p 進ガンマ関数の関数等式と連続性に関して

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Euler の Γ 関数

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt \quad (\Re(z) > 0)$$

$\Rightarrow \Gamma(z+1) = z\Gamma(z)$ (Difference equation) $\Rightarrow \Gamma(n) = (n-1)!$

$$\Gamma(dz) = \frac{d^{dz-\frac{1}{2}}}{(2\pi)^{\frac{d-1}{2}}} \prod_{k=0}^{d-1} \Gamma(z + \tfrac{k}{d}) \quad (d \in \mathbb{N})$$ (Multiplication formula)

$\Rightarrow 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) = \Gamma(z)\Gamma(z+\frac{1}{2}) \qquad \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}.$

特徴付け

- $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, C^1 級, (D), (M) (Artin)
- $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, C^1 級, (M), $\lim_{z\to 0} zf(z) = 1$ (Kairies)
- $\mathbb{C}$ 上有理型, $\mathbb{R}_{>0}$ 上対数凸, (D), $f(1) = 1$ (Bohr-Mollerup)

Lerch の公式

Hurwitz zeta 関数

$$\zeta(s,z) := \sum_{m\geq 0}(z+m)^{-s} \left(= \frac{\Gamma(1-s)}{2\pi i}\int_{\leftarrow\!\!\!\rightleftarrows\circ} \frac{t^{s-1}e^{zt}}{1-e^t}dt \right)$$

Lerch の公式

$$\Gamma(z) = \sqrt{2\pi}\exp(\zeta'(0,z))$$

※ “simple proof” by (Bohr-Mollerup), [Yoshida’s book, p17]

$$\zeta(s,z) = z^{-s} + \zeta(s,z+1) \quad \Rightarrow (D)\ \Gamma(z+1) = z\Gamma(z)$$

$$\zeta(s,dz) = d^{-s}\sum_{k=0}^{d-1}\zeta(s,z+\tfrac{k}{d}) \quad \Rightarrow (M)\ \Gamma(dz) = \frac{d^{dz-\frac{1}{2}}}{(2\pi)^{\frac{d-1}{2}}}\prod_{k=0}^{d-1}\Gamma(z+\tfrac{k}{d})$$

p 進 Γ 関数 $(p \neq 2)$

Morita の p 進 Γ 関数

$$\Gamma_p(z) := \lim_{\mathbb{N} \ni n \to z} (-1)^n \prod_{k=1,\ p \nmid k}^{n-1} k \quad (z \in \mathbb{Z}_p)$$

- (D_p) $\Gamma_p(z+1) = z^* \Gamma_p(z)$, $z^* = \begin{cases} -z & (p \nmid z) \\ -1 & (p \mid z) \end{cases}$
- (M_p) $\Gamma_p(dz) = i_{p,d} \cdot d^{dz-(dz)_1-1} \prod_{k=0}^{d-1} \Gamma_p(z + \frac{k}{d}) \quad (p \nmid d)$

  $i_{p,d} := \prod_{k=0}^{d-1} \Gamma_p(\frac{k}{d}) \in \mu_4$, $z = z_0 + p z_1$ $(z_0 \in \{1, 2, \ldots, p\})$
- (Lerch_p) $\log_p \Gamma_p(z) = \zeta_p'(0, z)$ (Ferrero-Greenberg)

  $\zeta_p(s, z) := \sum_{m \geq 0,\ p \nmid (z+m)} (z+m)^{-s}$ の "p 進補間", $\ker \log_p = \mu_\infty \cdot p^{\mathbb{Q}}$

p 進 Γ 関数 $(p \neq 2)$

$$\Gamma_p(z) := \lim_{\mathbb{N} \ni n \to z} (-1)^n \prod_{k=1,\ p \nmid k}^{n-1} k \quad (z \in \mathbb{Z}_p)$$

- (D_p) $\Gamma_p(z+1) = z^* \Gamma_p(z)$, $z^* = -z \ (p \nmid z)$ または $-1 \ (p \mid z)$
- (M_p) $\Gamma_p(dz) = i_{p,d} \cdot d^{dz-(dz)_1-1} \prod_{k=0}^{d-1} \Gamma_p(z + \frac{k}{d}) \quad (p \nmid d)$

