On periods and the absolute Frobenius on Fermat curves

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L-functions and Motives in Niseko 2024

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This talk is related to these papers, particularly [K2].

[K1] Fermat curves and a refinement of the reciprocity law on cyclotomic units. J. Reine Angew. Math. 741 (2018), 255–273

[K2] Note on Coleman's formula for the absolute Frobenius on Fermat curves, Annales de l'Institut Fourier (2024)

[K3] On a common refinement of Stark units and Gross-Stark units, preprint (arXiv:1706.03198)

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Two formulas concerning Fermat curves $F_n \colon x^n + y^n = 1$

Rohrlich's formula (in Appendix by Rohrlich of Gross' paper)

Consider differential forms $\eta_{r,s} = x^{r-1} y^{s-n} dx$ $(0 < r, s < n,\ r + s \neq n)$ of the first and second kind on F_n.

$$\int_\gamma \eta_{r,s} \equiv \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} = B(\tfrac{r}{n}, \tfrac{s}{n}) \mod \mathbb{Q}(\zeta_n)^\times$$

for any closed path γ with $\int_\gamma \eta_{r,s} \neq 0$, $\zeta_n := e^{\frac{2\pi i}{n}}$.

Coleman's formula (Frobenius matrix)

Consider the absolute Frobenius action $\mathrm{Fr}_p \curvearrowright H^1_{dR}(F_n, \mathbb{Q}_p)$.

$$\mathrm{Fr}_p(\eta_{r,s}) \equiv \frac{\Gamma_p(\frac{r'+s'}{n})}{\Gamma_p(\frac{r'}{n})\Gamma_p(\frac{s'}{n})} \cdot \eta_{r',s'} \mod \mathbb{Q}^\times \mu_\infty$$

for $p \nmid n$, $pr \equiv r' \bmod n$, μ_∞: group of roots of unity

Two formulas concerning Fermat curves $F_n: x^n + y^n = 1$

Rohrlich's formula

$$\int_\gamma \eta_{r,s} \equiv \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})} = B(\tfrac{r}{n}, \tfrac{s}{n}) \mod \mathbb{Q}(\zeta_n)^\times$$

$\eta_{r,s} = x^{r-1}y^{s-n}dx$ $(0 < r, s < n,\ r + s \neq n)$, γ: closed path.

Coleman's formula

$$\mathrm{Fr}_p(\eta_{r,s}) \equiv \frac{\Gamma_p(\frac{r'+s'}{n})}{\Gamma_p(\frac{r'}{n})\Gamma_p(\frac{s'}{n})} \cdot \eta_{r',s'} \mod \mathbb{Q}^\times \mu_\infty$$

abs.Frob.$\mathrm{Fr}_p \curvearrowright H^1_{dR}(F_n, \mathbb{Q}_p)$, $p \nmid n$, $pr \equiv r' \bmod n$, μ_∞: roots of unity

- Similar phenomenon occurs between Chowla-Selberg formula for the eta-function and Ogus' formula for Fr_p on elliptic curves with CM. In these cases, $\prod_{a=1}^{d-1} \Gamma_*(\frac{a}{d})^{\chi_d(a)}$ $(* = \emptyset, p)$ appears.
- The theme of my talk is "why does such a similarity occur?"

Period symbol

- Before we begin the main discussion, we introduce Shimura's period symbol, obtained by decomposing period integrals.
- Then we can "decompose" Rohrlich's formula $\int_\gamma \eta_{r,s} \equiv \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})}$ into (single) Γ-function version.
- (by CM-theory, solve linear simultaneous equations $\{\log\Gamma(\frac{r}{n}) + \log\Gamma(\frac{s}{n}) - \log\Gamma(\frac{r+s}{n}) \fallingdotseq \log\int_\gamma \eta_{r,s}\}_{r,s} \Rightarrow \log\Gamma(\frac{r}{n}) \fallingdotseq ?$)

Definition (Shimura's period symbol)

K: CM-field of degree $2n$, $A/\overline{\mathbb{Q}}$: abelian variety with CM by K. Then

$$K \cong \mathrm{End}(A) \otimes \mathbb{Q} \curvearrowright H^1_{dR}(A, \overline{\mathbb{Q}}) = \bigoplus_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} \overline{\mathbb{Q}} \cdot \eta_\sigma,$$

where K acts η_σ through σ. CM type $\Xi_A := \{\sigma \mid \eta_\sigma \text{ is holom.}\}$, $|\Xi_A| = n$.

