\overline{\mathbb{Q}_p})^{\mathbb{Q}}/\mu_\infty,\ \tau \curvearrowright \mathbb{Q} \cap (0,1)$.
- $\Gamma_p(z) := \exp_p\left(\frac{d}{ds}\right.$ p-adic interpolation of $\left.\sum_{k=0}^{\infty}(z+k)^{-s}|_{s=0}\right)$.

Theorem (Coleman, $p \neq 2$, $\frac{a}{N} \in (\mathbb{Q} - \mathbb{Z}_{(p)}) \cap (0,1)$)

$$\mathfrak{G}\left(\tfrac{a}{N}\right) := \frac{\Gamma_1(\frac{a}{N})(2\pi i)_p^{\frac{1}{2}-\langle\frac{a}{N}\rangle}\prod p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}}{(2\pi i)^{\frac{1}{2}-\langle\frac{a}{N}\rangle}\prod p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{N}\rangle}}$$

$$\Rightarrow \Phi_\tau\left(\frac{\mathfrak{G}(\frac{a}{N})}{\Gamma_p(\frac{a}{N})p^{e(\frac{1}{2}-\frac{a}{N})}}\right) \equiv \frac{\mathfrak{G}(\tau(\frac{a}{N}))}{\Gamma_p(\tau(\frac{a}{N}))p^{e(\frac{1}{2}-\tau(\frac{a}{N}))}}$$

$(e := \mathrm{ord}_p N)$. As a result,

$$\Gamma_{\mathrm{period},\tau}(z) := p^{e(z-\tau^{-1}(z))}\frac{\mathfrak{G}(z)}{\Phi_\tau(\tau^{-1}(z))} \quad (z \in (\mathbb{Q}-\mathbb{Z}_{(p)}) \cap (0,1))$$

is "continuous" at $(z, \tau^{-1}(z))$.

We need only the continuity!

A characterization of Γ_p (on-going)

Fix $\tau \in W_{\mathbb{Q}_p}$ with $\deg \tau = 1$.

$$z \in \mathbb{Z}_{(p)} \cap (0, \tfrac{1}{p}) \Rightarrow \frac{\Gamma_{\text{period}}(z)\Gamma_{\text{period},\tau}(z+\frac{1}{p})\cdots\Gamma_{\text{period},\tau}(z+\frac{p-1}{p})}{\Gamma_{\text{period}}(pz)}$$

$$\equiv \frac{p^{\frac{1}{2}-(z_1+1)}\frac{\mathfrak{G}(z)}{\Phi_{cris}(\mathfrak{G}(z_1+1))}p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-s_1-\cdots-s_{p-1}}\frac{\mathfrak{G}(z+\frac{1}{p})\cdots\mathfrak{G}(z+\frac{p-1}{p})}{\Phi_\tau(\mathfrak{G}(s_1)\cdots\mathfrak{G}(s_{p-1}))}}{p^{\frac{1}{2}-z}\frac{\mathfrak{G}(pz)}{\Phi_{cris}(\mathfrak{G}(z))}}$$

$$(\{s_1, \ldots, s_{p-1}\} := \{\tau^{-1}(z+\tfrac{1}{p}), \ldots, \tau^{-1}(z+\tfrac{p-1}{p})\})$$

$$\equiv p^{(p-1)z}\frac{\frac{\prod_{k=0}^{p-1}\Gamma_1(z+\frac{k}{p})}{\Gamma_1(pz)}}{\Phi_\tau\left(\frac{\prod_{k=0}^{p-1}\Gamma_1(\frac{z}{p}+\frac{k}{p}))}{\Gamma_1(z)}\right)} \equiv p^{(p-1)z}\frac{p^{\frac{1}{2}-pz}}{\Phi_\tau(p^{\frac{1}{2}-z})} \equiv 1.$$

