CM 周期，多重ガンマ関数，Stark 予想の関係と その p 進類似

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Dasgupta Kakde の最近の仕事とその周辺

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吉田予想と Stark 予想 (rank 1 abelian)

吉田予想 (absolute CM-periods)	**Stark 予想**
多重 Γ 関数 $\Leftrightarrow$ CM 周期	$\exp(\zeta'(0,\sigma)) \doteqdot$ Stark 単数

新谷公式: $\exp(\zeta'(0,\sigma)) \doteqdot$ 多重 Γ 関数の積, CM 周期の単項関係式

$\rightsquigarrow$ 吉田予想 $\Rightarrow \exp(\zeta'(0,\sigma)) \in \overline{\mathbb{Q}}$

p 進吉田予想	**Gross-Stark 予想**
p 進多重 Γ $\Leftrightarrow$ Frob. on CM ab.var.	$\exp_p(\zeta_p'(0,\sigma)) \doteqdot$ G-S 単数

p 進新谷公式, Frob. 作用の固有値 $\doteqdot$ G-S 単数

$\rightsquigarrow$ p 進吉田予想 $\Rightarrow$ G-S 予想

[CM 周期: p 進周期] $\in (\mathbb{C}^\times \times B_{dR}^\times)/\overline{\mathbb{Q}}^\times$

$\rightsquigarrow$ p 進周期環値 Γ 関数 (に関する予想) $\Rightarrow$ S 単数, G-S 単数

$\overset{(?)}{\Uparrow}$ 関数等式, p 進連続性

- CM 周期 (周期記号):
  Shimura, G., *Abelian varieties with complex multiplication and modular functions*, Princeton Math. Ser. 46 (1998).
- 絶対 CM 周期:
  Yoshida, H., *Absolute CM-Periods*, Math. Surveys Monogr. (2003).
  K-, On the algebraicity of some products of special values of Barnes' multiple gamma function. *Amer. J. Math.* (2018).
- p 進絶対 CM 周期:
  K- and Yoshida, H., On p-adic absolute CM-periods. I, II, *Amer. J. Math.* (2008), *Publ. Res. Inst. Math. Sci.* (2009).
- [CM 周期: p 進周期]:
  K-, Fermat curves and a refinement of the reciprocity law on cyclotomic units, *J. Reine Angew. Math.* (2018).
  K-, On a common refinement of Stark units and Gross-Stark units, preprint (arXiv:1706.03198).
  K-, Note on Coleman's formula for the absolute Frobenius on Fermat curves, preprint (arXiv:1904.02879)

CM 周期

- 円周率 $\pi = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = 3.1415\ldots$
  $\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{y}$ on $x^2+y^2=1$.
- レムニスケート周率 $\varpi = 2\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = 2.6220\ldots$
  $\int \frac{1}{\sqrt{1-x^4}}\,dx = \int \frac{dx}{y}$ on $x^4+y^2=1 \curvearrowleft [\sqrt{-1}$倍: $(x,y)\mapsto(\sqrt{-1}x,y)]$

- E: 楕円曲線 $/\overline{\mathbb{Q}}$, $\mathrm{End}(E)\otimes_{\mathbb{Z}}\mathbb{Q}=K$: 虚二次体
  $\Rightarrow p_K := \pi^{-1}\int_\gamma \frac{dx}{y} \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ ($\because$ E のモデル, 閉路 γ).

CM 周期

Theorem 1

$f(z)$: 楕円保型形式, フーリエ係数 $\in \overline{\mathbb{Q}} \Rightarrow \dfrac{f(\tau)}{{p_K}^{\mathrm{wt}\,f}} \in \overline{\mathbb{Q}}$ $(\tau \in K \cap \mathcal{H})$.