$$\exists p_K(\sigma,\tau) \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times \text{ s.t. } \prod_{\tau \in \Xi_A} p_K(\sigma,\tau) \equiv \pi^{-1} \int_\gamma \eta_\sigma \mod \overline{\mathbb{Q}}^\times.$$

Period symbol

Definition (Shimura's period symbol)

K: CM-field, $A/\overline{\mathbb{Q}}$: ab.var. with CM by K.

$$\exists p_K(\sigma,\tau) \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times \text{ s.t. } \prod_{\tau\in\Xi_A} p_K(\sigma,\tau) \equiv \pi^{-1}\int_\gamma \eta_\sigma \mod \overline{\mathbb{Q}}^\times.$$

e.g, $K=\mathbb{Q}(\zeta_n)$, $\zeta_n \curvearrowright F_n : x^n+y^n=1$, $\sigma_a : \mathbb{Q}(\zeta_n)\to\mathbb{C}$, $\zeta_n\mapsto\zeta_n^a$
Any CM-type is in the form $\Xi_{r,s} := \{\sigma_a \mid \langle\frac{ar}{n}\rangle + \langle\frac{as}{n}\rangle + \langle\frac{a(n-r-s)}{n}\rangle = 1\}$

$$\prod_{\tau\in\Xi_{r,s}} p_{\mathbb{Q}(\zeta_n)}(\mathrm{id},\tau) \equiv \pi^{-1}\int_\gamma \eta_{r,s} \mod \overline{\mathbb{Q}}^\times \quad (r+s<n).$$

Yoshida and K. "solved" Rohrlich's formula: $\int_\gamma \eta_{r,s} \equiv \frac{\Gamma(\frac{r}{n})\Gamma(\frac{s}{n})}{\Gamma(\frac{r+s}{n})}$ for $\Gamma(\frac{r}{n})$:

$$\Gamma(\tfrac{a}{n}) \equiv \pi^{1-\langle\frac{a}{n}\rangle} \prod_{b \bmod n\in(\mathbb{Z}/n\mathbb{Z})^\times} p_{\mathbb{Q}(\zeta_n)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle} \mod \overline{\mathbb{Q}}^\times.$$

p-adic analogue

- We may regard period integral: $H_1^{sing} \times H^1_{dR} \to \mathbb{C}$, $(\gamma,\eta) \mapsto \int_\gamma \eta$ as the dual of de Rham isom.: $H^1_{sing} \otimes \mathbb{C} \cong H^1_{dR} \otimes \mathbb{C}$
- Define "p-adic period integral" and "p-adic period symbol"

$$\int_{\gamma,p} \eta \in B_{dR}, \quad p_{K,p}(\sigma,\tau) \in B_{dR}^\times/\overline{\mathbb{Q}}^\times$$

  by replacing de Rham isom. with the composite of comparison isom.'s

$$H^1_{sing} \otimes B_{dR} \cong H^1_{et} \otimes B_{dR} \cong H^1_{dR} \otimes B_{dR}$$

- (In [K3], we used motives associated to alg.Hecke char.s instead of ab.var.s with CM, to avoid problems with "field of definition")
- Each symbol has ambiguity of $\bmod \overline{\mathbb{Q}}^\times$ since $\int_{\gamma,*} \eta$ $(* = \emptyset, p)$ depends on the choices of γ, η. Assuming that we take the same γ, η for both symbols, the following "ratio" is well-defined up tp μ_∞.

$$[p_K(\sigma,\tau) : p_{K,p}(\sigma,\tau)] \in (\mathbb{C}^\times \times B_{dR}^\times)/\overline{\mathbb{Q}}^\times$$

- CM $\Rightarrow$ pot.good $\Rightarrow \int_{\gamma,p} \eta \in B_{cris}\overline{\mathbb{Q}_p} \Rightarrow p_{K,p}(\sigma,\tau) \curvearrowleft$ abs.Frob. Fr_p
- Since Fr_p acts trivially on H^1_{et}, the action on η = that on $\int_{\gamma,p} \eta, p_{K,p}$

p-adic analogue

- We defined $p_{K,p}(\sigma,\tau) \in B_{dR}^{\times}/\overline{\mathbb{Q}}^{\times}$ with the action of Fr_p
- The "ratio" $[p_K(\sigma,\tau) : p_{K,p}(\sigma,\tau)] \in (\mathbb{C}^{\times} \times B_{dR}^{\times})/\overline{\mathbb{Q}}^{\times}$ is well-defined up tp μ_∞

