Hilbert-Siegel modular forms and automorphic forms on adele groups

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1 Hilbert-Siegel modular group

Let F be a totally real number field of degree d. We denote the ring of integers and the unit group of F by $\mathfrak{o} = \mathfrak{o}_F$ and $\mathfrak{o}^\times = \mathfrak{o}_F^\times$.

The embeddings of F into $\mathbb{R}$ are denoted by $\iota^{(1)}, \ldots, \iota^{(d)}$. For $a \in F$, we set $\iota^{(i)}(a) = a^{(i)}$ $(i = 1, \ldots, d)$. $a \in F$ is totally positive if $a^{(1)}, \ldots, a^{(d)} > 0$. We write $a \gg 0$ if $a \in F$ is totally positive. The set of totally real elements of F is denoted by $F_+^\times$. Put $\mathfrak{o}_{F,+}^\times = \mathfrak{o}_F^\times \cap F_+^\times$. This is a subgroup of $\mathfrak{o}_F^\times$ of finite index.

For a ring R, the symplectic group $\mathrm{Sp}_m(R)$ is given by

$$\mathrm{Sp}_m(R) = \{\gamma \in \mathrm{SL}_{2m}(R) \,|\, {}^t g w_m g = w_m\}, \quad w_m = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2m}(R)$,

$$g \in \mathrm{Sp}_m(R) \iff A\,{}^tB = B\,{}^tA,\ C\,{}^tC = D\,{}^tC,\ A\,{}^tD - B\,{}^tC = \mathbf{1}_m.$$

Definition 1. For an integral ideal $\mathfrak{n}$ of F, we put

$$\Gamma(\mathfrak{n}) = \{\gamma \in \mathrm{Sp}_m(\mathfrak{o}_F) \,|\, \gamma - \mathbf{1}_{2m} \in \mathrm{M}_{2m}(\mathfrak{n})\}.$$

$\Gamma(\mathfrak{n})$ is called the principal congruence subgroup of $\mathrm{Sp}_m(\mathfrak{o}_F)$ of level $\mathfrak{n}$.

Definition 2. A subgroup $\Gamma \subset \mathrm{Sp}_m(F)$ is called a congruence subgroup of $\mathrm{Sp}_m(F)$ if there is a integral ideal $\mathfrak{n}$ such that $\Gamma(\mathfrak{n}) \subset \Gamma$ and $[\Gamma : \Gamma(\mathfrak{n})] < \infty$.

Example: For fractional ideal $\mathfrak{a}$, $\mathfrak{b}$ of F such that $\mathfrak{ab} \subset \mathfrak{o}_F$,

$$\Gamma[\mathfrak{a}, \mathfrak{b}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_m^+(F) \,\middle|\, a, d \in \mathrm{M}_m(\mathfrak{o}),\ b \in \mathrm{M}_m(\mathfrak{a}),\ c \in \mathrm{M}_m(\mathfrak{b}) \right\}$$

is a congruence subgroup of $\mathrm{Sp}_m(F)$.

Let $\mathbb{A} = \mathbb{A}_F$ be the adele ring of F. It is known that the strong approximation property holds for Sp_m, i.e., for any place v of F, $\mathrm{Sp}_m(F)\mathrm{Sp}_m(F_v)$ is dense in $\mathrm{Sp}_m(\mathbb{A})$. In particular, $\mathrm{Sp}_m(F)$ is dense in $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$, where $\mathbb{A}_{\mathrm{fin}}$ is the finite adele ring of F, when $\mathrm{Sp}_m(F)$ is regarded as a subgroup of $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$.

As a fundamental system of neighbourhood of the unit element of $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$, one can choose

$$\mathbf{K}(\mathfrak{n}) = \{ g \in \mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}}) \mid g - \mathbf{1}_{2m} \in \mathrm{M}_{2m}(\mathfrak{n}\hat{\mathfrak{o}}) \} .$$

Here, $\mathfrak{n}$ extends over all integral ideals of F. By definition, we have $\mathbf{K}(\mathfrak{n}) \cap \mathrm{Sp}_m(F) = \Gamma(\mathfrak{n})$.

Theorem 1. *We think of* $\mathrm{Sp}_m(F)$ *as a subgroup of* $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$. *Let* $\Gamma \subset \mathrm{Sp}_m(F)$ *be a congruence subgroup. Then the closure* $\bar{\Gamma}$ *of* Γ *in* $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$ *is a compact open subgroup of* $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$.

Conversely, let $C \subset \mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$ *be a compact open subgroup. Then* $C \cap \mathrm{Sp}_m(F)$ *is a congruence subgroup of* $\mathrm{Sp}_m(F)$.

