On Yoshida's conjecture concerning CM-periods and multiple gamma functions.

Tomokazu Kashio*

September 17, 2018

Abstract

Yoshida formulated a conjecture which expresses Shimura's period symbol in terms of Barnes' multiple gamma functions, up to algebraic numbers, in around 2000. In this talk, we introduce its refinement and some partial results. We describe a common refinement of Yoshida's conjecture, Stark's conjecture, and its p-analogue by Gross.

1 Introduction

First, I would like to thank Professor Ikeda for giving me a chance to talk here. even though the topic is rather far from Hilbert-Siegel modular forms.

I shall announce some progress on Yoshida's conjecture concerning CM-periods. He was a regular member of this conference but retired a few years ago.

First we recall a version of

The Chowla-Selberg formula.

For an imaginary quadratic field K, we define $p_K \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ as follows.

- Let $E/\overline{\mathbb{Q}}$ be an elliptic curve with CM by K. Then we put

$$p_K \equiv \pi^{-1} \int_\gamma \omega_{\mathrm{hol}} \mod \overline{\mathbb{Q}}^\times.$$

  Here ω_{hol} is holomorphic differential one form on E (e.g., $\frac{dx}{y}$ for $E : y^2 = x^3 + ax + b$), γ is an arbitrary closed path on $E(\mathbb{C})$ satisfying $\int_\gamma \omega_{\mathrm{hol}} \neq 0$.

- Or equivalently, write $K = \mathbb{Q}(\tau)$ $(\mathrm{Im}(\tau) > 0)$. Then we put

$$p_K \equiv \eta(\tau)^2 \mod \overline{\mathbb{Q}}^\times.$$

  Here η is Dedekind eta function, which is the only one modular form in this talk.

*Tokyo University of Science, kashio_tomokazu@ma.noda.tus.ac.jp

Theorem 1 (A corollary of the Chowla-Selberg formula)**.**

$$p_K \equiv \pi^{-\frac{1}{2}} \prod_{a=1}^{d-1} \Gamma(\tfrac{a}{d})^{\frac{w\chi(\overline{a})}{4h}} \mod \overline{\mathbb{Q}}^{\times}.$$

Note that the original formula provides the explicit value of $|\eta(\tau)|$*.*

As is well known, this p_K is a special case of CM-periods:

Definition 1 (Shimura's period symbol)**.** *Let* K *be a CM-field. For complex embeddings* σ, τ *of* K*, we define*

$$p_K(\sigma,\tau) \in \mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}$$

by "decomposing" period integrals as

$$\prod_{\tau\in\Phi} p_K(\sigma,\tau) \equiv \begin{cases} \pi^{-1}\int_\gamma \omega_\sigma & (\sigma\in\Phi) \\ \int_\gamma \omega_\sigma & (\sigma\notin\Phi) \end{cases}$$

in a certain way. Here we take an abelian variety $A/\overline{\mathbb{Q}}$ *with CM of type* (K,Φ)*, differential one forms* ω_σ *where* K *acts as* $\sigma(K)$*. More precisely*

- $K \cong \mathrm{End}(A)\otimes\mathbb{Q} \curvearrowright H^1_{dR}(A/\overline{\mathbb{Q}}) \cong K\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$*, which is a* $\deg K$*-dimensional vector space.*
- *We can write* $H^1_{dR}(A/\overline{\mathbb{Q}}) = \bigoplus_{\tau\colon K\hookrightarrow\mathbb{C}} \overline{\mathbb{Q}}\omega_\tau$ *s.t.* $k^*\omega_\tau = \tau(k)\omega_\tau$*.*
- *It has a subspace of holomorphic one forms* $= \bigoplus_{i=1}^{n}\overline{\mathbb{Q}}\omega_{\tau_i}$*.*
- *Then* $\Phi = \{\tau_1,\dots,\tau_n\}$ *is called the CM-type of* A*, which is a half of all complex embeddings.*

When K is an imaginary quadratic field, the above $p_K = p_K(\mathrm{id},\mathrm{id})$.