Corollary 2 ($f = \Delta$ (モジュラー判別式) & Chowla-Selberg 公式)

$$p_K \equiv \pi^{-\frac{1}{2}} \prod_{a=1}^{d_K} \Gamma\left(\tfrac{a}{d_K}\right)^{\frac{w_K \chi_K(a)}{4h_K}} \mod \overline{\mathbb{Q}}^{\times}.$$

Example 3 ($K = \mathbb{Q}(\sqrt{-1})$)

$E\colon y^2 = x^3 - x$, $\mathrm{End}(E) = \mathbb{Z}[\sqrt{-1}]$, $\sqrt{-1}\colon (x, y) \mapsto (-x, iy)$.

$$p_{\mathbb{Q}(\sqrt{-1})} = 2\pi^{-1}\int_{-1}^{0} \frac{dx}{\sqrt{x^3-x}} \overset{x=-\sqrt{t}}{=} \pi^{-1}\int_0^1 \frac{dt}{t^{\frac{3}{4}}(1-t)^{\frac{1}{2}}} = \pi^{-1}B\left(\tfrac{1}{4}, \tfrac{1}{2}\right) = \pi^{\frac{-1}{2}} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}.$$

※ $\int_\gamma \frac{dx}{y} \Rightarrow h_K$?

※ $\int_\gamma \frac{dx}{y} \Rightarrow h_K$?

where w is the number of roots of unity contained in K and h is the class number of K. Except for the cases $K = \mathbf{Q}(\sqrt{-1})$ and $K = \mathbf{Q}(\sqrt{-3})$, there is no known way to derive (1.4) from the direct evaluation of an elliptic integral. We will give a geometric proof of (1.4) in the next section.

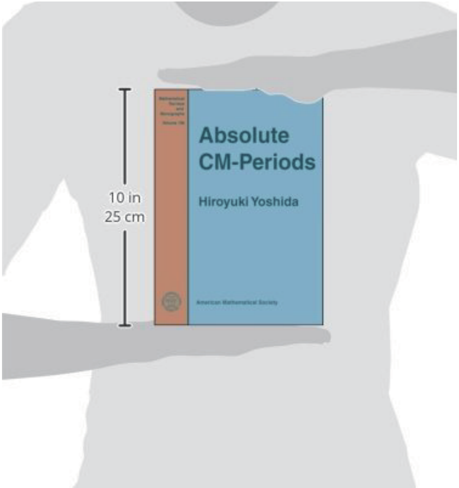
- $K = \mathbb{Q}(\sqrt{-d})$

$$\Rightarrow E_K : y^2 = x^3 + \cdots$$

$$\Rightarrow \int_\gamma \frac{dx}{y} = 2\int_{\cdots} \frac{dx}{\sqrt{x^3+\cdots}}$$

$$= \cdots$$

$$= c \cdot \pi^{\frac{1}{2}} \prod_{a=1}^{d} \Gamma\left(\frac{a}{d}\right)^{\frac{w_K \chi_K(a)}{4h_K}}.$$



CM 周期

K: 一般の CM 体

- 楕円曲線 ($\fallingdotseq \mathbb{C}/\mathcal{O}_K$) $\rightsquigarrow$ アーベル多様体 ($\fallingdotseq \mathbb{C}^n/\mathcal{O}_K$)
  $\Rightarrow$ 志村氏の周期記号: $p_K(\sigma,\tau)$ ($\sigma,\tau\colon K \hookrightarrow \mathbb{C}$).
- Hilbert 保型形式 at CM 点 $\overset{\text{志村}}{\Leftrightarrow}$ $p_K(\sigma,\tau)$
- $p_K(\sigma,\tau)$ $\overset{\text{吉田予想}}{\Leftrightarrow}$ 多重ガンマ関数

- A: アーベル多様体 $/\overline{\mathbb{Q}}$ s.t. $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}\cong K$: CM 体.
- ω: 正則 1 形式 $\overset{\sigma}{\curvearrowleft} \mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}=K$, γ: $A(\mathbb{C})$ の閉路
- $\displaystyle\int_\gamma \omega_\sigma \rightsquigarrow \overset{\text{“分解”}}{\cdots} \rightsquigarrow p_K(\sigma,\tau)\in\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ s.t. $\displaystyle\int_\gamma \omega_\sigma \equiv \pi \prod_{\tau\in\Xi_A} p_K(\sigma,\tau)$.

