# Problem of definition: Jordan vs Einstein frame, beyond slow-roll

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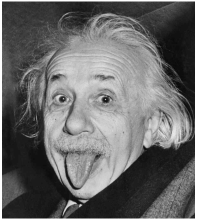
# Einstein's General Relativity



- In Einstein gravity, it is assumed matter is minimally coupled to gravity

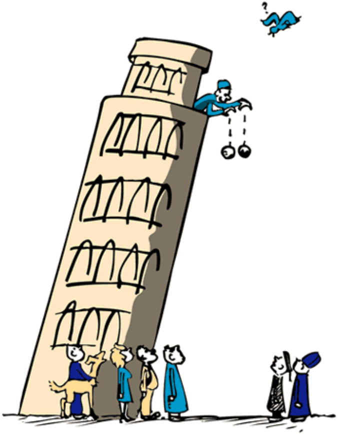
$$S = \int d^4x \sqrt{-g}(M_{\mathrm{p}}^2 R/2) + S_{\mathrm{m}}[g_{\mu\nu}, \phi]$$

- $R=$ Ricci Scalar, $S_m =$ matter action
- For example, single scalar field $\phi$

$$S_{\mathrm{m}} = \int d^4x \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi + V(\phi)\right]$$

# Einstein's General Relativity (con't)

- Strong/weak equivalence principle
- Strong: *The gravitational motion of a small test body depends only on its initial position in spacetime and velocity, and not on its constitution.*



# Non-minimal Coupled Models

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# Non-minimal Coupled Models

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- Example, introducing non-minimal coupling to gravity

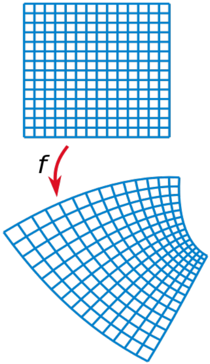
$$S_J = \int d^4x\sqrt{-g}(f(\psi)M_{\rm p}^2 R/2) + S_m[g_{\mu\nu}, \psi]$$

- generic in modified gravity and unified theories, such as string theory, f(R), Chameleons, TeVeS...
- conformally related to

$$S_E = \int d^4x\sqrt{-g_E}(M_{\rm p}^2 \tilde{R}/2) + \tilde{S}_m[(g_{\mu\nu})_E, \psi]$$

- by the conformal transformation $g_{\mu\nu} \to (g_{\mu\nu})_E = f(\psi)g_{\mu\nu}$
- They are mathematically equivalent
- Question: But are they physically equivalent?

# Physics should be frame independent!



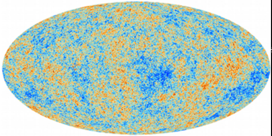

- Conformal transformation = field redefinition
- More precisely, conformal transformation = change of scale
- 1 meter is only meaningful with respect to a reference scale

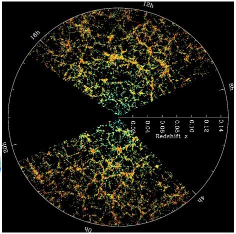
# But...

- In cosmology, density fluctuations are usually quantified in terms of $\zeta$ q

$$\zeta \equiv -\varphi + \frac{H\delta\rho}{\dot{\rho}}$$

- can be defined in both conformal frames, where $\rho$ is the effective energy density from $G_{\mu\nu} = T_{\mu\nu}/M_{\rm p}^2$
- For instance





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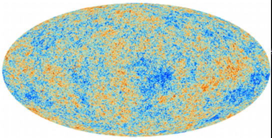

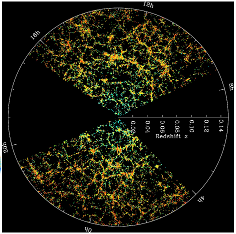


- dimensionless and gauge invariant, but *not frame independent* as we will see...

# Aside: Isocurvature perturbation

- Perturbation is purely adiabatic if $\delta P = \frac{\dot{P}}{\dot{\rho}}\delta\rho$. Not always true though...