By this correspondence, congruence subgroups of $\mathrm{Sp}_m(F)$ *correspond to compact open subgroups of* $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$ *in one-to-one way.*

***Proof*.** Suppose that C is a compact open subgroup of $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$ and that $C \supset \mathbf{K}(\mathfrak{n})$. Put $\Gamma = \mathrm{Sp}_m(F) \cap C$. Since C is an open and closed subgroup of $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$, we have $\overline{\Gamma} = C$ by the strong approximation property. In particular, Γ contains an element of any left coset of $\mathbf{K}(\mathfrak{n})$, and so $[C : \mathbf{K}(\mathfrak{n})] = [C : \Gamma \cdot \mathbf{K}(\mathfrak{n})] = [\Gamma : \Gamma \cap \mathbf{K}(\mathfrak{n})] = [\Gamma : \Gamma(\mathfrak{n})]$. It follows that Γ is a congruence subgroup of $\mathrm{Sp}_m(F)$. In particular, considering the case $C = \mathbf{K}(\mathfrak{n})$, we have $\overline{\Gamma(\mathfrak{n})} = \mathbf{K}(\mathfrak{n})$.

Conversely, suppose that $\Gamma \supset \Gamma(\mathfrak{n})$ is a congruence subgroup of $\mathrm{Sp}_m(F)$. Then the closure $\bar{\Gamma}$ contains $\overline{\Gamma(\mathfrak{n})} = \mathbf{K}(\mathfrak{n})$ as a subgroup of finite index. It follows that $\bar{\Gamma}$ is a compact open subgroup of $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$. Moreover, since $\Gamma \cap \mathbf{K}(\mathfrak{n}) = \Gamma(\mathfrak{n})$, we hve $[\Gamma : \Gamma(\mathfrak{n})] = [\bar{\Gamma} : \mathbf{K}(\mathfrak{n})]$. Hence we have $\bar{\Gamma} \cap \mathrm{Sp}_m(F) = \Gamma$. □

2 Hilbert-Siegel modular forms

A real symmetrix matrix $Y \in \mathrm{Sym}_m(\mathbb{R})$ of size m is positive definite if all the eigenvalues of Y are positive. Write $Y > 0$ if Y is positive definite. Put $\mathrm{Sym}_m^+(\mathbb{R}) = \{Y \in \mathrm{Sym}_m(\mathbb{R}) \mid Y > 0\}$. Let

$$\mathfrak{h}_m = \{X + \sqrt{-1}Y \in \mathrm{Sym}_m(\mathbb{C}) \mid X, Y \in \mathrm{Sym}_m(\mathbb{R}),\ Y > 0\}$$

be the Siegel upper half space of size m. The symplectic group $\mathrm{Sp}_m(\mathbb{R})$ acts on $\mathfrak{h}_m$ by

$$\gamma(Z) = (AZ+B)(CZ+D)^{-1} \qquad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_m(\mathbb{R}),\ Z \in \mathfrak{h}_m.$$

The automorphy factor $j(\gamma, Z)$ $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_m(\mathbb{R})$ and $Z \in \mathfrak{h}_m$ is defined by

$$j(\gamma, Z) = \det(CZ+D).$$

Let $\mathbf{k} = (k^{(1)}, k^{(2)}, \ldots, k^{(d)}) \in \mathbb{Z}^d$ be a multi-index of size m. Then the automorphy factor $j(\gamma, Z)^{\mathbf{k}}$ for $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathrm{Sp}_m(\mathbb{R})^d$ and $Z = (Z_1, \ldots, Z_d) \in \mathfrak{h}_m^d$ is defined by

$$j(\gamma, Z)^{\mathbf{k}} = \prod_{i=1}^{d} j(\gamma_i, Z_i)^{k^{(i)}}.$$

Suppose that $f(\mathbf{z})$ is a $\mathbb{C}$-valued function on $\mathfrak{h}_m^d$. Then, for $\gamma \in \mathrm{Sp}_m(\mathbb{R})^d$, put

$$(f|_{\mathbf{k}}\gamma)(Z) = f(\gamma(Z))j(\gamma, Z)^{-\mathbf{k}}.$$

Then we have

$$(f|_{\mathbf{k}}\gamma_1)|_{\mathbf{k}}\gamma_2 = f|_{\mathbf{k}}(\gamma_1\gamma_2) \qquad \text{for } \gamma_1, \gamma_2 \in \mathrm{Sp}_m(\mathbb{R})^d.$$

Definition 3. A holomorphic function $f(Z)$ on $\mathfrak{h}_m^d$ is a weak Hilbert-Siegel modular form for a congruence subgroup Γ if $f|_{\mathbf{k}}\gamma = f$ for any $\gamma \in \Gamma$.

Let f be a weak Hilbert-Siegel modular form for a congruence subgroup Γ. Let $\mathfrak{m}$ be an integral ideal such that $\Gamma(\mathfrak{m}) \subset \Gamma$. Then we have

$$f(Z+\mu) = f(Z) \qquad \forall \mu \in \mathrm{Sym}_m(\mathfrak{m}).$$

It follows that $f(Z)$ has a Fourier expansion

$$f(Z)=\sum_{\xi\in\mathrm{Sym}(\mathfrak{m})^\vee} a_f(\xi)\mathbf{e}(\xi Z).$$

Here,

$$\begin{aligned}\mathrm{Sym}_m(\mathfrak{m})^\vee =&\{\xi\in\mathrm{Sym}_m(F)\mid \mathrm{Tr}_{F/\mathbb{Q}}\mathrm{tr}(\xi\mu)\in\mathbb{Z},\ {}^\forall\mu\in\mathrm{Sym}_m(\mathfrak{m})\}\\ \mathbf{e}(\xi Z)=&\exp(2\pi\sqrt{-1}\sum_{i=1}^d \mathrm{tr}(\xi^{(i)}Z_i)).\end{aligned}$$

This Fourier expansion converges absolutely and uniformly on any compact subset of $\mathfrak{h}_m^d$. If $f(z)$ is a weak Hilbert-Siegel modular form of weight $\mathbf{k}$ for Γ, then $f|_{\mathbf{k}}\gamma$ is a weak Hilbert-Siegel modular form of weight $\mathbf{k}$ for $\gamma^{-1}\Gamma\gamma$.