関数等式での特徴付け？

例

$f : \mathbb{Z}_p \to \mathbb{C}_p$: 連続, $f(z+1) = f(z) \Rightarrow f(z)$: 定数 $(\because \mathbb{N} \overset{\text{稠密}}{\subset} \mathbb{Z}_p)$
$\therefore G(z)$ s.t. $(D_p) \Rightarrow G(z)/\Gamma_p(z)$: 定数 $\Rightarrow G(z) = \Gamma_p(z) \cdot$ 定数

p 進 Γ 関数 $(p \neq 2)$

(M_p) $\Gamma_p(dz) = i_{p,d} \cdot d^{dz-(dz)_1-1} \prod_{k=0}^{d-1} \Gamma_p(z + \frac{k}{d})$ $(p \nmid d)$ での特徴付け？

命題 ([arXiv:1904.02879, Proposition 3.2])

$f\colon \mathbb{Z}_p \to \mathbb{C}_p$: 連続, $f(dz) = \displaystyle\prod_{k=0}^{d-1} f(z + \tfrac{k}{d})$ $(p \nmid {}^\forall d)$

$$\Rightarrow {}^\exists \alpha_k \text{ s.t. } f(1 + \textstyle\sum_{k=0}^{\infty} x_k p^k) = \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}}$$

1. $c_k := \frac{f(z+1)}{f(z)}$ は $k = \mathrm{ord}_p\, z$ のみによる, $\alpha_k = c_k \displaystyle\prod_{i=0}^{k-1} c_i^{(p-1)p^{k-1-i}}$
2. $c_0 = c_1 = \cdots \Rightarrow f(z) = c_0^{z-\frac{1}{2}}$
3. $c_1 = c_2 = \cdots \Rightarrow f(z) = c_0^{z-\frac{1}{2}} (c_1/c_0)^{z_1 + \frac{1}{2}}$

p 進 Γ 関数 $(p \neq 2)$

f: $\mathbb{Z}_p$ 上連続, $f(dz) = \prod_{k=0}^{d-1} f(z + \frac{k}{d})$ $(p \nmid {}^\forall d)$, $c_k := \frac{f(z+1)}{f(z)}$ $(\mathrm{ord}_p\, z = k)$

2. $c_0 = c_1 = \cdots \Rightarrow f(z) = c_0^{z-\frac{1}{2}}$
3. $c_1 = c_2 = \cdots \Rightarrow f(z) = c_0^{z-\frac{1}{2}}(c_1/c_0)^{z_1+\frac{1}{2}}$

- "p 倍公式": $\begin{cases} f(pz) = \prod_{k=0}^{p-1} f(z + \frac{k}{p}) \\ f(p(z + \frac{1}{p})) = \prod_{k=1}^{p} f(z + \frac{k}{p}) \end{cases}$

  $\Rightarrow \frac{f(pz)}{f(pz+1)} = \frac{f(z)}{f(z+1)} \Leftrightarrow c_k = c_{k+1}$
- $c_0 = 1 \Leftrightarrow f(2) = f(1) \Leftrightarrow \Gamma_p(2) = -\Gamma_p(1)$
- [thesis], [crelle, Lemma 4.2] $\Rightarrow$ ${}^\exists\Gamma_p$: $\mathbb{Q}_p - \mathbb{Z}_p$ 上連続, $\exists!$ mod μ_∞, s.t.

$$\Gamma_p(z+1) \equiv z\Gamma_p(z),\ \Gamma_p(2z) \equiv 2^{2z-\frac{1}{2}}\Gamma_p(z)\Gamma_p(z+\frac{1}{2}) \bmod \mu_\infty$$

代数的整数論

K/k: abel.ext., $\sigma \in \mathrm{Gal}(K/k) \Rightarrow \zeta(s,\sigma) = \sum_{\mathrm{Artin}:\, \mathfrak{a} \mapsto \sigma} N\mathfrak{a}^{-s}$ (部分 ζ 関数)