We can "solve" Coleman's formula

$$\mathrm{Fr}_p(\eta_{r,s}) \equiv \frac{\Gamma_p(\frac{r'+s'}{n})}{\Gamma_p(\frac{r'}{n})\Gamma_p(\frac{s'}{n})} \cdot \eta_{r',s'} \mod \mathbb{Q}^{\times}\mu_\infty \quad (p \nmid n)$$

for $\Gamma_p(z)$, by a similar manner to the case of Rohrlich's formula:

Corollary ([K2, K3])

We put $P(\frac{a}{n}) := \frac{\Gamma(\frac{a}{n})\pi_p^{\frac{1}{2}-\langle\frac{a}{n}\rangle}\prod_b p_{\mathbb{Q}(\zeta_n),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle}}{\pi^{1-\langle\frac{a}{n}\rangle}\prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle}} \in B_{dR}^{\times}/\mu_\infty$. Then

$$\Gamma_p(\tfrac{a'}{n}) \equiv p^{\frac{1}{2}-\langle\frac{a}{n}\rangle}\frac{P(\frac{a'}{n})}{\mathrm{Fr}_p(P(\frac{a}{n}))} \mod \mu_\infty. \quad (a' \equiv pa \bmod n)$$

summary 1

By "linear algebra", we may rewrite two formulas as follows:

Rohrlich's formula (in terms of period symbol)

$$\Gamma(\tfrac{a}{n}) \equiv \pi^{1-\langle \frac{a}{n} \rangle} \prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{n} \rangle} \mod \overline{\mathbb{Q}}^\times.$$

$$\Rightarrow P(\tfrac{a}{n}) := \frac{\Gamma(\frac{a}{n})\pi_p^{\frac{1}{2}-\langle \frac{a}{n} \rangle} \prod_b p_{\mathbb{Q}(\zeta_n),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{n} \rangle}}{\pi^{1-\langle \frac{a}{n} \rangle} \prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle \frac{ab}{n} \rangle}} \in B_{dR}^\times/\mu_\infty$$

Coleman's formula (in terms of period symbols)

$$\Gamma_p(\tfrac{a'}{n}) \equiv p^{\frac{1}{2}-\langle \frac{a}{n} \rangle} \frac{P(\frac{a'}{n})}{\mathrm{Fr}_p(P(\frac{a}{n}))} \mod \mu_\infty.$$

- These also imply Chowla-Selberg formula and Ogus' formula up to $\mathrm{mod}\overline{\mathbb{Q}}^\times$.
- Why Γ and Γ_p appear?

functional equations vs. monomial relations

Γ-function is characterized by some functional equations and smoothness:

(D) $\Gamma(z+1) = z\Gamma(z)$ (Difference equation)

(M) $\Gamma(dz) = \frac{d^{dz-\frac{1}{2}}}{(2\pi)^{\frac{d-1}{2}}} \prod_{k=0}^{d-1} \Gamma(z+\frac{k}{d})$ $(d \in \mathbb{N})$ (Multiplication formula)

characterized by $[C^1, (D), (M)]$ or $[C^1, (M), \lim_{z\to 0} z\Gamma(z) = 1]$ or $\cdots$.

C^1: a function with continuous derivatives.

Proposition

RHS of Rohrlich's formula $\Gamma(\frac{a}{n}) \equiv \pi^{1-\langle\frac{a}{n}\rangle} \prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle} =: G(\frac{a}{n})$ satisfies the "same fun.eq." as (M):

(M') $G(\frac{da}{n}) \equiv \frac{1}{(2\pi)^{\frac{d-1}{2}}} \prod_{k=0}^{d-1} G(\frac{a}{n}+\frac{k}{d}) \mod \overline{\mathbb{Q}}^\times$

It is trivial since LHS $\equiv$ RHS. I mean we can provide an alternative proof.

functional equations vs. monomial relations

(D) $\Gamma(z+1) = z\Gamma(z)$

(M) $\Gamma(dz) = \frac{d^{dz-\frac{1}{2}}}{(2\pi)^{\frac{d-1}{2}}} \prod_{k=0}^{d-1} \Gamma(z+\frac{k}{d})$ $(d \in \mathbb{N})$

characterized by $[C^1, (D), (M)]$ or $[C^1, (M), \lim_{z\to 0} z\Gamma(z) = 1]$ or $\cdots$.