A symmetrix matrix $\xi\in\mathrm{Sym}_m(F)$ is totally positive definite if $\iota^{(1)}(\xi),\dots,\iota^{(d)}(\xi)$ are all positive definite. We write $\xi>0$ if ξ is totally positive definite. A symmetrix matrix $\xi\in\mathrm{Sym}_m(F)$ is totally positive semi-definite if $\iota^{(1)}(\xi),\dots,\iota^{(d)}(\xi)$ are all positive semi-definite. We write $\xi\geq 0$ if ξ is totally positive semi-definite.

Definition 4. A weak Hilbert-Siegel modular form $f(Z)$ of weight $\mathbf{k}$ for Γ is a Hilbert-Siegel modular form if $f(Z)$ has a Fourier expansion of the form

$$(f|_{\mathbf{k}}\gamma)(Z)=\sum_{\substack{\xi\in\mathrm{Sym}(F)\\ \xi\geq 0}} a_{f,\gamma}(\xi)\mathbf{e}(\xi Z)$$

for any $\gamma\in\mathrm{Sp}_m(F)$. The space of all Hilbert-Siegel modular form of weight $\mathbf{k}$ for Γ is denoted by $\mathcal{M}_{\mathbf{k}}(\Gamma)$.

Definition 5. A Hilbert-Siegel modular form $f(Z)$ of weight $\mathbf{k}$ for Γ is a Hilbert-Siegel cusp form if $f(Z)$ has a Fourier expansion of the form

$$(f|_{\mathbf{k}}\gamma)(Z)=\sum_{\substack{\xi\in\mathrm{Sym}(F)\\ \xi> 0}} a_{f,\gamma}(\xi)\mathbf{e}(\xi Z)$$

for any $\gamma\in\mathrm{Sp}_m(F)$. The space of all Hilbert-Siegel modular form of weight $\mathbf{k}$ for Γ is denoted by $\mathcal{S}_{\mathbf{k}}(\Gamma)$.

Theorem 2 (Koecher principle)**.** *Suppose that $dm\geq 2$. Then a weak Hilbert-Siegel modular form of weight $\mathbf{k}$ for Γ is automatically a Hilbert-Siegel modular form.*

Proof. First we consider the case $m=1$ and $d>1$. Choose an integral ideal $\mathfrak{m}$ such that $\Gamma(\mathfrak{m}) \subset \Gamma$. Then $f(\mathbf{z})$ is stable under a translation $\mathbf{z} \mapsto \mathbf{z}+\mu$ for $\mu \in \mathfrak{m}$. Fix $\mathbf{y}=(y_1,\ldots,y_d) \in (\mathbb{R}_+^\times)^d$. Then as a function of $\mathbf{x} \in \mathbb{R}^d$, $f(\mathbf{x}+\sqrt{-1}\mathbf{y})$ is a periodic function with period $\mathfrak{m}$. It follows that there is a positive number $M>0$ such that

$$|f(\mathbf{x}+\sqrt{-1}\mathbf{y})| < M, \qquad {}^\forall \mathbf{x} \in \mathbb{R}^d.$$

Since

$$a_f(\xi) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/\mathfrak{n})} \int_{\mathbb{R}^d/\mathfrak{n}} f(\mathbf{x}+\sqrt{-1}\mathbf{y})\mathbf{e}(-\xi(\mathbf{x}+\sqrt{-1}\mathbf{y}))\,d\mathbf{x},$$

we have

$$|a_f(\xi)| \le M e^{2\pi\cdot\mathrm{tr}(\xi\mathbf{y})}.$$

It follows that for any $\varepsilon \in \mathfrak{o}_{F,+}^\times$, $\varepsilon \equiv 1 \bmod \mathfrak{m}$ and and $n \in \mathbb{Z}_{>0}$,

$$a_f(\xi) \le \varepsilon^{-n\mathbf{k}} M e^{2\pi\cdot\mathrm{tr}(\varepsilon^{2n}\xi\mathbf{y})} \qquad {}^\forall \xi \in \mathfrak{m}^\vee.$$

Suppose that $\xi \notin F_+^\times \cup \{0\}$. We may assume $\xi^{(1)}<0$. Then by Dirichlet unit theorem, there exists $\varepsilon \in \mathfrak{o}_+^\times$ such that $\varepsilon^{(1)}>1$, $\varepsilon^{(2)},\ldots,\varepsilon^{(d)}<1$. We may assume $\varepsilon \equiv 1 \bmod \mathfrak{m}$. Then for $n \longrightarrow +\infty$, we have

$$\varepsilon^{-n\mathbf{k}} e^{2\pi\cdot\mathrm{tr}(\varepsilon^{2n}\xi\mathbf{y})} \longrightarrow 0.$$

Hence we have $a_f(\xi)=0$.