- $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, $\sigma_a : \zeta_n \mapsto \zeta_n^a \Rightarrow \zeta(s,\sigma_a) = m^{-s}\zeta(s, \frac{a}{m})$
  (L) $\Rightarrow \zeta'(0,\sigma_a) = \log\Gamma(\frac{a}{n}) - (\frac{1}{2} - \frac{a}{n})\log n - \frac{1}{2}\log 2\pi.$
- $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n)^+/\mathbb{Q})$, $\mathbb{Q}(\zeta_n)^+ := \mathbb{Q}(\zeta_n + \zeta_n^{-1})$
  $\sigma_a, \sigma_{n-a} \mapsto \overline{\sigma_a} : \zeta_n + \zeta_n^{-1} \mapsto \zeta_n^a + \zeta_n^{-a}$
  $\Rightarrow \zeta(s,\overline{\sigma_a}) = \zeta(s,\sigma_a) + \zeta(s,\sigma_{n-a})$
  $\Rightarrow \exp(-2\zeta'(0,\overline{\sigma_a})) = \left(\frac{\Gamma(\frac{a}{n})\Gamma(\frac{n-a}{n})}{2\pi}\right)^{-2} \overset{\text{Euler}}{=} (2\sin(\frac{a}{n}\pi))^2$
  $\fallingdotseq$ <u>円単数</u> $\in \mathbb{Q}(\zeta_n)^+$

予想 (Stark 予想 (実素点, rank one abelian))

$K \overset{\exists}{\hookrightarrow} \mathbb{R} \Rightarrow u(\sigma) := \exp(-2\zeta'(0,\sigma)) \in K$, $\tau(u(\sigma)) = u(\tau\sigma), \ldots$

$$K/k = \mathbb{Q}(\zeta_n)^+/\mathbb{Q} \quad \Rightarrow \quad \overline{\sigma_b}(\sin(\tfrac{a}{n}\pi)) = \pm\sin(\tfrac{ab}{n}\pi)$$

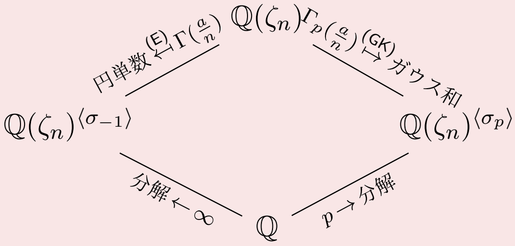
※ p 進版？ (Euler$_p$) $\Gamma_p(z)\Gamma_p(1-z) \in \mu_4$

$\{z, 1-z\} \leftrightarrow \sigma_{-1} \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \leftrightarrow \mathbb{Q}(\zeta_n)^+ = \mathbb{Q}(\zeta_n)^{\langle \sigma_{-1} \rangle}$

$$p \nmid n \Rightarrow \prod_{k=0}^{f-1} \Gamma_p(\langle \frac{ap^k}{n} \rangle) \fallingdotseq \underline{\text{ガウス和}} \in \mathbb{Q}(\zeta_n)^{\langle \sigma_p \rangle} \text{ (Gross-Koblitz)}$$

※ $\rightsquigarrow$ Gross-Stark 予想 (Dasgupta-Darmon-Pollack, Ventullo)

"まとめ"



Fermat 曲線

- $F_n\colon x^n + y^n = 1$, 種数 $\frac{(n-1)(n-2)}{2}$
- $\langle \omega_{\frac{r}{n},\frac{s}{n}} := x^{r-1}y^{s-n}dx \mid 0 < r, s < n,\ r+s \neq n \rangle = H^1_{dR}(F_n)$

$$\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}} \overset{t=x^n}{=} \int t^{\frac{r}{n}}(1-t)^{\frac{s}{n}}\,dt \risingdotseq B(\tfrac{r}{n},\tfrac{s}{n}) = \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} \quad \text{(Rohrlich)}$$

※ 線形代数 (“志村の周期記号”) $\Rightarrow \Gamma(\frac{a}{n}) = *** \prod(\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}})^{***}$

e.g., $B(\frac{1}{3},\frac{1}{3})^2 B(\frac{2}{3},\frac{2}{3}) = \frac{\Gamma(\frac{1}{3})^2\Gamma(\frac{1}{3})^2}{\Gamma(\frac{2}{3})^2}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})} = \frac{1}{3}\Gamma(\frac{1}{3})^3$