※ CM 周期 $\fallingdotseq [H^1_{dR}(A)\times H_1^B(A(\mathbb{C}))\to\mathbb{C},\ (\omega,\gamma)\mapsto \int_\gamma \omega]$

$\Leftrightarrow$ de Rham 同型 : $H^1_B(A(\mathbb{C}))\otimes\mathbb{C}\cong H^1_{dR}(A)\otimes\mathbb{C}$

吉田予想: CM 周期 $\Leftrightarrow$ 多重 Γ 関数

Definition 4 (Barnes の多重 Γ 関数)

$$\Gamma(x,(\omega_1,\dots,\omega_r)) := \exp\left(\frac{d}{ds}\sum_{m_1,\dots,m_r\geq 0}(x+m_1\omega_1+\cdots+m_r\omega_r)^{-s}\Big|_{s=0}\right).$$

Example 5

$$\Gamma(x,(1)) = \exp\left(\frac{d}{ds}\sum_{m\geq 0}(x+m)^{-s}\Big|_{s=0}\right) \overset{\text{Lerch}}{=} \frac{\Gamma(x)}{\sqrt{2\pi}}.$$

Conjecture 6 (吉田予想, 絶対 CM 周期)

$$p_K(\sigma,\tau) \equiv \pi^r \times \prod \Gamma(x,\boldsymbol{\omega})^a \times \prod b^c \bmod \overline{\mathbb{Q}}^\times \text{ の形.}$$

※ $x,\omega_i,b,c \in \widetilde{K}\cap\mathbb{R}$, $r,a\in\mathbb{Q}$ は明示的だが “新谷基本領域” 等に依存.

※ 虚二次体なら C-S 公式 $\Rightarrow p_K(\mathrm{id},\mathrm{id}) \equiv \pi^{-\frac{1}{2}}\prod_{a=1}^{d_K}\Gamma(\frac{a}{d})^{\frac{w_K\chi_K(a)}{4h_K}}$.

Example 7 (円分体の場合)

$F_n\colon x^n + y^n = 1$ ($\rightsquigarrow$ ヤコビ多様体の既約成分が CM ab. var.)

$$\int_\gamma x^{r-1} y^{s-n} dx = B(\tfrac{r}{n}, \tfrac{s}{n}) = \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} \quad (0 < r, s, r+s < n).$$

線形代数 $\Rightarrow \Gamma(\frac{a}{n}) \equiv \pi^{1-\langle \frac{a}{n} \rangle} \prod_{(b,n)=1} p_{\mathbb{Q}(\zeta_n)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{n} \rangle} \bmod \overline{\mathbb{Q}}^\times$.

Example 8 ($K = \mathbb{Q}(\sqrt{2\sqrt{5}-26})$)

$C\colon y^2 = \frac{7+\sqrt{41}}{2}x^6 + (-10-2\sqrt{41})x^5 + 10x^4 + \frac{41+\sqrt{41}}{2}x^3 + (3-2\sqrt{41})x^2 + \frac{7-\sqrt{41}}{2}x + 1$,
$\omega_{\mathrm{id}} := \frac{2dx}{y} + \frac{(\sqrt{5}-1)x dx}{y}$, $C', \omega'_{\mathrm{id}}$: 共役 ($\sqrt{41} \mapsto -\sqrt{41}$) なもの

$$\pi^{-1} \int_\gamma \omega_{\mathrm{id}} \int_{\gamma'} \omega'_{\mathrm{id}} \fallingdotseq \prod_{(x_1,x_2)\in R} \Gamma(x_1 + \tfrac{3-\sqrt{5}}{2}x_2, (1, \tfrac{3-\sqrt{5}}{2}))$$
$$\times \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{19\sqrt{5}+42}{123}} \frac{(\sqrt{5}+13)\sqrt{-8\sqrt{5}+20+(\sqrt{5}+15)\sqrt{2\sqrt{5}-26}}}{6560}.$$