Next, we consider the case $m \ge 2$. Let N be an integer such that $\Gamma(N\mathfrak{o}) \subset \Gamma$. Put$L = \mathrm{Sym}(N\mathfrak{o})$ and $L^\vee = \mathrm{Sym}(N\mathfrak{o})^\vee$. Then $f(\mathbf{z})$is stable under a translation $Z \mapsto Z+\mu$ by $\mu \in L$. Fix

$$\begin{aligned} Y =& (Y^{(1)},\ldots,Y^{(d)}) \in \mathrm{Sym}_m^+(\mathbb{R})^d, \\ Y^{(i)} =& \mathrm{diag}(y_1^{(i)},\ldots,y_m^{(i)}), \quad y_1^{(i)},\ldots,y_m^{(i)}>0, \quad (i=1,\ldots,d). \end{aligned}$$

Then as a function of $X \in \mathrm{Sym}(\mathbb{R})^d$, $f(X+\sqrt{-1}Y)$ is periodic with period L. It follows that there exists a positive number $M>0$ such that

$$|f(X+\sqrt{-1}Y)| < M, \qquad {}^\forall X \in \mathrm{Sym}_m(\mathbb{R})^d$$

Since

$$a_f(\xi) = \frac{1}{\mathrm{Vol}(\mathrm{Sym}_m(\mathbb{R})^d/L)} \int_{(\mathrm{Sym}_m(\mathbb{R})^d/L} f(X+\sqrt{-1}Y)\mathbf{e}(-\xi(X+\sqrt{-1}Y))\,dX,$$

we have

$$|a_f(\xi)| \leq M \exp(2\pi \cdot \mathrm{tr}(\xi \mathbf{y})).$$

It follows that for any $A \in \mathrm{SL}_m(\mathfrak{o})$ such that $A \equiv \mathbf{1}_m \bmod N$, we have

$$a_f(\xi) \leq M \exp(2\pi \cdot \mathrm{tr}(A\xi\, {}^t\! AY)) \qquad {}^\forall \xi \in L^\vee.$$

Suppose that $\xi = (\xi_{ij}) \in \mathrm{Sym}_m(F)$ is not totally positive semi-definite. We need show that $a_f(\xi) = 0$. By assumption, $\iota^{(1)}(\xi), \ldots, \iota^{(d)}(\xi)$ are not all positive semi-definite. We may assume $\iota^{(1)}(\xi)$ is not positive semi-definite. Then there exists $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ such that $v\iota^{(1)}(\xi) \cdot {}^t v < 0$. Since $\mathbb{Q}^m$ is dense in $\mathbb{R}^m$, we may assume $v \in \mathbb{Q}^m$. Moreover, multiplying an integer, we may assume $v \in \mathbb{Z}^m$. Let K be a GCD of $v_1 - v_2, v_1 - v_3, \ldots, v_1 - v_m$. By replacing v by $(1, 0, \ldots, 0) + KNv$, if necessary, we may assume v satisfies the following conditions:

- $v \equiv (1, 0, \ldots, 0) \bmod N$.
- $v_1, \ldots, v_m$ are coprime.

It follows that there exists $A \in \mathrm{SL}_m(\mathbb{Z})$, $A \equiv \mathbf{1}_m \bmod N$ such that the first column of A is v. By replacing ξ by $A\xi\, {}^t\! A$, we may assume that $\iota^{(1)}(\xi_{11}) < 0$. Put

$$B = \begin{pmatrix} 1 & N & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \text{\huge 0} \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \text{\huge 0} & & 1 \end{pmatrix}.$$

Then we have $B \equiv \mathbf{1}_m \bmod N$ and

$$\mathrm{tr}(B^n \xi \cdot {}^t B^n) = \sum_{i=1}^{d} (\xi_{11}^{(i)} N^2 n^2 + 2\xi_{12}^{(i)} Nn) y_2^{(i)} + (\xi_{11}^{(i)} y_1^{(i)} + \xi_{22}^{(i)} y_2^{(i)} + \cdots + \xi_{mm}^{(i)} y_m^{(i)}).$$

One can choosing Y such that $\sum_{i=1}^{d} \xi_{11}^{(i)} y_2^{(i)} < 0$. Then for $n \to \infty$, we have $\mathrm{tr}(B^n \xi \cdot {}^t B^n) \to -\infty$. Hence we have $a_f(\xi) = 0$, as desired. $\square$

The following are known:

- $\mathcal{M}_{\mathbf{k}}(\Gamma)$, $\mathcal{S}_{\mathbf{k}}(\Gamma)$ are finite-dimensional.
- If $F \neq \mathbb{Q}$ and $k^{(i)} \neq k^{(j)}$ for some $i, j \in \{1, \ldots, d\}$, then we have $\mathcal{M}_{\mathbf{k}}(\Gamma) = \mathcal{S}_{\mathbf{k}}(\Gamma)$.