When $K = \mathbb{Q}(\zeta_n)$.

Theorem 2 (Yoshida)**.** *For* $K=\mathbb{Q}(\zeta_n)$*, we have*

$$p_{\mathbb{Q}(\zeta_n)}(\mathrm{id},\sigma) \equiv \pi^{-\frac{\delta_\sigma}{2}} \prod_{\eta\in\hat{G}_-} \prod_{c=1,(c,n)=1}^{n-1} \Gamma(\tfrac{c}{n})^{\frac{\eta(\sigma c)}{L(0,\eta)\varphi(n)}}.$$

Here we identify $G := \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^\times$*.* $\hat{G}_-$ *is the set of all odd characters. We put* $\delta_\sigma := 1,-1,0$ *for* $\sigma=\mathrm{id}$*, complex conjugation, otherwise, respectively.*

Proof. Yoshida derived this formula from two formulas:

- Rohrlich's formula: Consider Fermat curves $F_n\colon x^n + y^n = 1$ and differential one forms $\eta_{r,s} = x^r y^{s-n} \frac{dx}{x}$. We have

$$\int_\gamma \eta_{r,s} \doteqdot \int_0^1 t^{\frac{r}{n}-1}(1-t)^{\frac{s}{n}-1} dt =: B(\tfrac{r}{n}, \tfrac{s}{n}) = \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})}.$$

  Since $J(F_n)$ has CM by $\mathbb{Q}(\zeta_n)$, these values provides $\int_\gamma \omega_\sigma$.

- Euler's reflection formula provides "monomial relations"

$$\Gamma(\tfrac{c}{n})\Gamma(\tfrac{n-c}{n}) = \frac{\pi}{\sin(\frac{c}{n}\pi)} \equiv \pi \mod \overline{\mathbb{Q}}^\times.$$

□

Problem

What happens for general K?

$\Rightarrow$ Yoshida's conjecture (1996, 1998, 2003).

General CM-fields K.

Yoshida defined a class invariant $\Gamma(c, \iota)$, which is $\exp(X(\iota(c)))$ in his book "Absolute CM-Periods".

Definition 2. *Let F be a totally real field, $C_\mathfrak{f}$ the narrow ideal class group modulo $\mathfrak{f}$, $c \in C_\mathfrak{f}$, ι a real embedding of F. Take a fractional ideal $\mathfrak{a} \in c$ and a Shintani's fundamental domain D of $F_+^\times/\mathcal{O}_{F,+}^\times$, which is a disjoint union of cones. Then we put*

$$\Gamma(c, \iota; D, \mathfrak{a}) := \exp\left(\frac{d}{ds} \sum_{z \in \mathfrak{a}^{-1} \cap D,\ z\mathfrak{a} \in c} \iota(z)^{-s}\Bigg|_{s=0}\right) \times \text{ a correction term.}$$

- *The "correction term" is in the form of*

$$\prod_{i=1}^k \iota(\alpha_i)^{\iota(\beta_i)}\ (\alpha_i, \beta_i \in F).$$

  We will omit the details although this term is "troublesome" in practice.

- $\Gamma(c, \iota; D, \mathfrak{a}) \bmod \overline{\mathbb{Q}}^\times$ *depends only on c, ι, not on $\mathfrak{a}, D$. Hence we write $\Gamma(c, \iota)$.*

- *When $F = \mathbb{Q}$, $\mathfrak{f} = (d)$, $c = \overline{(a)}$, we see that*

$$\Gamma(c, \mathrm{id}) = \exp\left(\frac{d}{ds} \sum_{\mathbb{N} \ni k \equiv a \bmod n} k^{-s}\Bigg|_{s=0}\right) = \Gamma(\tfrac{a}{n}) n^{\frac{a}{n}-\frac{1}{2}} (2\pi)^{-\frac{1}{2}}.$$

Then he formulated a conjecture in the form of

Conjecture 1 (Yoshida)**.**

$$p_K(\sigma,\tau) \equiv \text{ a product of } \Gamma(c,\iota)\text{'s.}$$

The following one is the "reverse version".