$R = \{(\frac{1}{41}, \frac{5}{41}), (\frac{2}{41}, \frac{10}{41}), (\frac{4}{41}, \frac{20}{41}), (\frac{5}{41}, \frac{25}{41}), (\frac{8}{41}, \frac{40}{41}), (\frac{9}{41}, \frac{4}{41}), (\frac{10}{41}, \frac{9}{41}), (\frac{16}{41}, \frac{39}{41}), (\frac{18}{41}, \frac{8}{41}), (\frac{20}{41}, \frac{18}{41}),$
$(\frac{21}{41}, \frac{23}{41}), (\frac{23}{41}, \frac{33}{41}), (\frac{25}{41}, \frac{2}{41}), (\frac{31}{41}, \frac{32}{41}), (\frac{32}{41}, \frac{37}{41}), (\frac{33}{41}, \frac{1}{41}), (\frac{36}{41}, \frac{16}{41}), (\frac{37}{41}, \frac{21}{41}), (\frac{39}{41}, \frac{31}{41}), (\frac{40}{41}, \frac{36}{41})\}$

吉田予想

Conjecture 9

K/F: アーベル拡大, K: CM 体, F: 総実体, $\tau \in \mathrm{Gal}(K/F)$.

$$\Gamma_F(\tau) \equiv \pi^{\zeta(0,\tau)} \prod_{\sigma \in \mathrm{Gal}(K/F)} p_K(\mathrm{id}, \sigma)^{\zeta(0,\sigma\tau)} \mod \overline{\mathbb{Q}}^\times.$$

- $\zeta(s,\tau) = \sum_{\left(\frac{K/F}{\mathfrak{a}}\right)=\tau} N\mathfrak{a}^{-s}$.
- RHS $= \pi^{\zeta(0,\tau)} p_K(\tau, \sum_{\sigma \in \mathrm{Gal}(K/F)} \zeta(0,\sigma)\sigma)$.
- $\Gamma_F(\tau) = \prod_{x,\boldsymbol{\omega}} \Gamma(x, \boldsymbol{\omega}) \times \prod_{b,c} b^c$ ($x, \omega_i, b, c \in F$) の形.
- $\exp(\zeta'(0,\tau)) \overset{\text{新谷公式}}{=} \prod_{\iota\colon F \hookrightarrow \mathbb{R}} \Gamma_{\iota(F)}(\iota \circ \tau \circ \iota^{-1})$
- absolute CM-periods $\cdots$ $p_K(\mathrm{id}, \sigma) \equiv \prod_\tau \Gamma_F(\tau)^{r_\tau}$ ($r_\tau \in \mathbb{Q}$) の形.

吉田予想 ⇔ Stark 予想

- F: 総実体, K/F: アーベル拡大, $\tau \in \mathrm{Gal}(K/F)$

$$\Rightarrow \zeta(s,\tau) := \sum_{\left(\frac{K/F}{\mathfrak{a}}\right)=\tau} N\mathfrak{a}^{-s} \overset{\text{新谷基本領域} \fallingdotseq F^\times/\mathcal{O}_F^\times}{=} \sum_{z\in \text{“錐”}} N(z)^{-s}$$

新谷公式: $$\exp(\zeta'(0,\tau)) = \prod_{\iota\colon F\hookrightarrow\mathbb{R}} \left(\prod_{x,\boldsymbol{\omega}} \Gamma(\iota(x),\iota(\boldsymbol{\omega})) \cdot \prod_{b,c} \iota(b)^{\iota(c)} \right)$$

Conjecture 10 (Stark 予想 (rank 1 abelian, 実素点))