3 Vector-valued Hilbert-Siegel modular forms

Put

$$\mathbf{i} = (\sqrt{-1} \cdot \mathbf{1}_m, \dots, \sqrt{-1} \cdot \mathbf{1}_m) \in \mathfrak{h}_m^d.$$

The stabilizer of $\mathbf{i}$ in $\mathrm{Sp}_m(F_\infty) \simeq \mathrm{Sp}_m(\mathbb{R})^d$ is denoted by K_∞. Then K_∞ is isomorphic to $\mathrm{U}(m)^d$. Here, the unitary group $\mathrm{U}(m)$ is considered as a subgroup of $\mathrm{Sp}_m(\mathbb{R})$ by

$$A + \sqrt{-1}B \ \mapsto \ \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

For $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$ such that $k_1 \geq \cdots \geq k_m$, the irreducible representation of $\mathrm{U}(m)$ with highest weight $(k_1, \dots, k_m)$ is denoted by ρ_k. For a multi-index weight $\mathbf{k} = (k^{(1)}, \dots, k^{(d)})$, $k^{(i)} = (k_1^{(i)}, \dots, k_m^{(i)}) \in \mathbb{Z}^m$ such that $k_1^{(i)} \geq \cdots \geq k_m^{(i)}$ for each i, the irreducible representation $\rho_\mathbf{k}$ of $\mathrm{U}(m)^d$ is defined by $\rho_\mathbf{k} = \rho_{k^{(1)}} \otimes \cdots \otimes \rho_{k^{(d)}}$. The representation space of $\rho_\mathbf{k}$ is denoted by $V_{\rho_\mathbf{k}}$.

The canonical automorphy factor $J(\gamma, Z) \in \mathrm{GL}_m(\mathbb{C})^d$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_m(\mathbb{R})$ and $Z \in \mathfrak{h}_m$ is defined by

$$J(\gamma, Z) = CZ + D.$$

Let $\mathbf{k} = (k^{(1)}, k^{(2)}, \dots, k^{(d)})$, $k^{(i)} = (k_1^{(i)}, \dots, k_m^{(i)})$, $k_1^{(i)} \geq \cdots \geq k_m^{(i)}$ be a multi-index weight of size m. Then $\rho_\mathbf{k}(J(\gamma, Z)) \in \mathrm{GL}(V_{\rho_\mathbf{k}})$ for $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathrm{Sp}_m(\mathbb{R})^d$ and $Z = (Z_1, \dots, Z_d) \in \mathfrak{h}_m^d$ is an automorphy factor. Suppose that $f(\mathbf{z})$ is a $V_{\rho_\mathbf{k}}$-valued function on $\mathfrak{h}_m^d$. Then, for $\gamma \in \mathrm{Sp}_m(\mathbb{R})^d$, put

$$(f|_{\rho_\mathbf{k}}\gamma)(Z) = \rho_\mathbf{k}(J(\gamma, Z))^{-1} f(\gamma(Z))$$

Then we have

$$(f|_{\rho_\mathbf{k}}\gamma_1)|_{\rho_\mathbf{k}}\gamma_2 = f|_{\rho_\mathbf{k}}(\gamma_1\gamma_2) \qquad \text{for } \gamma_1, \gamma_2 \in \mathrm{Sp}_m(\mathbb{R})^d.$$

Definition 6. A $V_{\rho_\mathbf{k}}$-valued holomorphic function $f(Z)$ on $\mathfrak{h}_m^d$ is a weak Hilbert-Siegel modular form of vector weight $\rho_\mathbf{k}$ for a congruence subgroup Γ if $f|_{\rho_\mathbf{k}}\gamma = f$ for any $\gamma \in \Gamma$.

Remark 1. $\rho = \rho_\mathbf{k}$ is usually called the weight $\rho_\mathbf{k}$. But, since the word "weight" is somewhat confusing, we use the word "vector weight" here.

Let f be a weak Hilbert-Siegel modular form for a congruence subgroup Γ of vector weight $\rho_{\mathbf{k}}$. Let $\mathfrak{m}$ be an integral ideal such that $\Gamma(\mathfrak{m}) \subset \Gamma$. Then we have

$$f(Z+\mu) = f(Z) \qquad \forall \mu \in \mathrm{Sym}_m(\mathfrak{m}).$$

It follows that $f(Z)$ has a Fourier expansion

$$f(Z) = \sum_{\xi \in \mathrm{Sym}(\mathfrak{m})^\vee} a_f(\xi)\mathbf{e}(\xi Z).$$

This Fourier expansion converges absolutely and uniformly on any compact subset of $\mathfrak{h}_m^d$. If $f(z)$ is a weak Hilbert-Siegel modular form of vector weight $\rho_{\mathbf{k}}$ for Γ, then $f|_{\rho_{\mathbf{k}}}\gamma$ is a weak Hilbert-Siegel modular form of vector weight $\rho_{\mathbf{k}}$ for $\gamma^{-1}\Gamma\gamma$.

Definition 7. A weak Hilbert-Siegel modular form $f(Z)$ of vector weight $\rho_{\mathbf{k}}$ for Γ is a Hilbert-Siegel modular form if $f(Z)$ has a Fourier expansion of the form

$$(f|_{\rho_{\mathbf{k}}}\gamma)(Z) = \sum_{\substack{\xi \in \mathrm{Sym}(F) \\ \xi \geq 0}} a_{f,\gamma}(\xi)\mathbf{e}(\xi Z)$$

for any $\gamma \in \mathrm{Sp}_m(F)$. The space of all Hilbert-Siegel modular form of vector weight $\rho_{\mathbf{k}}$ for Γ is denoted by $\mathcal{M}_{\rho_{\mathbf{k}}}(\Gamma)$.