$H^1_{dR}(F_n)\otimes\overline{\mathbb{Q}_p} \cong H^1_{cris}(F_n\times\mathbb{F}_p)\otimes\overline{\mathbb{Q}_p} \curvearrowleft \Phi_\tau$: “絶対フロベニウス作用”

$$\Phi_\tau(\omega_{\frac{r}{n},\frac{s}{n}}) \risingdotseq \begin{cases} \frac{1}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))}\omega_{\tau(\frac{r}{n}),\tau(\frac{s}{n})} & (\frac{r}{n},\frac{s}{n}\in\mathbb{Z}_p) \\ \frac{B_p(\frac{r}{n},\frac{s}{n})}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))}\omega_{\tau(\frac{r}{n}),\tau(\frac{s}{n})} & (\frac{r}{n},\frac{s}{n}\notin\mathbb{Z}_p) \end{cases} \quad \text{(Coleman)}$$

Fermat 曲線

$H^1_{dR}(F_n) \otimes \overline{\mathbb{Q}_p} \cong H^1_{cris}(F_n \times \mathbb{F}_p) \otimes \overline{\mathbb{Q}_p} \curvearrowleft \Phi_\tau$: "絶対フロベニウス作用"

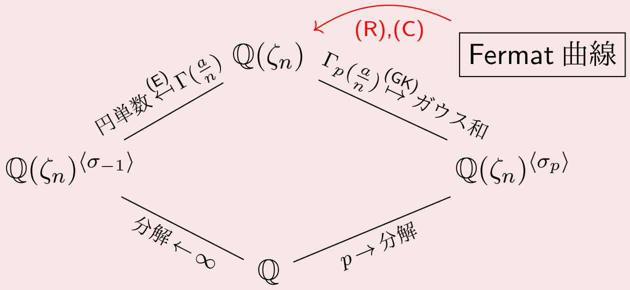
$$\Phi_\tau(\omega_{\frac{r}{n},\frac{s}{n}}) \doteqdot \begin{cases} \frac{1}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))}\omega_{\tau(\frac{r}{n}),\tau(\frac{s}{n})} & (\frac{r}{n}, \frac{s}{n} \in \mathbb{Z}_p) \\ \frac{B_p(\frac{r}{n},\frac{s}{n})}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))}\omega_{\tau(\frac{r}{n}),\tau(\frac{s}{n})} & (\frac{r}{n}, \frac{s}{n} \notin \mathbb{Z}_p) \end{cases} \quad \text{(Coleman)}$$

- $B_p(\alpha, \beta) = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}$
- $F_n \times \mathbb{F}_p \curvearrowleft \Phi_{cris}$: 絶対ロベニウス
- $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ s.t. $\tau|_{\mathbb{Q}_p^{ur}} = \sigma_p^{\deg \tau}$ with $\deg \tau \in \mathbb{Z}_p$

  $\Rightarrow H^1_{cris}(F_n/\mathbb{F}_p) \curvearrowleft \Phi_\tau := \Phi_{cris}^{\deg \tau} \otimes \tau$
- $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \curvearrowright \mu_\infty \overset{\zeta_n^a \leftrightarrow \frac{a}{n}}{=} [0,1) \cap \mathbb{Q}$
- (Coleman) $\Rightarrow$ (Gross-Koblitz) $p \nmid n \Rightarrow \prod_{k=0}^{f-1} \Gamma_p(\langle \frac{ap^k}{n} \rangle) \doteqdot$ ガウス和

- (R) $\cdots \displaystyle\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}} \doteqdot B(\tfrac{r}{n},\tfrac{s}{n})$
- (C) $\cdots \Phi_\tau(\omega_{\frac{r}{n},\frac{s}{n}}) \doteqdot \begin{cases} \dfrac{1}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))}\omega_{\tau(\frac{r}{n}),\tau(\frac{s}{n})} & (\frac{r}{n},\frac{s}{n}\in\mathbb{Z}_p) \\ \dfrac{B_p(\frac{r}{n},\frac{s}{n})}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))}\omega_{\tau(\frac{r}{n}),\tau(\frac{s}{n})} & (\frac{r}{n},\frac{s}{n}\notin\mathbb{Z}_p) \end{cases}$

"まとめ"



p 進周期環値 β 関数 ([crelle])