$K^{\exists} \hookrightarrow \mathbb{R}$ のとき

$$\exp(\zeta'(0,\tau))^2 \in K^\times (\subset \mathbb{R}),\ \tau'(\exp(\zeta'(0,\tau))^2) = \exp(\zeta'(0,\tau\tau'))^2,\ \ldots$$

- 吉田予想 $\Rightarrow \exp(\zeta'(0,\tau)) \equiv \prod_{\sigma,\tau} p_K(\sigma,\tau)^a$
  & 周期の単項関係式 $\Rightarrow \exp(\zeta'(0,\tau)) \in \overline{\mathbb{Q}}$
- 実際は吉田予想を “改造”

Example 11 ($\mathbb{Q}(\zeta_n)^+/\mathbb{Q}$)

$F_n\colon x^n + y^n = 1$, $\int_\gamma x^{r-1}y^{s-n}dx = \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})}$.

$$\text{カップ積}\colon H^1(F_n) \times H^1(F_n) \to H^2(F_n) \cong H^1(\mathbb{G}_m)$$
$$(x^{r-1}y^{s-n}dx, x^{n-r-1}y^{n-s}dx) \mapsto c \cdot \frac{dx}{x}$$

$\overset{\text{Ex.7}}{\Rightarrow} \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} \frac{\Gamma(\frac{n-r}{n})\Gamma(\frac{n-s}{n})}{\Gamma(\frac{n-r-s}{n})} \equiv \pi \Rightarrow \exp(\zeta'(0,\sigma_a)) \overset{\text{Lerch}}{=} \frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi} \in \overline{\mathbb{Q}}$.

p 進類似 $\cdots$ de Rham の同型 $\Rightarrow$ p 進 Hodge の比較同型

- 周期記号: $H^1_{dR}(A) \times H^B_1(A) \to \mathbb{C}$, $(\omega_\sigma, \gamma) \mapsto \int_\gamma \omega_\sigma \overset{\text{“分解”}}{\rightsquigarrow} p_K(\sigma, \tau)$.
- $H^1_{dR}(A) \times H^B_1(A) \to B_{dR}$, $(\omega_\sigma, \gamma) \mapsto \int_{\gamma,p} \omega_\sigma$

  $\overset{\text{“分解”}}{\rightsquigarrow}$ p 進周期記号: $p_{K,p}(\sigma, \tau) \in (B_{cris}\overline{\mathbb{Q}}_p)^{\mathbb{Q}}/\overline{\mathbb{Q}}^\times$ ($\because$ pot. good red.)
- $[p_K(\sigma, \tau) : p_{K,p}(\sigma, \tau)] \in (\mathbb{C}^\times \times B^\times_{dR})/(\mu_\infty \times \mu_\infty)\overline{\mathbb{Q}}^\times$
- p 進吉田予想:

$$\text{abs.Frob. } \Phi_{cris} \curvearrowright [p_K(\sigma, \tau) : p_{K,p}(\sigma, \tau)] \Leftrightarrow \Gamma(x, \boldsymbol{\omega}), \Gamma_p(x, \boldsymbol{\omega})$$

$$\Gamma_p(x, \boldsymbol{\omega}) := \exp_p\left(\frac{d}{ds} \Big[\sum_{m_1, \ldots, m_r \geq 0} (x + m_1\omega_1 + \cdots + m_r\omega_r)^{-s} \Big]_{p\text{ 進補完}} \Big|_{s=0} \right).$$

p 進類似

c.f. 吉田予想

K/F: アーベル拡大, K: CM 体, F: 総実体, $\tau \in \mathrm{Gal}(K/F)$.

$$\Gamma_F(\tau) \equiv \pi^{\zeta(0,\tau)} \prod_{\sigma \in \mathrm{Gal}(K/F)} p_K(\mathrm{id}, \sigma)^{\zeta(0,\sigma\sigma')} \bmod \overline{\mathbb{Q}}^{\times}.$$