Definition 8. A Hilbert-Siegel modular form $f(Z)$ of vector weight $\rho_{\mathbf{k}}$ for Γ is a Hilbert-Siegel cusp form if $f(Z)$ has a Fourier expansion of the form

$$(f|_{\rho_{\mathbf{k}}}\gamma)(Z) = \sum_{\substack{\xi \in \mathrm{Sym}(F) \\ \xi > 0}} a_{f,\gamma}(\xi)\mathbf{e}(\xi Z)$$

for any $\gamma \in \mathrm{Sp}_m(F)$. The space of all Hilbert-Siegel modular form of vector weight $\rho_{\mathbf{k}}$ for Γ is denoted by $\mathcal{S}_{\rho_{\mathbf{k}}}(\Gamma)$.

It is known that if $k_m^{(i)} < 0$ for some i, then $\mathcal{M}_{\rho_{\mathbf{k}}}(\Gamma) = 0$. The Koecher principle holds for vector-valued Hilbert-Siegel modular forms. It is also known that $\mathcal{M}_{\rho_{\mathbf{k}}}(\Gamma)$ and $\mathcal{S}_{\rho_{\mathbf{k}}}(\Gamma)$ are finite-dimensional vector spaces.

4 Hilbert-Siegel modular forms on $\mathrm{Sp}_m(\mathbb{A})$

The adele group $\mathrm{Sp}_m(\mathbb{A})$ of Sp_m is given by

$$\mathrm{Sp}_m(\mathbb{A}) = \bigcup_{\mathfrak{S}} \left(\prod_{v \notin \mathfrak{S}} \mathrm{Sp}_m(\mathfrak{o}_v) \right) \times \left(\prod_{v \in \mathfrak{S}} \mathrm{Sp}_m(F_v) \right),$$

where $\mathfrak{S}$ extends over finite sets of places of F containing all archimedean places.

Let Γ be a congruence subgroup of $\mathrm{Sp}_m(F)$ and $\mathbf{K}_\Gamma$ its closure in $\mathrm{Sp}_m(\mathbb{A}_{\mathrm{fin}})$. Then the strong approximation property, we have

$$\mathrm{Sp}_m(\mathbb{A}) = \mathrm{Sp}_m(F) \cdot \mathbf{K}_\Gamma \cdot \mathrm{Sp}_m(F_\infty).$$

Here, $F_\infty \simeq \mathbb{R}^d$ is the infinite part of $\mathbb{A}$. When $\mathrm{Sp}_m(F)$ is regarded as a subgroup of $\mathrm{Sp}_m(\mathbb{A})$, we have

$$\mathrm{Sp}_m(F) \cap \mathbf{K}_\Gamma \cdot \mathrm{Sp}_m(F_\infty) = \Gamma.$$

Let $f(Z)$ be a Hilbert-Siegel modular form of vector weight $\rho = \rho_{\mathbf{k}}$ for Γ. Then we can construct a V_ρ-valued function $\phi_f(g)$ as follows.

For $g \in \mathrm{Sp}_m(\mathbb{A})$, choose a decomposition

$$g = \gamma \cdot u \cdot h, \qquad \gamma \in \mathrm{Sp}_m(F),\ u \in \mathbf{K}_\Gamma,\ h \in \mathrm{Sp}_m(F_\infty)$$

and put

$$\phi_f(g) = (f|_\rho h)(\mathbf{i}).$$

Then we have a well-defined V_ρ-valued function ϕ_f on $\mathrm{Sp}_m(\mathbb{A})$. In fact, let

$$g = \gamma' \cdot u' \cdot h', \qquad \gamma' \in \mathrm{Sp}_m(F),\ u' \in \mathbf{K}_\Gamma,\ h' \in \mathrm{Sp}_m(F_\infty)$$

be another decomposition. Then $\delta := \gamma {\gamma'}^{-1} \in \Gamma$. Let δ_{fin} and δ_∞ be the finite and infinite part of δ. Then we have

$$u' = \delta_{\mathrm{fin}} u, \qquad h' = \delta_\infty h$$

and so

$$f|_{\mathbf{k}} h' = f|_\rho \delta_\infty h = f|_\rho h.$$

Hence $\phi_f(g)$ is well-defined.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexification of the Lie algebra of $\mathrm{Sp}_m(F_\infty)$ and K_∞, respectively. Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ such that

$$[\mathfrak{k}, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm, \quad [\mathfrak{p}^+, \mathfrak{p}^+] = [\mathfrak{p}^-, \mathfrak{p}^-] = 0, \quad [\mathfrak{p}^+, \mathfrak{p}^-] = \mathfrak{k}.$$