- c.f. $H_1^{sing}(F_n) \times H^1_{dR}(F_n) \to \mathbb{C}$, $(\gamma, \omega) \mapsto \int_\gamma \omega$

  p 進 Hodge 理論 $\Rightarrow$ $H_1^{sing}(F_n) \times H^1_{dR}(F_n) \to B_{dR}$, $(\gamma, \omega) \mapsto \int_{p,\gamma} \omega$

定義

$$\mathfrak{B}(\tfrac{r}{n}, \tfrac{s}{n}) := \begin{cases} \dfrac{B(\frac{r}{n},\frac{s}{n})}{\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}}} \langle \frac{r}{n} + \frac{s}{n} \rangle^{\lfloor \frac{r}{n} + \frac{s}{n} \rfloor} \int_{p,\gamma} \omega_{\frac{r}{n},\frac{s}{n}} & (\frac{r}{n}, \frac{s}{n} \in \mathbb{Z}_p) \\ \dfrac{B(\frac{r}{n},\frac{s}{n})}{\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}}} \dfrac{\int_{p,\gamma} \omega_{\frac{r}{n},\frac{s}{n}}}{B_p(\frac{r}{n},\frac{s}{n})} & (\frac{r}{n}, \frac{s}{n} \notin \mathbb{Z}_p) \end{cases} \in B_{cris} \overline{\mathbb{Q}_p}$$

(Coleman) $\Phi_\tau \curvearrowright H^1_{cris}(F_n/\mathbb{F}_p)$

$$\Rightarrow \frac{\Phi_\tau(\mathfrak{B}(\frac{r}{n},\frac{s}{n}))}{\mathfrak{B}(\tau(\frac{r}{n}),\tau(\frac{s}{n}))} = \begin{cases} \dfrac{(-1)^{\lfloor \tau(\frac{r}{n}) + \tau(\frac{s}{n}) \rfloor} p^{1 - \lfloor \frac{r}{n} + \frac{s}{n} \rfloor}}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))} & (\frac{r}{n}, \frac{s}{n} \in \mathbb{Z}_p,\ \deg \tau = 1) \\ \text{a root of unity} \cdot p^{\frac{\deg \tau}{2}} & (\frac{r}{n}, \frac{s}{n} \notin \mathbb{Z}_p) \end{cases}$$

$\Rightarrow \sigma_b(\sin(\frac{a}{n}\pi)) \equiv \sin(\frac{ab}{n}\pi) \bmod \mu_\infty$ if $\sigma_b(\zeta_n) = \zeta_n^b$

p 進周期環値 β 関数 ([crelle])

$$\mathfrak{B}(\tfrac{r}{n},\tfrac{s}{n}) := \begin{cases} \dfrac{B(\frac{r}{n},\frac{s}{n})}{\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}}} \langle \frac{r}{n}+\frac{s}{n}\rangle^{\lfloor \frac{r}{n}+\frac{s}{n}\rfloor} \int_{p,\gamma}\omega_{\frac{r}{n},\frac{s}{n}} & (\frac{r}{n},\frac{s}{n}\in\mathbb{Z}_p) \\ \dfrac{B(\frac{r}{n},\frac{s}{n})}{\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}}} \dfrac{\int_{p,\gamma}\omega_{\frac{r}{n},\frac{s}{n}}}{B_p(\frac{r}{n},\frac{s}{n})} & (\frac{r}{n},\frac{s}{n}\notin\mathbb{Z}_p) \end{cases} \in B_{cris}\overline{\mathbb{Q}_p}$$

(C) $\cdots$ $$\frac{\Phi_\tau(\mathfrak{B}(\frac{r}{n},\frac{s}{n}))}{\mathfrak{B}(\tau(\frac{r}{n}),\tau(\frac{s}{n}))} = \begin{cases} \dfrac{(-1)^{\lfloor \tau(\frac{r}{n})+\tau(\frac{s}{n})\rfloor} p^{1-\lfloor \frac{r}{n}+\frac{s}{n}\rfloor}}{B_p(\tau(\frac{r}{n}),\tau(\frac{s}{n}))} & (\frac{r}{n},\frac{s}{n}\in\mathbb{Z}_p,\ \deg\tau=1) \\ \text{a root of unity}\cdot p^{\frac{\deg\tau}{2}} & (\frac{r}{n},\frac{s}{n}\notin\mathbb{Z}_p) \end{cases}$$