Conjecture 12 (p 進吉田予想, $\mathfrak{p}_F \nmid \mathfrak{f}_{K/F}$-版)

$$G(\tau) := \frac{\Gamma_F(\tau) \cdot \pi_p^{\zeta(0,\tau)} \prod_{\sigma \in \mathrm{Gal}(K/F)} p_{K,p}(\mathrm{id}, \sigma)^{\zeta(0,\sigma\sigma')}}{\pi^{\zeta(0,\tau)} \prod_{\sigma \in \mathrm{Gal}(K/F)} p_K(\mathrm{id}, \sigma)^{\zeta(0,\sigma\sigma')}} \in (B_{cris}\overline{\mathbb{Q}}_p)^{\mathbb{Q}}/\mu_\infty.$$

$$\Rightarrow \frac{G((\frac{K/F}{\mathfrak{p}_F})\tau))}{\Phi_{cris}(G(\tau))} \equiv \Gamma_{F,p}(\tau) \bmod \mu_\infty.$$

- $\exists \mathfrak{p}_F \mid \mathfrak{f}_{K/F}$-版.
- p-adic abs. CM-periods $\cdots$ $\Phi_{cris} \curvearrowright H^1_{cris}(A), H_{cris}(M(\chi)) \Leftrightarrow \Gamma_{F,p}$.

p 進類似

Example 13 (円分体の場合, $p \nmid 2n$)

$$G(\tfrac{a}{n}) := \frac{\frac{\Gamma(\frac{a}{n})}{\sqrt{2\pi}} \pi_p^{\frac{1}{2}-\langle\frac{a}{n}\rangle} \prod_{(b,n)=1} p_{\mathbb{Q}(\zeta_n),p}(id,\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle}}{\pi^{\frac{1}{2}-\langle\frac{a}{n}\rangle} \prod_{(b,n)=1} p_{\mathbb{Q}(\zeta_n)}(id,\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle}} \overset{\text{Ex.7}}{\in} (B_{cris}\overline{\mathbb{Q}}_p)^{\mathbb{Q}}/\mu_\infty$$

Coleman: $\Phi_{cris} \curvearrowright F_n/\mathbb{F}_p \Rightarrow p^{\frac{1}{2}-\langle\frac{a}{n}\rangle} \frac{G(\langle\frac{pa}{n}\rangle)}{\Phi_{\mathrm{cris}} G(\frac{a}{n})} \equiv \Gamma_p(\langle\frac{pa}{n}\rangle) \mod \mu_\infty$.

Theorem 14 ("common refinement")

Cnj.12 $\Rightarrow$ rank 1 abel. Gross-Stark 予想 (解決), Stark 単数へのガロア作用

Example 15 ($\mathbb{Q}(\zeta_n)^+/\mathbb{Q}$, $p \nmid 2n$)

$$G(\tfrac{a}{n})G(\tfrac{n-a}{n}) = \frac{\Gamma(\frac{a}{n})}{\sqrt{2\pi}} \frac{\Gamma(\frac{n-a}{n})}{\sqrt{2\pi}} = \exp(\zeta'(0,\sigma_a)) \overset{\text{Ex.11}}{\in} \overline{\mathbb{Q}}$$

Ex. 13 $\circlearrowleft$ Φ_{cris}, $\Phi_{cris}|_{\overline{\mathbb{Q}}} =$ Frob. at p.

関数等式と連続性

$$G(\tfrac{a}{n}) := \frac{\frac{\Gamma(\frac{a}{n})}{\sqrt{2\pi}} \pi_p^{\frac{1}{2}-\langle \frac{a}{n} \rangle} \prod_b p_{K,p}(id,\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{n} \rangle}}{\pi^{\frac{1}{2}-\langle \frac{a}{n} \rangle} \prod_b p_K(id,\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{n} \rangle}}, \ \Gamma_{\mathrm{period}}(\langle \tfrac{pa}{n} \rangle) := p^{\frac{1}{2}-\langle \frac{a}{n} \rangle} \frac{G(\langle \frac{pa}{n} \rangle)}{\Phi_{\mathrm{cris}} G(\frac{a}{n})}$$

Coleman の公式 $\overset{\text{Ex.13}}{\Rightarrow}$ $\Gamma_{\mathrm{period}}(\langle \frac{pa}{n} \rangle) \equiv \Gamma_p(\langle \frac{pa}{n} \rangle) \mod \mu_\infty$.