Conjecture 2 (AJM 140 (2018), no. 3)**.**

$$\Gamma(c,\mathrm{id}) \equiv (2\pi i)^{\zeta(0,c)} \prod_{c'\in C_{\mathfrak{f}}} p_K(c|_K, c'|_K)^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}},$$

where K denotes the maximal CM subfield of the narrow ray class field $H_{\mathfrak{f}}$ modulo $\mathfrak{f}$. We identify $C_{\mathfrak{f}} = \mathrm{Gal}(H_{\mathfrak{f}}/F)$.

Remark 1. *This is not just a restatement.*

(i) *Conjecture 1 is equivalent to "$\prod_{c|_K=\sigma}$ of Conjecture 2" for $\sigma \in \mathrm{Gal}(K/F)$:*

$$\prod_{c|_K=\sigma} \Gamma(c,\mathrm{id}) \equiv \prod_{c|_K=\sigma} (2\pi i)^{\zeta(0,c)} \prod_{c'\in C_{\mathfrak{f}}} p_K(c|_K, c'|_K)^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}.$$

To prove this equivalence, need some monomial relations between $\Gamma(c,\iota)$'s which are the main results in op. cit.

(ii) *When $F = \mathbb{Q}$, Conjecture 1 coincides with Theorem 2. Recall that it follows from Rohrlich's formula and Euler's reflection formula.*

(iii) *When $F = \mathbb{Q}$, Conjecture 2 follows from Rohrlich's formula, without Euler's reflection formula.*

Remark 2. *The rank one abelian Stark conjecture w.r.t real places states that*

Assume that K is an abelian extension of a totally real field F which has a real place ι. Then, for $\tau \in \mathrm{Gal}(K/F)$, $\exp(2\zeta'(0,\tau))$ is in $\iota(K)^\times$ and satisfies some properties; "unitness", "reciprocity law", "abelian condition".

Here $\zeta(s,\tau)$ is the partial zeta function defined by

$$\zeta(s,\tau) := \sum_{\mathcal{O}_F \supset \mathfrak{a} \overset{Artin\ map}{\mapsto} \tau} N\mathfrak{a}^{-s}.$$

Note that Shintani's formula states that

$$\exp(\zeta'(0,\tau)) = \prod_{C_{\mathfrak{f}_{K/F}} \ni c \overset{Artin\ map}{\mapsto} \tau} \prod_{\iota:\, F\hookrightarrow\mathbb{R}} \Gamma(c,\iota).$$

On day, Professor Yoshida was asked the relation between his conjecture and Stark's conjecture in a workshop. He answered, roughly speaking, "I don't know. However, both conjectures will be proved at the same time". In fact, we see that Conjecture 2 implies

$$\Gamma(c, \mathrm{id})\Gamma(cs, \mathrm{id}) \equiv 1$$

for $s \in C_{\mathfrak{f}}$ *corresponding to complex conjugations on* $H_{\mathfrak{f}}$*, since* p_K *satisfies the relation*

$$p_K(\sigma, \tau)p_K(\rho\sigma, \tau) \equiv 1 \quad (\rho\text{: complex conjugation}).$$

This implies the algebraicity $\exp(\zeta'(0, \tau)) \in \overline{\mathbb{Q}}^{\times}$*, since "K has a real place" means that the kernel of the Artin map contains such an* s*.*

2 p-adic analogues

Recall that the de Rham isomorphism

$$H^1_B(A) \otimes \mathbb{C} \cong H^1_{dR}(A) \otimes \mathbb{C}$$

and the duality

$$H^1_B(A) \times H^B_1(A) \to \mathbb{Q}$$

provide the usual period integral

$$H^B_1(A) \times H^1_{dR}(A) \to \mathbb{C}, \ (\gamma, \omega) \mapsto \int_\gamma \omega.$$