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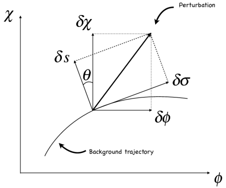
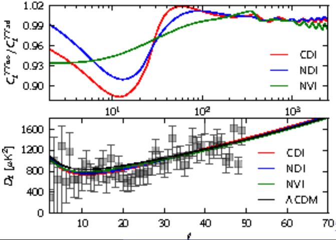
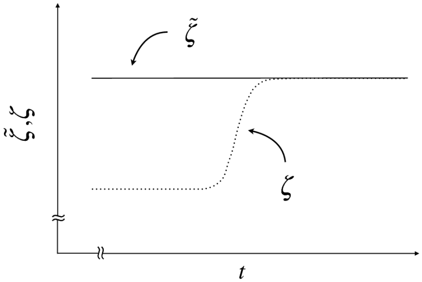
- Perturbation is purely adiabatic if $\delta P = \frac{\dot{P}}{\dot{\rho}}\delta\rho$. Not always true though...
- Entropic/isocurvature perturbations
  = perturbations $\perp$ background trajectory, natural in multifield inflation models




- some hints in 2013 Planck results



# Inequivalence of $\zeta$ in Einstein and Jordan frames




- It was found that $\zeta$ is frame-dependent in the presence of isocurvature perturbation [White et al. 12, arXiv:1205.0656], [Chiba and Yamaguchi 13]
- reason: isocurvature perturbation is frame-dependent (artificial)

# Inequivalence of $\zeta$ in Einstein and Jordan frames

- Examples, in multifield models

$$\zeta - \tilde{\zeta} \approx \mathcal{A}_{JK}\mathcal{K}^{JK} + \mathcal{B}_{JK}\dot{\mathcal{K}}^{JK}\,, \quad \mathcal{K}^{JK} \equiv \delta\phi^J\dot{\phi}^K - \delta\phi^K\dot{\phi}^J$$

where

$$\mathcal{A}_{JK} = \frac{1}{\mathcal{C}}\left\{\left[\left(\frac{G_{PQ}\dot{\phi}^P\dot{\phi}^Q + 2\left(\ddot{f} - H\dot{f}\right)}{2f} - 2H^2\right) f_K G_{IJ} + 2H\dot{\phi}^L f_{KL} G_{IJ} - 2H\dot{f}G_{IJ,K}\right]\dot{\phi}^I - 2H f_K G_{IJ}\ddot{\phi}^I\right\},$$

$$\mathcal{B}_{JK} = \frac{2Hf_K G_{IJ}\dot{\phi}^I}{\mathcal{C}} \quad \text{and}$$

$$\mathcal{C} = 2\kappa^2 f S_{MN}\dot{\phi}^M\dot{\phi}^N\left(G_{PQ}\dot{\phi}^P\dot{\phi}^Q + 2\left(\ddot{f} - H\dot{f}\right)\right).$$

- $\mathcal{K}^{JK}$ is a measure of the isocurvature perturbation
- $\zeta - \tilde{\zeta} \to 0$ only if isocurvature vanishes in general

# Relation between $\zeta$ and $\tilde{\zeta}$

- linear and non-linear order
- using the separate universe assumption, we can write $\zeta$ and $\tilde{\zeta}$ in terms of the $\delta N$ formalism [White, Minamitsuji and Sasaki et al.]

$$\zeta = \mathcal{N}_I \delta\phi^I + \mathcal{N}_{IJ}\delta\phi^I_{\tilde{R}}\delta\phi^J_{\tilde{R}} +$$
$$\tilde{\zeta} = \tilde{N}_I \delta\phi^I + \tilde{N}_{IJ}\delta\phi^I_{\tilde{R}}\delta\phi^J_{\tilde{R}} + \ldots$$

- $\delta\phi^I_{\tilde{R}}$ = flat-gauge field perturbations in Einstein frame
- observables can be expressed in terms of $\delta N$ coefficients, e.g. in Jordan frame

$$n_s - 1 = -2(\tilde{\epsilon}_H)_* - \frac{2}{\mathcal{N}_I\mathcal{N}^I} + \frac{2}{3\tilde{H}_*^2}\frac{\mathcal{N}^K\mathcal{N}^L}{\mathcal{N}_I\mathcal{N}_J S_*^{IJ}}\left[\tilde{\nabla}_K\tilde{\nabla}_L\tilde{V} - \tilde{R}_{KLPQ}\frac{d\phi^P}{d\tilde{t}}\frac{d\phi^Q}{d\tilde{t}}\right]_*$$

$$\mathcal{P}_\zeta = \mathcal{N}^I\mathcal{N}_I\left(\frac{\tilde{H}}{2\pi}\right)^2_* \quad \text{and} \quad r = \frac{8}{\mathcal{N}^I\mathcal{N}_I}$$