Rohrlich: $\Gamma(\frac{a}{n}) \equiv G(\frac{a}{n}) := \pi^{1-\langle \frac{a}{n} \rangle} \prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id}, \sigma_b)^{\frac{1}{2}-\langle \frac{ab}{n} \rangle} \mod \overline{\mathbb{Q}}^\times$

(M') $G(\frac{da}{n}) \equiv \frac{1}{(2\pi)^{\frac{d-1}{2}}} \prod_{k=0}^{d-1} G(\frac{a}{n}+\frac{k}{d}) \mod \overline{\mathbb{Q}}^\times$

(Proof) Period (symbol) satisfies some monomial relations including

(E) $p_K(\widetilde{\sigma}|_K, \tau) \equiv \prod_{\widetilde{\tau}|_K=\tau} p_L(\widetilde{\sigma}, \widetilde{\tau}) \mod \overline{\mathbb{Q}}^\times$ for extension $K \subset L$

The case of $\mathbb{Q}(\zeta_n) \subset \mathbb{Q}(\zeta_{dn})$ relates $G(\frac{*}{nd})$ to $G(\frac{*}{n})$. □

- $(E) \Leftrightarrow L(s, \chi \circ N_{L/K}) = L(s, \chi)$, $M(\chi \circ N_{L/K}) = M(\chi) \otimes_K L$
- We may say(?) Γ-function, characterized by (M), appears naturally
- p-adic case is more concrete

Main result ([K2])

(D) $\Gamma(z+1) = z\Gamma(z)$

(M) $\Gamma(dz) = \frac{d^{dz-\frac{1}{2}}}{(2\pi)^{\frac{d-1}{2}}} \prod_{k=0}^{d-1} \Gamma(z+\frac{k}{d})$ $(d \in \mathbb{N})$

characterized by $[C^1, (D), (M)]$ or $[C^1, (M), \lim_{z\to 0} z\Gamma(z) = 1]$ or $\cdots$.

The following p-adic analogues are well-known:

(D_p) $\Gamma_p(z+1) = z^*\Gamma_p(z)$, $z^* = -z$ $(p \nmid z)$, -1 $(p \mid z)$

(M_p) $\Gamma_p(dz) = i_{p,d} \cdot d^{dz-(dz)_1-1} \prod_{k=0}^{d-1} \Gamma_p(z+\frac{k}{d})$ $(p \nmid d,\ i_{p,d} \in \mu_4)$

characterized by $[C^0, (D_p), \Gamma_p(1) = -1]$.

C^0: a continuous function in the p-adic topology

For $z \in \mathbb{Z}_p$, we define $z_0 \in \{1, \ldots, p\}$, $z_1 \in \mathbb{Z}_p$ by $z = z_0 + pz_1$.

Main result ([K2])

(D_p) $\Gamma_p(z+1) = z^*\Gamma_p(z)$, $z^* = -z$ $(p \nmid z)$, -1 $(p \mid z)$

(M_p) $\Gamma_p(dz) = i_{p,d} \cdot d^{dz-(dz)_1-1} \prod_{k=0}^{d-1} \Gamma_p(z+\frac{k}{d})$ $(p \nmid d,\ i_{p,d} \in \mu_4)$

characterized by $[C^0,\ (D_p),\ \Gamma_p(1) = -1]$.

The following provides another characterization:

Proposition

$f \in C^0(\mathbb{Z}_p)$, $f(dz) = \prod_{k=0}^{d-1} f(z+\frac{k}{d})$ $(p \nmid d)$, $\frac{f(p^k+1)}{f(p^k)} = \frac{f(p^{k+1}+1)}{f(p^{k+1})}$ $(k \in \mathbb{N})$

$\Rightarrow f(z) = a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}}$ $({}^\exists a, b$: constants$)$.

- Namely, (M_p) and $\frac{\Gamma_p(p^k+1)}{\Gamma_p(p^k)} = -1$ $(k \geq 1)$ characterize $\Gamma_p(z)$ up to "p-adic $\exp$-functions"
- $\frac{f(p^k+1)}{f(p^k)} = \frac{f(p^{k+1}+1)}{f(p^{k+1})}$ corresponds to "p-multiplication formula"

- $\frac{f(p^k+1)}{f(p^k)} = \frac{f(p^{k+1}+1)}{f(p^{k+1})}$ corresponds to "p-multiplication formula"

Let us assume p-multiplication formula holds,

$$f(pz) = \prod_{k=0}^{p-1} f(z + \tfrac{k}{p})$$

although it doesn't actually make sense because k/p are not p-adic integers.
However, if it were to hold, shifting the equation by $1/p$ would give us the next equation.

$$f(p(z + \tfrac{1}{p})) = \prod_{k=1}^{p} f(z + \tfrac{k}{p})$$

Then, by dividing both sides, we obtain this equation.

$$\frac{f(pz)}{f(pz+1)} = \frac{f(z)}{f(z+1)},$$

which do make sense since $pz, pz+1, z, z+1 \in \mathbb{Z}_p$. So the condition $\frac{f(p^k+1)}{f(p^k)} = \frac{f(p^{k+1}+1)}{f(p^{k+1})}$ may be regarded as a modification of p-multiplication formula.