Let B_{dR} be Fontaine's p-adic period ring. We define p-adic CM-periods $p_{K,p}$ similarly with p_K, by replacing the de Rham isomorphism with comparison isomorphism of p-adic Hodge theory

$$H^1_B(A) \otimes B_{dR} \cong H^1_{dR}(A) \otimes B_{dR}.$$

Proposition 1. *The ratio between the CM-period and is p-adic analogue is well-defined, at least up to roots of unity. That is,*

$$[p_K(\sigma, \tau) : p_{K,p}(\sigma, \tau)] \in (\mathbb{C}^\times \times B^\times_{dR})/(\mu_\infty \times \mu_\infty)\overline{\mathbb{Q}}^{\times}$$

does not depend on the choices of models of abelian varieties A with CM, etc.

Recall that $\Gamma(c, \iota) = \Gamma(c, \iota; D, \mathfrak{a})$ depends on the choices of an ideal $\mathfrak{a} \in c$ and a fundamental domain D of $F^\times_+/\mathcal{O}^\times_{F,+}$.

Proposition 2. *The ratio between "multiple gamma function" and "p-adic analogue" is well-defined, at least up to roots of unity. That is,*

$$[\Gamma(c, \iota) : \Gamma_p(c, \iota)] \in (\mathbb{C}^\times \times \mathbb{C}^\times_p)/(\mu_\infty \times \mu_\infty)\overline{\mathbb{Q}}^{\times}$$

is well-defined, whenever $\mathfrak{p}_\iota \mid \mathfrak{f}$. *Here* $\mathfrak{p}_\iota$ *is the prime ideal corresponding to* $F \overset{\iota}{\subset} \overline{\mathbb{Q}} \overset{\text{fixed}}{\subset} \mathbb{C}_p$ *and we put*

$$\Gamma_p(c,\iota) := \exp_p\left(\frac{d}{ds}\left[\sum_{z\in\mathfrak{a}^{-1}\cap D,\ z\mathfrak{a}\in c} \iota(z)^{-s}\right]_{p\text{-adic int.}}\Bigg|_{s=0}\right) \times \text{ a correction term.}$$

The assumption $\mathfrak{p}_\iota \mid \mathfrak{f}$ *is needed for the p-adic interpolation of the series.*

Let $c \in C_{\mathfrak{f}}$. For $* = \emptyset, p$, we put

$$\mathrm{P}_*(c) := (2\pi i)_*^{\zeta(0,c)} \prod_{c'\in C_{\mathfrak{f}}} p_{K,*}(c|_K, c'|_K)^{\frac{\zeta(0,c')}{[H_{\mathfrak{f}}:K]}}.$$

For simplicity, consider the case $\iota = \mathrm{id}$, $\mathfrak{p}_{\mathrm{id}} \mid \mathfrak{f}$. Then, under Conjecture 2, we put

$$\Gamma(c) := \frac{\Gamma(c,\mathrm{id})}{\mathrm{P}(c)} \frac{\mathrm{P}_p(c)}{\Gamma_p(c,\mathrm{id})} \mod \mu_\infty.$$

Since abelian varieties with CM have potentially good reductions, $\Gamma(c)$ takes values in $B_{\mathrm{cris}}\overline{\mathbb{Q}_p}$, where B_{cris} is a subring of B_{dR} equipped with the absolute Frobenius action. We define the action of an element τ in the p-adic Weil group as

$$\Phi_\tau := (\text{abs. Frob})^{\deg\tau} \otimes \tau \curvearrowright B_{\mathrm{cris}}\overline{\mathbb{Q}_p} = B_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}_p}.$$

Conjecture 3 (arXiv:1706.03198)**.** *For* $\tau \in W_{F_{\mathfrak{p}_{\mathrm{id}}}}$ *(Weil group), we have*

$$\Phi_\tau(\Gamma(c)) \equiv \Gamma(c_\tau c) \mod \mu_\infty.$$

Here $c_\tau \in C_{\mathfrak{f}}$ *denotes the ideal class corresponding to* $\tau|_{H_{\mathfrak{f}}}$ *via the Artin map:*

$$\tau \in W_{F_{\mathfrak{p}_{\mathrm{id}}}} \subset \mathrm{Gal}(\overline{\mathbb{Q}_p}/F_{\mathfrak{p}_{\mathrm{id}}}) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F) \twoheadrightarrow \mathrm{Gal}(H_{\mathfrak{f}}/F) \cong C_{\mathfrak{f}}.$$