# Relation between $\zeta$ and $\tilde{\zeta}$ (con't)

- general relation between $N$ and $\tilde{N}$

$$N = \int_{\mathcal{R}}^{\omega} \mathcal{H} \mathrm{d}\eta = \int_{\mathcal{R}}^{\omega} \left( \tilde{\mathcal{H}} - \frac{f'}{2f} \right) \mathrm{d}\eta = \tilde{N}(\omega, \mathcal{R}) - \frac{1}{2} \ln \left( \frac{f_\omega}{f_\mathcal{R}} \right)$$

- consider a simplified case where $f' \approx 0$ at the time of interest (late time)
- using the $\delta N$ formalism, the first and second order $\delta N$ coefficients are related by

$$\mathcal{N}_I \approx \tilde{N}_I - \left( \frac{1}{2} + c \right) \left( \frac{f_J}{f} \right)_\diamond \left( \frac{\partial \phi_\diamond^J}{\partial \phi_*^I} \right)_\omega$$

$$\mathcal{N}_{IJ} \approx \tilde{N}_{IJ} - \left( \frac{1}{2} + c \right) \left[ \left( \frac{f_{KL}}{f} - \frac{f_K f_L}{f^2} \right)_\diamond \left( \frac{\partial \phi_\diamond^K}{\partial \phi_*^I} \right)_\omega \left( \frac{\partial \phi_\diamond^L}{\partial \phi_*^J} \right)_\omega + \left( \frac{f_K}{f} \right)_\diamond \left( \frac{\partial^2 \phi_\diamond^K}{\partial \phi_*^I \partial \phi_*^J} \right)_\omega \right]$$

- we have assumed $\epsilon_f \equiv |f'/\mathcal{H} f| \ll 1$
- $c \equiv \frac{\tilde{\mathcal{H}}}{\tilde{\rho}' \tilde{\rho}}$, equals to $-1/3$ (matter era) and $-1/4$ (radiation era)
- $\zeta - \tilde{\zeta}$ can be arbitrarily large depending on $f$, but how about observables?

# Difference between primordial observables beyond slowroll

- Defining the fractional difference between the power spectra amplitudes and the spectral indices

$$\Delta\mathcal{P}_\zeta \equiv \frac{\mathcal{P}_\zeta - \tilde{\mathcal{P}}_\zeta}{\tilde{\mathcal{P}}_\zeta}$$

- and slightly different definition for $n_s$ and $f_{\rm NL}$

$$\frac{n_s - 1 + 2(\tilde{\epsilon}_H)_*}{\tilde{n}_s - 1 + 2(\tilde{\epsilon}_H)_*} = (1+\Delta\mathcal{P}_\zeta)^{-1}(1+\Delta n_s)$$
$$\frac{f_{\rm NL}}{\tilde{f}_{\rm NL}} = (1+\Delta\mathcal{P}_\zeta)^{-2}(1+\Delta f_{\rm NL})$$

- using the asymptotic relation between the $\delta N$ coefficients, we therefore have

$$\Delta\mathcal{P}_\zeta = -\frac{1}{\tilde{N}_P\tilde{N}^P}\left[(1+2c)\left(\frac{f_K}{f}\right)_\diamond \left(\frac{\partial\phi^K_\diamond}{\partial\phi^I_*}\right)_\omega \tilde{N}^I - \left(\frac{1}{2}+c\right)^2 \left(\frac{f_K f_L}{f^2}\right)_\diamond \left(\frac{\partial\phi^K_\diamond}{\partial\phi^I_*}\right)_\omega \left(\frac{\partial\phi^L_\diamond}{\partial\phi^J_*}\right)_\omega S^{IJ}_*\right]$$

- and similarly for $\Delta n_s$ and $\Delta f_{\rm NL}$

# Model considered

- we consider the multifield model with the following action in the Jordan frame

$$S_J = \int \mathrm{d}^4x\sqrt{-g}\left\{f(\Phi)R - \frac{1}{2}G_{IJ}(\Phi)g^{\mu\nu}\partial\phi^I\partial\phi^J - V(\Phi)\right\}$$