Main result ([K2])

Proposition

$f \in C^0(\mathbb{Z}_p)$ satisfying

- prime-to-p multiplication formula: $f(dz) = \prod_{k=0}^{d-1} f(z + \frac{k}{d})$ $(p \nmid d)$
- "p-multiplication formula": $\frac{f(p^k+1)}{f(p^k)} = \frac{f(p^{k+1}+1)}{f(p^{k+1})}$ $(k \in \mathbb{N})$

$\Rightarrow f(z) = a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}}$ $({}^\exists a, b$: constants).

Coleman's formula: $\Gamma_p(\frac{a'}{n}) \equiv p^{\frac{1}{2}-\langle \frac{a}{n} \rangle} \frac{P(\frac{a'}{n})}{\mathrm{Fr}_p(P(\frac{a}{n}))} \mod \mu_\infty$.

Theorem

Under an assumption of "p-adic continuity of Frobenius action" ($\fallingdotseq$ the RHS of Coleman's formula is continuous in p-adic topology, including the case of $p \mid n$), Coleman's formula holds automatically, up to p-adic $\exp$-functions.

Proposition

$f \in C^0(\mathbb{Z}_p)$ satisfying

- prime-to-p multiplication formula: $f(dz) = \prod_{k=0}^{d-1} f(z + \frac{k}{d})$ $(p \nmid d)$
- "p-multiplication formula": $\frac{f(p^k+1)}{f(p^k)} = \frac{f(p^{k+1}+1)}{f(p^{k+1})}$ $(k \in \mathbb{N})$

$\Rightarrow f(z) = a^{z-\frac{1}{2}} b^{z_1+\frac{1}{2}}$ $({}^\exists a, b$: constants).

Coleman's formula: $\Gamma_p(\frac{a'}{n}) \equiv p^{\frac{1}{2}-\langle\frac{a}{n}\rangle} \frac{P(\frac{a'}{n})}{\mathrm{Fr}_p(P(\frac{a}{n}))} \mod \mu_\infty$.

Theorem

Under an assumption of "p-adic continuity of Frobenius action", we have

1. $f(\frac{a'}{n}) :=$ RHS/LHS satisfies conditions of Proposition.
2. In particular, (without using Coelman's formula) we see that ${}^\exists a, b$ s.t.
$$\Gamma_p(\tfrac{a}{n}) \equiv a^{\frac{a}{n}-\frac{1}{2}} b^{(\frac{a}{n})_1+\frac{1}{2}} \cdot p^{\frac{1}{2}-\langle\frac{a}{n}\rangle} \frac{P(\frac{a'}{n})}{\mathrm{Fr}_p(P(\frac{a}{n}))} \mod \mu_\infty.$$

(proof) Every part of $P(\frac{a}{n}) = \frac{\Gamma(\frac{a}{n})\pi_p^{\frac{1}{2}-\langle\frac{a}{n}\rangle} \prod_b p_{\mathbb{Q}(\zeta_n),p}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle}}{\pi^{1-\langle\frac{a}{n}\rangle} \prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id},\sigma_b)^{\frac{1}{2}-\langle\frac{ab}{n}\rangle}}$ and LHS satisfy multiplication formula or its variants: (M) or (M_p) or (M'). □