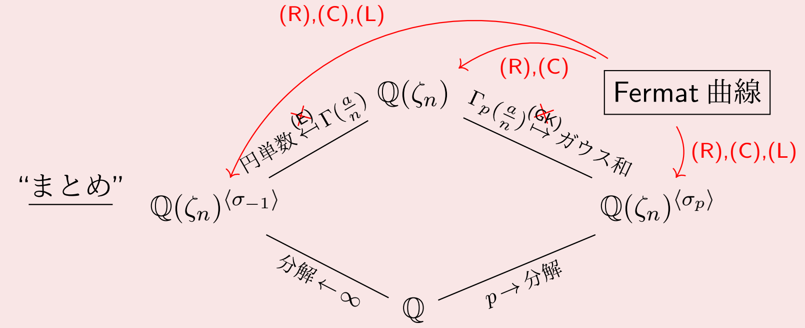
$^\forall p \Rightarrow \sigma_b(\sin(\frac{a}{n}\pi)) \equiv \sin(\frac{ab}{n}\pi) \bmod \mu_\infty$ if $\sigma_b(\zeta_n)=\zeta_n^b$

$$\because \cup : H^1\times H^1 \to H^2 \Rightarrow \frac{\int_{p,\gamma}\omega_{\frac{r}{n},\frac{s}{n}}}{\int_\gamma \omega_{\frac{r}{n},\frac{s}{n}}}\frac{\int_{p,\gamma'}\omega_{\frac{n-r}{n},\frac{n-s}{n}}}{\int_{\gamma'}\omega_{\frac{n-r}{n},\frac{n-s}{n}}} = \frac{\pi_p}{\pi},\ (\text{Euler}_p)$$

$$\Rightarrow \mathfrak{B}(\tfrac{r}{n},\tfrac{s}{n})\mathfrak{B}(\tfrac{n-r}{n},\tfrac{n-s}{n}) \doteqdot \frac{\sin(\frac{r}{n}\pi)\sin(\frac{s}{n}\pi)}{\sin(\frac{r+s}{n}\pi)},\ \Phi_\tau|_{\overline{\mathbb{Q}}} = \sigma_p^{\deg\tau}$$ □

※ $H^1\times H^1 \to H^2 \Rightarrow \int_\gamma \omega_{\frac{r}{n},\frac{s}{n}}\int_{\gamma'}\omega_{\frac{n-r}{n},\frac{n-s}{n}} \doteqdot \pi \overset{(L)}{\Rightarrow} \sin(\frac{a}{n}\pi)\in\overline{\mathbb{Q}}$

p 進周期環値 β 関数 ([crelle])



- 要精密版 (ねらい目？): $\mathrm{mod}\,\mu_\infty$, 単数性, (S,T) 版
- 一般化: $/\mathbb{Q} \Rightarrow$ / 総実体, ガンマ関数 ⇒ 多重ガンマ関数, 円単数, ガウス和 ⇒Stark 単数, Gross-Stark 単数, (Lerch) ⇒ 新谷公式, (Rohrlich) ⇒ 吉田予想 [Yoshida's book], (Coleman) ⇒ 予想 [arXiv:1706.03198]
- Fermat 曲線 ⇒ ???

Γ 関数版 ([arXiv:1904.02879])

$$I = I_{\mathbb{Q}(\zeta_n)} := \bigoplus_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})} \mathbb{Q}\sigma = \bigoplus_{b \in (\mathbb{Z}/n\mathbb{Z})^\times} \mathbb{Q}\sigma_b$$

定義 (志村の周期記号, [Yoshida's book, §2, Chap.III])

$[p : p_p] \colon I \to \overline{\mathbb{Q}}^\times \backslash (\mathbb{C}^\times \times B_{dR}^\times)/(\mu_\infty \times \mu_\infty)$: 線形 s.t.