- or in the Einstein frame

$$S_E = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left\{\frac{M_p^2 R}{2} - \frac{1}{2}\mathcal{S}_{IJ}(\Phi)\tilde{g}^{\mu\nu}\partial\phi^I\partial\phi^J - \tilde{V}(\Phi)\right\}$$

- note that the field space metric and scalar potential are related by

$$\mathcal{S}_{IJ} = \frac{M_p^2}{2f}\left(G_{IJ} + 3\frac{f_I f_J}{f}\right) \quad \text{and} \quad \tilde{V} = \frac{VM_p^4}{4f^2}$$

- after inflation ends, reheating is modeled by adding a friction to the field EOM in Einstein frame

$$(\phi^I)'' + \tilde{\Gamma}^I_{JK}(\phi^J)'(\phi^K)' + 2(\tilde{\mathcal{H}} + \tilde{a}^{(I)}\Gamma)(\phi^I)' + \tilde{a}^2\mathcal{S}^{IJ}\tilde{V}_J = 0$$

$$\rho_\gamma' + 4\mathcal{H}\rho_\gamma = \frac{^{(I)}\Gamma}{\tilde{a}}\mathcal{S}_{IJ}(\phi^I)'(\phi^J)'$$

# Simple example: Non-minimal coupling $f' = 0$ always

- to illustrate, we consider the class of two-field models
- simple explicit example: $f = f(\chi)$ and $\chi$ is a frozen spectator field

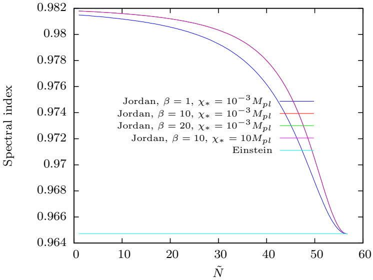
$$\mathcal{S}_{\chi\chi} \equiv \frac{M_p^2}{2f}\left(G_{\chi\chi} + 3\frac{f_\chi^2}{f}\right) \quad \text{and} \quad \tilde{V} = V(\phi)M_p^2$$

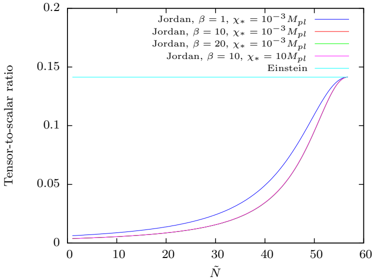
- we further assume $G_{\phi\chi} = 0$ and $G_{\phi\phi} = f$ such that there is no mixing in the kinetic term
- Einstein frame results = simple chaotic inflation

$$\tilde{n}_s - 1 = -6\tilde{\epsilon}_* + 2\tilde{\eta}_* \,, \quad \tilde{r} = 16\tilde{\epsilon}_*$$

# Simple example: Non-minimal coupling $f' = 0$ always (con't)

- results during slow-roll inflationary regime, consistent with previous analytic work




- model choice: $\tilde{V}(\phi) = \frac{1}{2} m^2 \phi^2$, $2f/M_p^2 = e^{-\beta\chi/M_p}$ with $\phi_* = 15.0 M_p$
- observables seem coincide at the end of inflation
- however, $\zeta$ still evolve in Jordan frame...

# Simple example: Non-minimal coupling $f' = 0$ always (con't)

- how about beyond slow-roll, particularly after reheating?

## Simple example: Non-minimal coupling $f' = 0$ always (con't)

- how about beyond slow-roll, particularly after reheating?
- for this particular model, since the non-minimal coupled field $\chi$ is frozen, things simplify
- the fractional difference

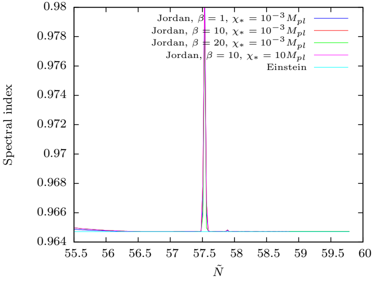
$$\Delta\mathcal{P}_\zeta = \frac{1}{16}\left(\frac{1}{\tilde{N}_\phi^2}\right)\left(\frac{f_\chi}{f}\right)_*^2 S_*^{\chi\chi}$$