Note that in the case $d = 1$, we have

$$\begin{aligned}
\mathfrak{g} &= \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{pmatrix} \,\middle|\, X_1, X_2, X_3 \in \mathrm{M}_m(\mathbb{C}),\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\}, \\
\mathfrak{k} &= \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\middle|\, A, B \in \mathrm{M}_m(\mathbb{C}),\ -A = {}^tA,\ B = {}^tB \right\}, \\
\mathfrak{p}^+ &= \left\{ \begin{pmatrix} Z & \sqrt{-1}Z \\ \sqrt{-1}Z & -Z \end{pmatrix} \,\middle|\, Z \in \mathrm{M}_m(\mathbb{C}),\ Z = {}^tZ \right\}, \\
\mathfrak{p}^- &= \left\{ \begin{pmatrix} Z & -\sqrt{-1}Z \\ -\sqrt{-1}Z & -Z \end{pmatrix} \,\middle|\, Z \in \mathrm{M}_m(\mathbb{C}),\ Z = {}^tZ \right\}.
\end{aligned}$$

The element of $\mathfrak{p}^-$ acts on a function on $\mathfrak{h}_m^d$ by an anti-holomorphic differential operator. Since f is a holomorphic function on $\mathfrak{h}_m^d$, we have $\mathfrak{p}^- \cdot \phi_f = 0$.

Let $\mathcal{U}(\mathfrak{g})$ and $\mathcal{Z}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and its center, respectively.

An element of $\mathcal{U}(\mathfrak{g})$ (resp. $\mathcal{Z}(\mathfrak{g})$) acts on $C^\infty(\mathrm{Sp}_m(F_\infty))$ as a left invariant (resp. bi-invariant) differential operator. A ring homomorphism $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is called an infinitesimal character.

Let $\mathfrak{h} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ be the Weyl group. By the Harish-Chandra isomorphism, we have $\mathcal{Z}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}]^W$. The Cartan subalgebra is isomorphic to $\underbrace{\mathbb{C}^m \times \cdots \times \mathbb{C}^m}_{d \text{ times}}$. Let $\chi_{\mathbf{k}}$ be the infinitesimal character determined by

$$(H_1^{(1)}, H_2^{(1)}, \ldots, H_m^{(1)}, \ldots\ldots, H_1^{(d)}, H_2^{(d)}, \ldots, H_m^{(d)}) \mapsto \sum_{i=1}^{d} \sum_{j=1}^{m} (k_j^{(i)} - j) H_j^{(d)}.$$

Proposition 1. *We have $z \cdot \phi_f = \chi_{\mathbf{k}}(z)\phi_f$ for any $f \in \mathcal{M}_{\rho_{\mathbf{k}}}(\Gamma)$.*

Proof. Let $\mathfrak{n}_{\mathfrak{k}}^+$ and $\mathfrak{n}_{\mathfrak{k}}^-$ be the maximal nilotent subalgebras of $\mathfrak{k}$, corresponding to positive and negative root systems, respectively. Since $\phi_f(gk) = \rho(k)\phi_f(g)$ for $k \in K_\infty$, we may assume ϕ_f is a highest weight vector, i.e., $N \cdot \phi_f = 0$ for any $N \in \mathfrak{n}_{\mathfrak{k}}^+$.

Put $\mathfrak{n} = \mathfrak{p}^- + \mathfrak{n}_{\mathfrak{k}}^+$. Then $\mathfrak{n}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. Then we have

$$\mathfrak{n} \cdot \phi_f = 0,$$
$$H \cdot \phi_f = \left(\sum_{i=1}^{d} \sum_{j=1}^{m} k_j^{(i)} \right) \phi_f$$

for

$$H = (H_1^{(1)}, H_2^{(1)}, \ldots, H_m^{(1)}, \ldots\ldots, H_1^{(d)}, H_2^{(d)}, \ldots, H_m^{(d)}) \in \mathfrak{h}.$$

Let δ be the half the sum of roots in $\mathfrak{n}$. Then we have

$$\delta(H) = -\sum_{i=1}^{d} \sum_{j=1}^{m} j H_j^{(i)}.$$

By the Poincare-Birkoff-Witt theorem, we have $\mathbb{C}[\mathfrak{h}] \cap \mathcal{U}(\mathfrak{g})\mathfrak{n} = 0$ and $\mathcal{Z}(\mathfrak{g}) \subset \mathbb{C}[\mathfrak{h}] + \mathcal{U}(\mathfrak{g})\mathfrak{n}$. Let $\gamma' : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}]$ be the first projection. Let $\sigma : \mathfrak{h} \to \mathbb{C}[\mathfrak{h}]$ be the map given by $\sigma(H) = H - \delta(H)$. Then the Harish-Chandra isomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}]^W$ is induced by $\gamma = \sigma \circ \gamma'$. Since

$$\gamma^{-1}(H) \cdot \phi_f = \left(\sum_{i=1}^{d} \sum_{j=1}^{m} (k_j^{(i)} - j) \right) \phi_f,$$

We obtain the proposition. □

Let $\mathcal{M}_\rho(\mathrm{Sp}_m(F)\backslash \mathrm{Sp}_m(\mathbb{A})/\mathbf{K}_\Gamma)$ be the space of all ϕ_f such that $f \in \mathcal{M}_\rho(\Gamma)$. Then it is known that an element $\phi \in \mathcal{M}_\rho(\mathrm{Sp}_m(F)\backslash \mathrm{Sp}_m(\mathbb{A})/\mathbf{K}_\Gamma)$ is characterized as a function on $\mathrm{Sp}_m(\mathbb{A})$ with the properties:

(1) ϕ is left $\mathrm{Sp}_m(F)$ invariant.
(2) ϕ is right $\mathbf{K}_\Gamma$ invariant.
(3) For $h \in K_\infty \simeq \mathrm{U}(m)^d$, we have $\phi(gh) = \rho(h)\phi(g)$.
(4) $\mathfrak{p}^-\phi = 0$.
(5) $z \cdot \phi_f = \chi_{\mathbf{k}}(z)\phi_f$ for any $z \in \mathcal{Z}(\mathfrak{g})$.
(6) ϕ is slowly increasing on $\mathrm{Sp}_m(\mathbb{A})$.