$$\sum_{\lfloor \langle \frac{br}{n} \rangle + \langle \frac{bs}{n} \rangle \rfloor = 0} \sigma_b \mapsto [\pi : \pi_p]^{\lfloor \frac{s}{n} + \frac{r}{n} \rfloor - 1} [\textstyle\int_\gamma \omega_{\frac{s}{n},\frac{r}{n}} : \int_{p,\gamma} \omega_{\frac{s}{n},\frac{r}{n}}]$$

$$\sigma_b + \sigma_{-b} \mapsto [1 : 1]$$

(R) $\Rightarrow P(\frac{a}{n}) := \frac{\Gamma(\frac{a}{n})}{\sqrt{2\pi}} \left(\frac{\pi_p}{\pi}\right)^{\frac{1}{2} - \frac{a}{n}} \frac{p_p(\sum_{(b,n)=1} (\frac{1}{2} - \langle \frac{ab}{n} \rangle)\sigma_b)}{p(\sum_{(b,n)=1} (\frac{1}{2} - \langle \frac{ab}{n} \rangle)\sigma_b)} \in B_{cris}\overline{\mathbb{Q}_p}/\mu_\infty.$

(C) $\Leftrightarrow \begin{cases} \Gamma_p(\tau(\frac{a}{n})) \equiv p^{\frac{1}{2} - \frac{a}{n}} \frac{P(\tau(\frac{a}{n}))}{\Phi_\tau(P(\frac{a}{n}))} \bmod \mu_\infty & (\frac{a}{n} \in \mathbb{Z}_p, \deg \tau = 1) \\ \frac{\Gamma_p(\tau(\frac{a}{n}))}{\Gamma_p(\frac{a}{n})} \equiv p^{(\frac{a}{n} - \tau(\frac{a}{n}))\mathrm{ord}_p \frac{a}{n}} \frac{P(\tau(\frac{a}{n}))}{\Phi_\tau(P(\frac{a}{n}))} \bmod \mu_\infty & (\frac{a}{n} \notin \mathbb{Z}_p) \end{cases}$

“関数等式”

$$P(\tfrac{a}{n}) := \frac{\Gamma(\frac{a}{n})}{\sqrt{2\pi}}\left(\frac{\pi_p}{\pi}\right)^{\frac{1}{2}-\frac{a}{n}}\frac{p_p(\sum_{(b,n)=1}(\frac{1}{2}-\langle\frac{ab}{n}\rangle)\sigma_b)}{p(\sum_{(b,n)=1}(\frac{1}{2}-\langle\frac{ab}{n}\rangle)\sigma_b)} \in B_{cris}\overline{\mathbb{Q}_p}/\mu_\infty.$$

(C) $\Gamma_p(\frac{a}{n}) \equiv p^{\frac{1}{2}-\frac{a}{n}}\frac{P(\frac{a}{n})}{\Phi_\tau(P(\tau^{-1}(\frac{a}{n})))} \bmod \mu_\infty \ (\frac{a}{n}\in\mathbb{Z}_p, \deg\tau = 1)$

$$\inf\colon I_{\mathbb{Q}(\zeta_n)} \to I_{\mathbb{Q}(\zeta_{dn})},\ \sigma \mapsto \sum_{\tau|_{\mathbb{Q}(\zeta_n)}=\sigma}\tau \Rightarrow [p:p_p](\Xi) = [p:p_p](\inf(\Xi))$$

※ $F_{dn}\colon x^{dn}+y^{dn}=1 \to F_n\colon x^n+y^n=1,\ (x,y)\mapsto(x^d,y^d)$

$$\prod_{k=0}^{d-1}P(\tfrac{a}{n}+\tfrac{k}{d}) \leftrightarrow \sum_{k \bmod d}\ \sum_{b \bmod dn}(\tfrac{1}{2}-\langle\tfrac{b(ad+nk)}{dn}\rangle)\sigma_b \overset{B_1(dx)=\sum_{k=0}^{d-1}B_1(x+\frac{k}{d})}{=}$$

$$\sum_{b \bmod dn}(\tfrac{1}{2}-\langle\tfrac{bda}{n}\rangle)\sigma_b = \inf(\sum_{b \bmod n}(\tfrac{1}{2}-\langle\tfrac{bda}{n}\rangle)\sigma_b) \leftrightarrow P(\tfrac{da}{n})$$

$\Rightarrow$ 周期記号 (右辺) の “d 倍公式” ($^\forall d\in\mathbb{N}$, $d=p$ も OK)

Recall

命題 ([arXiv:1904.02879, Proposition 3.2])