Proposition 16 (乗法公式, $p \nmid d$)

$$\prod_{k=0}^{d-1} \Gamma_*(z + \tfrac{k}{d}) \equiv d^{1-dz+(dz)_1} \Gamma_*(dz) \mod \mu_\infty \ (* = p, \mathrm{period})$$

$z_1 \in \mathbb{Z}_p$ s.t. $z = z_0 + pz_1$, $z_0 \in \{1, 2, \ldots, p\}$.

"Proof" for Γ_{period}.

$z \in (0, \frac{1}{d}) \Rightarrow \dfrac{\prod_{k=0}^{d-1} \Gamma_{\mathrm{period}}(z+\frac{k}{d})}{\Gamma_{\mathrm{period}}(dz)} = \Gamma$ 関数 $\cdot \ \frac{\pi}{\pi_p} \cdot p$ 冪 $\cdot$ 周期記号

周期の単項関係式 $\Rightarrow \equiv d^{\cdots}$, Γ_{period} の p 進連続性 $\Rightarrow \forall z$. □

関数等式と連続性

乗法公式での特徴付け (?) $\Rightarrow$ Coleman の公式の別証明 $\Rightarrow$ 一般化

Proposition 17

$\mathbb{Z}_p$ 上の連続関数 $f(z)$ が $\displaystyle\prod_{k=0}^{d-1} f(z+\tfrac{k}{d}) \equiv f(dz)\ (p \nmid d)$ を満たす

$\Leftrightarrow \exists \alpha_k$ s.t. $f(1+\sum_{k=0}^{\infty} x_k p^k) \equiv \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}}\ (0 \leq x_k \leq p-1)$.

※ 無限個の自由度

Proposition 18

$\mathbb{Z}_p$ 上の連続関数 $f(z)$ s.t. $\prod_{k=0}^{d-1} f(z+\frac{k}{d}) \equiv f(dz)\ (p \nmid d)$. $c_n := \frac{f(p^n+1)}{f(p^n)}$.

1. $c_0 \equiv c_1 \equiv \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}}$.
2. $c_1 \equiv c_2 \equiv \cdots \Leftrightarrow f(z) \equiv c_0^{z-\frac{1}{2}} (c_1/c_0)^{z_1+\frac{1}{2}}\ (z = z_0 + pz_1,\ 1 \leq z_0 \leq p)$.

関数等式と連続性

$\Gamma_{\mathrm{period}}(z)$ の p 進連続性の仮定の下で

Proposition 19

$$c_{*,1} \equiv c_{*,2} \equiv \cdots \quad \left(c_{*,n} := \frac{\Gamma_*(p^n+1)}{\Gamma_*(p^n)}\right) \ (* = p, \mathrm{period}).$$

※ "p 倍公式" より従う.

Corollary 20

$$\exists a, b \text{ s.t. } \Gamma_{\mathrm{period}}(z) \equiv a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}} \Gamma_p(z).$$

絶対フロベニウス作用 $\Phi_{cris} \curvearrowright H^1_{cris}(A)$ の p 進連続性？

Example 21 ($\Phi_{cris} \curvearrowright H^1_{cris}(F_n)$)

$\Phi_{cris}\colon x^{r-1}y^{s-n}dx \mapsto c(r,s) \cdot x^{r'-1}y^{s'-n}dx$

$\Rightarrow c(r,s)$ は $\frac{r}{n}, \frac{s}{n}$ に関して p 進連続的.