summary 2

1. $\Gamma(z)$ is characterized by (M) $\Gamma(dz) = \cdots \prod_{k=0}^{d-1} \Gamma(z + \frac{k}{d})$ and $\cdots$.
2. Period symbol satisfies (E) $p_K(\widetilde{\sigma}|_K, \tau) \equiv \prod_{\widetilde{\tau}|_K = \tau} p_L(\widetilde{\sigma}, \widetilde{\tau}) \mod \overline{\mathbb{Q}}^\times$.
3. (E) transforms into (M') $G(\frac{da}{n}) \equiv \frac{1}{(2\pi)^{\frac{d-1}{2}}} \prod_{k=0}^{d-1} G(\frac{a}{n} + \frac{k}{d})$ for
   $G(\frac{a}{n}) := \pi^{1-\langle \frac{a}{n} \rangle} \prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id}, \sigma_b)^{\frac{1}{2} - \langle \frac{ab}{n} \rangle}$
4. Thus it is natural(?) that $\Gamma(z)$ appears in Rohrlich's formula for $G(z)$.
5. $\Gamma_p(z)$ is almost characterized by "multiplication formulas" (M_p), $\frac{f(p^k+1)}{f(p^k)} = \frac{f(p^{k+1}+1)}{f(p^{k+1})}$.
6. p-adic period symbol also satisfies (E) and (M')
7. Hence $\Gamma_p(z)$ appears automatically in Coleman's formula for
   $p^{\frac{1}{2} - \langle \frac{a}{n} \rangle} \frac{P(\frac{a'}{n})}{\mathrm{Fr}_p(P(\frac{a}{n}))}$ with $P(\frac{a}{n}) = \frac{\Gamma(\frac{a}{n}) \pi_p^{\frac{1}{2} - \langle \frac{a}{n} \rangle} \prod_b p_{\mathbb{Q}(\zeta_n),p}(\mathrm{id}, \sigma_b)^{\frac{1}{2} - \langle \frac{ab}{n} \rangle}}{\pi^{1-\langle \frac{a}{n} \rangle} \prod_b p_{\mathbb{Q}(\zeta_n)}(\mathrm{id}, \sigma_b)^{\frac{1}{2} - \langle \frac{ab}{n} \rangle}}$,
   under the assumption of "p-adic continuity of Frobenius action".
8. A more canonical formulation (and its proof) of "p-adic continuity of $\cdots$" in a general setting seems to be an interesting problem.

p-adic continuity of Frobenius action

For Fermat curves,

- $H^1_{dR}(F_n, \mathbb{Q}_p) = \langle \eta_{r,s,n} = x^{r-1}y^{s-n}dx \mid 0 < r, s < n,\ r+s \neq n \rangle_{\mathbb{Q}_p}$
- $\mathrm{Fr}_p \curvearrowright H^1_{dR}(F_n, \mathbb{Q}_p)$, $\exists \alpha_{r,s,n} \in \mathbb{Q}_p$ s.t. $\mathrm{Fr}_p(\eta_{r,s,n}) = \alpha_{r,s,n}\eta_{r',s',n}$ $(r' \equiv pr \mod n)$

"p-adic continuity of Frobenius action":

$$\left|\frac{r_1'}{n_1} - \frac{r_2'}{n_2}\right|_p, \left|\frac{s_1'}{n_1} - \frac{s_2'}{n_2}\right|_p \to 0 \Rightarrow |\alpha_{r_1,s_1,n_1} - \alpha_{r_2,s_2,n_2}|_p \to 0$$

$\because$ Coefficients of Laurent series' expansions of $\eta_{r_1,s_1,n_1}, \eta_{r_2,s_2,n_2}$ are close. □

Can these be generalized in the following cases?

- (A certain family of) algebraic curves C with $J(C)$ having CM
- Abelian varieties with CM
- Motives for Algebraic Hecke characters

[K1] Fermat curves and a refinement of the reciprocity law on cyclotomic units. J. Reine Angew. Math. 741 (2018), 255–273

[K2] Note on Coleman's formula for the absolute Frobenius on Fermat curves, Annales de l'Institut Fourier (2024)

[K3] On a common refinement of Stark units and Gross-Stark units, preprint (arXiv:1706.03198)

[Y] H.Yoshida, Absolute CM-Periods, Math.Surveys Monogr.106, (2003)

summary of [K1]

1. Define a p-adic-period-ring-valued beta function by

$$\mathfrak{B}(\tfrac{r}{n}, \tfrac{s}{n}) := \frac{B(\frac{r}{n}, \frac{s}{n}) \times \int_{\gamma,p} \eta_{r,s}}{\int_{\gamma} \eta_{r,s}}$$

2. Write a cyclotomic unit: $u_{a \bmod n} := (2\sin(\frac{a}{n}\pi))^2$ as a product of $\mathfrak{B}$.
3. Colman's formula is a refinement of the reciprocity law on cyclotomic units: $\sigma_p(u_{a \bmod n}) = u_{pa \bmod n}$.

summary of [Y], [K3]

Conjectural generalization, form $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ to CM-field $\overset{\text{ab.ex.}}{/}$ totally real field.