$f: \mathbb{Z}_p \to \mathbb{C}_p$: 連続, $f(dz) = \prod_{k=0}^{d-1} f(z + \frac{k}{d})$ $(p \nmid {}^\forall d)$

$$\Rightarrow {}^\exists \alpha_k \text{ s.t. } f(1 + \sum_{k=0}^{\infty} x_k p^k) = \prod_{k=0}^{\infty} \alpha_k^{x_k - \frac{p-1}{2}}.$$

1. $c_k = \frac{f(z+1)}{f(z)}$ $(\mathrm{ord}_p\, z = k)$ は一定, $\alpha_k = c_k \prod_{i=0}^{k-1} c_i^{(p-1)p^{k-1-i}}$
2. $c_0 = c_1 = \cdots \Rightarrow f(z) = c_0^{z - \frac{1}{2}}$
3. $c_1 = c_2 = \cdots \Rightarrow f(z) = c_0^{z - \frac{1}{2}} (c_1/c_0)^{z_1 + \frac{1}{2}}$

注意

p 倍公式 $\Rightarrow c_k = c_{k+1}$

“関数等式”

$$P(\tfrac{a}{n}) := \frac{\Gamma(\frac{a}{n})}{\sqrt{2\pi}} \left(\frac{\pi_p}{\pi}\right)^{\frac{1}{2}-\frac{a}{n}} \frac{p_p(\sum_{(b,n)=1}(\frac{1}{2}-\langle\frac{ab}{n}\rangle)\sigma_b)}{p(\sum_{(b,n)=1}(\frac{1}{2}-\langle\frac{ab}{n}\rangle)\sigma_b)} \in B_{cris}\overline{\mathbb{Q}_p}/\mu_\infty.$$

(C) $\Gamma_p(\frac{a}{n}) \equiv p^{\frac{1}{2}-\frac{a}{n}} \dfrac{P(\frac{a}{n})}{\Phi_\tau(P(\tau^{-1}(\frac{a}{n})))} \bmod \mu_\infty$ $(\frac{a}{n} \in \mathbb{Z}_p, \deg\tau = 1)$

$$\prod_{k=0}^{d-1} P(\tfrac{a}{n}+\tfrac{k}{d}) \leftrightarrow \sum_{k \bmod d}\ \sum_{b \bmod dn} (\tfrac{1}{2} - \langle\tfrac{b(ad+nk)}{dn}\rangle)\sigma_b \leftrightarrow P(\tfrac{da}{n}) \ (^\forall d \in \mathbb{N})$$

定理 (弱 Coleman の公式の “別証明”)

“連続性” の仮定の元で $^\exists c_0, c_1$ s.t.

$$\Gamma_p(\tfrac{a}{n}) \equiv c_0^{\frac{a}{n}-\frac{1}{2}} (c_1/c_0)^{(\frac{a}{n})_1+\frac{1}{2}} \cdot p^{\frac{1}{2}-\frac{a}{n}} \frac{P(\frac{a}{n})}{\Phi_\tau(P(\tau^{-1}(\frac{a}{n})))} \bmod \mu_\infty$$

絶対フロベニウス作用の “p 進連続性”

- $p^{\frac{1}{2}-\frac{a}{n}}\frac{P(\frac{a}{n})}{\Phi_\tau(P(\tau^{-1}(\frac{a}{n})))}$ ((C) の右辺) が $\frac{a}{n}\in\mathbb{Z}_p$ に対して連続的
  $\Rightarrow$ 乗法公式, $c_1=c_2=\cdots$
- $p^{(\frac{a}{n}-\tau(\frac{a}{n}))\mathrm{ord}_p\frac{a}{n}}\frac{P(\tau(\frac{a}{n}))}{\Phi_\tau(P(\frac{a}{n}))}$ (もう一つの (C) の右辺) の p 進連続性
  $\overset{?}{\Rightarrow}$ $c_0=c_1$
- $=\Gamma_p(z)$ なんだから p 進連続的なのは間違いない
  “直接証明できるか” が問題
- “$\frac{\Phi_{cris}(\omega_{\frac{r}{n},\frac{s}{n}})}{\omega_{\langle\frac{pr}{n}\rangle,\langle\frac{ps}{n}\rangle}}$ が $\langle\frac{pr}{n}\rangle,\langle\frac{ps}{n}\rangle\in\mathbb{Z}_p$ に対して連続的” は直接示せる
  $(\leftrightarrow B_p(\alpha,\beta))$

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