$$z \in \mathbb{Z}_{(p)} \cap (-\tfrac{1}{p}, 0) \Rightarrow \frac{\Gamma_{\text{period},\tau}(z+\frac{1}{p})\cdots\Gamma_{\text{period},\tau}(z+\frac{p-1}{p})\Gamma_{\text{period}}(z+1)}{\Gamma_{\text{period}}(pz+1)}$$

$$\equiv \frac{p^{(z+\frac{1}{p})+\cdots+(z+\frac{p-1}{p})-s_1-\cdots-s_{p-1}}\frac{\mathfrak{G}(z+\frac{1}{p})\ldots\mathfrak{G}(z+\frac{p-1}{p})}{\Phi_\tau(\mathfrak{G}(s_1)\ldots\mathfrak{G}(s_{p-1}))}p^{\frac{1}{2}-((z+1)_1+1)}\frac{\mathfrak{G}(z+1)}{\Phi_{cris}(\mathfrak{G}((z+1)_1+1))}}{p^{\frac{1}{2}-(z+1)}\frac{\mathfrak{G}(pz+1)}{\Phi_{cris}(\mathfrak{G}(z+1))}}$$

$$\equiv 1.$$

A characterization of Γ_p (on-going)

Remark

$$\{s_1,\ldots,s_{p-1}\} := \{\tau^{-1}(z+\tfrac{1}{p}),\ldots,\tau^{-1}(z+\tfrac{p-1}{p})\}$$

$$= \begin{cases} \{\frac{z+k}{p} \mid k=0,\ldots,p-1\} - \{\frac{z+p-z_0}{p}\} & (z \in (0,\frac{1}{p})), \\ \{\frac{z+k+1}{p} \mid k=0,\ldots,p-1\} - \{\frac{z+1+p-(z+1)_0}{p}\} & (z \in (-\frac{1}{p},0)). \end{cases}$$

$p \mid z$, i.e., $z_0 = p$, $(z+1)_0 = 1 \Rightarrow$ both sets $= \{\frac{z+k}{p} \mid k = 1,\ldots,p-1\}$

$$z \in \mathbb{Z}_{(p)} \cap (0,\tfrac{1}{p}) \Rightarrow \frac{\Gamma_{\mathrm{period}}(pz)}{\Gamma_{\mathrm{period}}(z)} \equiv p^{\frac{(p-1)^2}{p}z} \frac{\mathfrak{G}(z+\frac{1}{p})\ldots\mathfrak{G}(z+\frac{p-1}{p})}{\Phi_\tau(\mathfrak{G}(\frac{z+1}{p})\ldots\mathfrak{G}(\frac{z+p-1}{p}))}$$

$$z \in \mathbb{Z}_{(p)} \cap (-\tfrac{1}{p},0) \Rightarrow \frac{\Gamma_{\mathrm{period}}(pz+1)}{\Gamma_{\mathrm{period}}(z+1)} \equiv p^{\frac{(p-1)^2}{p}z} \frac{\mathfrak{G}(z+\frac{1}{p})\ldots\mathfrak{G}(z+\frac{p-1}{p})}{\Phi_\tau(\mathfrak{G}(\frac{z+1}{p})\ldots\mathfrak{G}(\frac{z+p-1}{p}))}$$

continuity $\Rightarrow \frac{\Gamma_{\mathrm{period}}(pz)}{\Gamma_{\mathrm{period}}(z)} \equiv \frac{\Gamma_{\mathrm{period}}(pz+1)}{\Gamma_{\mathrm{period}}(z+1)} \Rightarrow \frac{\Gamma_{\mathrm{period}}(z+1)}{\Gamma_{\mathrm{period}}(z)} \equiv \frac{\Gamma_{\mathrm{period}}(pz+1)}{\Gamma_{\mathrm{period}}(pz)}$.