Remark 3. (i) *The case when* $\mathfrak{p}_{\mathrm{id}} \nmid \mathfrak{f}$ *is rather complicated.*

(ii) *Yoshida and I had formulated several versions of conjectures on* Φ_τ *in terms of* $\Gamma(c,\iota), \Gamma_p(c,\iota)$. *We did not have the idea using CM-periods and its p-adic analogue at the same time. The above one is a refinement of them.*

Theorem 3. *Conjectures 2, 3 implies a "large part" of the rank one abelian Stark conjecture w.r.t real places and its p-adic analogue by Gross.*

Proof. • Shintani's formula and its p-adic analogue state that Stark units and Gross-Stark units can be expressed as a product of $\Gamma(c,\iota)$ and $\Gamma_p(c,\iota)$ respectively.

• In the setting of Stark's conjecture or the Gross-Stark conjecture, we have $\zeta(0,\tau) = \sum_{c\mapsto\tau} \zeta(0,c) = 0$. That is, the "period-part" of $\prod_{c\mapsto\tau} \Gamma(c)$ does not appear.

Then we can write Stark units and Gross-Stark units as a product of $\Gamma(c)$'s. Therefore the above conjectures imply the algebraicity property and the reciprocity law of Stark' units or Gross-Stark units. □

Theorem 4. *When* $H_{\mathfrak{f}}/\mathbb{Q}$ *is abelian and* $p \neq 2$, *Conjecture 3 holds true.*

Proof. By Yoshida's technique in his book, we can reduce to the case $F = \mathbb{Q}$. This case is proved in [Crelle 741 (2018)] by using Rohrlich's formula and its p-adic analogue by Coleman. □

3 A new approach.

Theorem 5 (in preparation)**.** *Colman's formula ($\doteqdot$ Conjecture 3 in the case $F = \mathbb{Q}$) follows from, roughly speaking,*

(i) *Rohrlich's formula.*

(ii) *Monomial relations:*

$$p_K(\tilde{\sigma}|_K, \tau) = p_L(\tilde{\sigma}, \mathrm{Inf}(\tau)) := \prod_{\tilde{\tau} \in \mathrm{Hom}(L,\mathbb{C}), \tilde{\tau}|K = \tau} p_L(\tilde{\sigma}, \tilde{\tau}),$$
$$(K \subset L,\ \tilde{\sigma} \in \mathrm{Hom}(L, \mathbb{C}),\ \tau \in \mathrm{Hom}(K, \mathbb{C})).$$

and their p-adic analogues.

(iii) *A kind of "p-adic continuity" of the absolute Frobenius action.*

Proof. We can rewrite Conjecture 3 with $F = \mathbb{Q}$ to the form of

special values of p-adic Γ-function
= a product of special values of Γ-function, CM-periods, and p-adic CM-periods.

p-adic continuity and multiplication formulas

$$\prod_{k=0}^{d-1} \Gamma_p(z + \tfrac{k}{d}) = (\text{a root of unity}) \cdot d^{\frac{1}{2} - dz} \Gamma_p(dz).$$

characterize the p-adic Γ-function. It suffices to show that the same holds for the right-hand side: In this setting, monomial relations (ii) with $K = \mathbb{Q}(\zeta_n) \subset L = \mathbb{Q}(\zeta_{dn})$ turn out to be multiplication formulas. □

A future problem

For general totally real fields F,

- Archimedean conjecture (Conjecture 2)
- Monomial relations of Shimura's period symbol
- "p-adic continuity" of the absolute Frobenius action

imply p-adic conjecture (Conjecture 3)?