When $dm > 1$, the condition (5) is not necessary by the Koecher principle.

For $\delta \in \mathrm{Sp}_m(F)$, we have $f|_{\mathbf{k}}\delta_\infty \in M_{\mathbf{k}}(\delta^{-1}\Gamma\delta)$. Moreover, we have

$$\phi_{(f|_{\mathbf{k}}\delta_\infty)}(g) = \phi_f(g\delta_{\mathrm{fin}}^{-1}).$$

Here, $\delta_{\rm fin}$ and δ_∞ are the finite and infinite part of δ, respectively. In fact, let

$$g = \gamma \cdot u \cdot h_\infty, \qquad \gamma \in \mathrm{Sp}_m(F), \quad u \in \delta^{-1}\mathbf{K}_\Gamma\delta, \quad h_\infty \in \mathrm{Sp}_m(F_\infty)$$

be a decomposition. Then we have

$$\phi_{(f|_{\mathbf{k}}\delta_\infty)}(g) = ((f|_{\mathbf{k}}\delta_\infty)|_{\mathbf{k}}h_\infty)(\mathbf{i}) = (f|_{\mathbf{k}}\delta_\infty h_\infty)(\mathbf{i}).$$

Note that a decomposition of $f\delta_{\rm fin}^{-1}$ is given by

$$g\delta_{\rm fin}^{-1} = (\gamma\delta^{-1})\cdot(\delta u\delta^{-1})\cdot(\delta_\infty h_\infty), \qquad \gamma\delta^{-1} \in \mathrm{Sp}_m(F),\ \delta u\delta^{-1} \in \mathbf{K}_\Gamma,\ \delta_\infty h_\infty \in \mathrm{Sp}_m(F_\infty).$$

Hence we have $\phi_f(g\delta_{\rm f}^{-1}) = (f|_{\mathbf{k}}\delta_\infty h_\infty)(\mathbf{i})$. It follows that

$$\phi_{(f|_{\mathbf{k}}\delta)}(g) = (f|_{\mathbf{k}}\delta_\infty h_\infty)(\mathbf{i}) = \phi_f(g\delta_{\rm fin}^{-1}).$$

Definition 9.

$$\mathcal{M}_\rho(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A})) = \bigcup_\Gamma \mathcal{M}_\rho(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A})/\mathbf{K}_\Gamma)$$
$$\mathcal{S}_\rho(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A})) = \bigcup_\Gamma \mathcal{S}_\rho(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A})/\mathbf{K}_\Gamma)$$

By what we have seen as above, the finite adele group $\mathrm{Sp}_m(\mathbb{A}_{\rm fin})$ acts on $\mathcal{M}_\rho(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A}))$ and $\mathcal{S}_\rho(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A}))$ by right translation.

Definition 10. A ($\mathbb{C}$-valued) function φ on $\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A})$ is an automophic form on $\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A})$ of infinitesimal character χ and K_∞-type ρ if the following conditions hold:

(1) φ is left $\mathrm{Sp}_m(F)$ invariant.
(2) φ is right $\mathbf{K}$ invariant for some compact open subgroup $\mathbf{K} \subset \mathrm{Sp}_m(\mathbb{A}_{\rm fin})$.
(3) φ has an infinitesimal character χ, i.e., $z \cdot \varphi = \chi(z)\varphi$ for any $z \in \mathcal{Z}(\mathfrak{g})$.
(4) By the action of K_∞ by the right translation, φ has a K_∞-type ρ.
(5) φ is slowly increasing on $\mathrm{Sp}_m(\mathbb{A})$.

The space of automorphic form on $\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A})$ of infinitesimal character χ and K_∞ type ρ is denoted by $\mathcal{A}(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A});\ \chi;\ \rho)$.

Put

$$\mathcal{A}(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A});\ \chi;\ \rho)^{\mathfrak{p}^-} = \{\phi \in \mathcal{A}(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A});\ \chi;\ \rho) \mid \mathfrak{p}^-\phi = 0\}.$$

Then we have

$$\mathcal{M}_\rho(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A}))\otimes V_\rho^\vee = \mathcal{A}(\mathrm{Sp}_m(F)\backslash\mathrm{Sp}_m(\mathbb{A});\ \chi;\ \rho)^{\mathfrak{p}^-}.$$

Here, $V_\rho^\vee$ is the dual space of V_ρ. This isomorphism is given by

$$f\otimes v^\vee \mapsto \langle \phi_f, v^\vee\rangle$$