A characterization of Γ_p (on-going)

Remark

$$\{s_1,\ldots,s_{p-1}\} := \{\tau^{-1}(z+\tfrac{1}{p}),\ldots,\tau^{-1}(z+\tfrac{p-1}{p})\}$$

$$= \begin{cases} \{\frac{z+k}{p} \mid k=0,\ldots,p-1\} - \{\frac{z+p-z_0}{p}\} & (z\in(0,\frac{1}{p})), \\ \{\frac{z+k+1}{p} \mid k=0,\ldots,p-1\} - \{\frac{z+1+p-(z+1)_0}{p}\} & (z\in(-\frac{1}{p},0)). \end{cases}$$

$p \nmid z$, e.g., $z_0 = 1 \Rightarrow \{\frac{z}{p}, \frac{z+1}{p},\ldots,\frac{z+p-2}{p}\} \neq \{\frac{z+1}{p},\ldots,\frac{z+p-2}{p},\frac{z+p}{p}\}$

$$z\in\mathbb{Z}_{(p)}\cap(0,\tfrac{1}{p}) \Rightarrow \frac{\Gamma_{\mathrm{period}}(pz)}{\Gamma_{\mathrm{period}}(z)} \equiv p^{\frac{(p-1)^2}{p}z+\frac{p-1}{p}} \frac{\mathfrak{G}(z+\frac{1}{p})\ldots\mathfrak{G}(z+\frac{p-1}{p})}{\Phi_\tau(\mathfrak{G}(\frac{z}{p})\mathfrak{G}(\frac{z+1}{p})\ldots\mathfrak{G}(\frac{z+p-2}{p}))}$$

$$z\in\mathbb{Z}_{(p)}\cap(-\tfrac{1}{p},0) \Rightarrow \frac{\Gamma_{\mathrm{period}}(pz+1)}{\Gamma_{\mathrm{period}}(z+1)} \equiv p^{\frac{(p-1)^2}{p}z-\frac{1}{p}} \frac{\mathfrak{G}(z+\frac{1}{p})\ldots\mathfrak{G}(z+\frac{p-1}{p})}{\Phi_\tau(\mathfrak{G}(\frac{z+1}{p})\ldots\mathfrak{G}(\frac{z+p-2}{p})\mathfrak{G}(\frac{z+p}{p}))}$$

continuity $\Rightarrow \dfrac{\Gamma_{\mathrm{period}}(z+1)}{\Gamma_{\mathrm{period}}(z)} \equiv p\Phi_\tau\left(\text{“}\dfrac{\mathfrak{G}(\frac{z+p}{p})}{\mathfrak{G}(\frac{z}{p})}\text{”}\right)\dfrac{\Gamma_{\mathrm{period}}(pz+1)}{\Gamma_{\mathrm{period}}(pz)}$.

However $\mathfrak{G}(z)$ is defined only on $z\in(0,1)$.

$\rightsquigarrow$ p-adic continuity(?), functional equations, e.t.c. when $\mathrm{ord}_p z = -1$.

A characterization of Γ_p (on-going)

Proposition

Coninuous functions $\Gamma_p(z), \Gamma_{\mathrm{period}}(z)$ satisfies

$$\prod_{k=0}^{d-1} \Gamma_*(z + \tfrac{k}{d}) \equiv \Gamma_*(dz) \quad (p \nmid d),$$

$$c_{*,1} \equiv c_{*,2} \equiv \cdots \quad (c_{*,n} := \tfrac{\Gamma_*(p^n+1)}{\Gamma_*(p^n)}).$$

Corollary

$\exists$ constants a, b satisfying

$$\Gamma_{\mathrm{period}}(z) \equiv a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}} \Gamma_p(z)$$

In particular, by computing the absolute Frobenius automorphism on only one Fermat curve, we obtain Coleman's formula for any $z \in \mathbb{Z}_p$ (e.g., $p = 3$, $F_5 \Rightarrow z = \frac{1}{5}, \frac{2}{5} \Rightarrow a^{\frac{-3}{10}} b^{\frac{-1}{10}} \equiv a^{\frac{-1}{10}} b^{\frac{-3}{10}} \equiv 1 \Rightarrow a \equiv b \equiv 1$).

Summary

- "Nice" properties of period symbols ⇒ Coleman's formula

$$\Gamma_{\mathrm{period}}(z) \equiv a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}} \Gamma_p(z) \mod \mu_\infty$$

  up to two parameters, under
  - the Archimedean formula

$$\Gamma_1(\tfrac{a}{N}) \equiv \pi^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N}\rangle} \mod \overline{\mathbb{Q}}^\times .$$

  - the p-adic continuity of

$$\Gamma_{\mathrm{period}}(z) := p^{\frac{1}{2}-z_1} \frac{\mathfrak{G}(z)}{\Phi_{cris}(\mathfrak{G}(z_1+1))} \quad (z \in \mathbb{Z}_p),$$

$$\Gamma_{\mathrm{period},\tau}(z) := p^{e(z-\tau^{-1}(z))} \frac{\mathfrak{G}(z)}{\Phi_\tau(\tau^{-1}(z))} \quad (z \in p^{-1}\mathbb{Z}_p^\times)$$

  with $\mathfrak{G}(\frac{a}{N}) := \dfrac{\Gamma_1(\frac{a}{N})\cdot(2\pi i)_p^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N}\rangle}}{(2\pi i)^{\frac{1}{2}-\langle \frac{a}{N}\rangle} \prod_{(b,N)=1} p_{\mathbb{Q}(\zeta_N)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{N}\rangle}}.$

- Then, calculation on only one Fermat curve produces Coleman's formula $\Gamma_{\mathrm{period}}(z) \equiv \Gamma_p(z) \mod \mu_\infty$ for any $z \in \mathbb{Z}_p$, which is equivalent to "$\frac{\mathfrak{G}(z+1)}{\mathfrak{G}(z)}$" $\equiv z \mod \mu_\infty$ $(z \in p^{-1}\mathbb{Z}_p^\times)$.

A generalization

Conjecture (Yoshida, K., arXiv:1706.03198)

Let F be a totally real field, H/F a finite abelian extension with K the maximal CM-subfield of H, $\sigma \in \mathrm{Gal}(H/F)$. Then

1. $\exp(X(\sigma)) \equiv \pi^{\zeta(0,\sigma)} \prod_{\sigma' \in \mathrm{Gal}(K/F)} p_K(\mathrm{id}, \sigma')^{\zeta(0,\sigma|_K \sigma')} \mod \overline{\mathbb{Q}}^\times$,
   where $\exp(X(\sigma)) \fallingdotseq$ a finite product of multiple Γ-functions.
2. Under ❶, put $\mathfrak{G}(\sigma) := \frac{\exp(X(\sigma)) \cdot (2\pi i)_p^{\zeta(0,\tau)} \prod p_{K,p}((\mathrm{id},\sigma')^{\zeta(0,\sigma|_K\sigma')}}{(2\pi i)^{\zeta(0,\tau)} \prod p_K((\mathrm{id},\sigma')^{\zeta(0,\sigma|_K\sigma')}}$.
   Let $\tau \in W_{F_\mathfrak{p}} \subset \mathrm{Gal}(\overline{F_\mathfrak{p}}/F_\mathfrak{p})$. Then $\frac{\mathfrak{G}(\tau|_H\sigma)}{\Phi_\tau(\mathfrak{G}(\sigma))} \equiv *** \mod \mu_\infty$, where $***$ in the RHS is defined in terms of p-adic multiple Γ-functions.

Problem

- In general, ❶, "p-adic continuity", "functional equatins" $\Rightarrow$? $\Rightarrow$ ❷.
- When $\mathbb{Q} \subset F \subset H \subset \mathbb{Q}(\zeta_N)$, Rohrlich-Bannai-Otsubo-Yoshida $\Rightarrow$ ❶,
  Coleman $\Rightarrow$ p-adic continuity, ~~❷~~ in some cases.
  In such cases, p-adic continuity $\Rightarrow$? $\Rightarrow$ ❷.