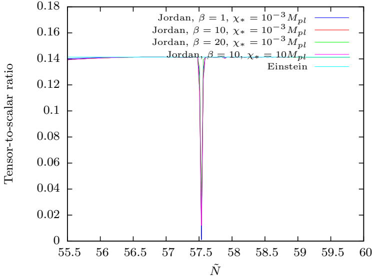
$$\frac{\tilde{n}_s - n_s}{n_s} = \Delta\mathcal{P}_\zeta\left[\frac{n_s - 1 + 2(\tilde{\epsilon}_H)_*}{n_s}\right]$$

- difference can only be large if $\Delta\mathcal{P}_\zeta \gg \mathcal{O}(1)$

# Simple example: Non-minimal coupling $f' = 0$ always (con't)

- Beyond slow-roll regime




- with reheating parameter $\Gamma_\phi = 0.1(m/\sqrt{2})$
- evolution terminates when $\Omega_\gamma > 0.9999$

# Difference is negligible after reheating

- we see the fractional difference between observables are negligible even $\zeta - \tilde{\zeta}$ is large
- why?

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- why? recall

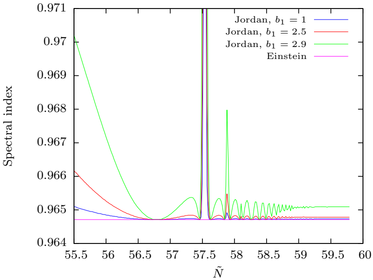
$$\Delta\mathcal{P}_\zeta = -\frac{1}{\tilde{N}_P\tilde{N}^P}\left[(1+2c)\left(\frac{f_K}{f}\right)_\diamond\left(\frac{\partial\phi^K_\diamond}{\partial\phi^I_*}\right)_\omega\tilde{N}^I - \left(\frac{1}{2}+c\right)^2\left(\frac{f_K f_L}{f^2}\right)_\diamond\left(\frac{\partial\phi^K_\diamond}{\partial\phi^I_*}\right)_\omega\left(\frac{\partial\phi^L_\diamond}{\partial\phi^J_*}\right)_\omega S^{IJ}_*\right]$$

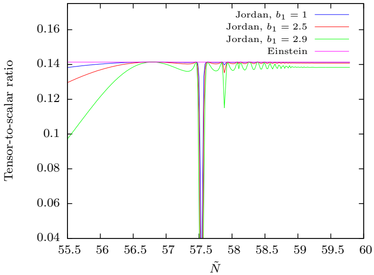
- reason: Einstein frame field space metric $S^{IJ}_*$ also depends on $f$

$$\mathcal{S}_{\chi\chi} \equiv \frac{M_p^2}{2f}\left(G_{\chi\chi} + 3\frac{f_\chi^2}{f}\right)$$

# Caveat I: negative Jordan frame field space metric

- We may tune $(G_{\chi\chi})_* \to (f_\chi^2/f)_*$, with $S_{\chi\chi}$ remains positive
- Example: $G_{\chi\chi} = -b_1(f_\chi^2/f)$, $\tilde{V}(\phi) = \frac{1}{2}m^2\phi^2$ and $2f/M_p^2 = e^{-\beta\chi/M_p}$, with $b_1 \leq 3$

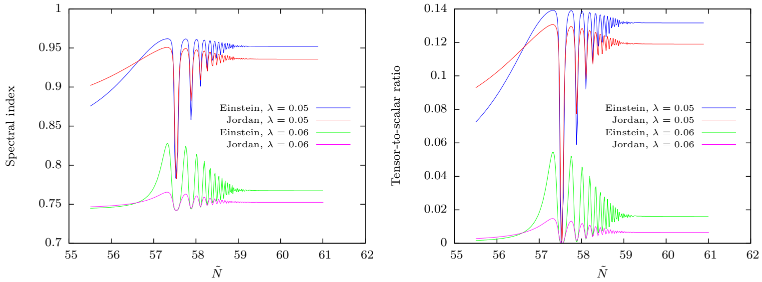



- only in the very fine-tuned limit the difference becomes significant

# Caveat II: non-frozen $f$

- more generic case: $f$ evolves
- the model choice: $\tilde{V}(\phi) = \frac{1}{2} m^2 \phi^2 \exp(-\lambda \chi^2 / M_p^2)$, $\mathcal{S}_{\chi\chi} = \mathcal{S}_{\phi\phi} = 1$ and $2f/M_p^2 = \exp(-0.5 \lambda \chi^2 / M_p^2)$.
- $\lambda = \{0.05, 0.06\}$, initial conditions $\chi_* = 10^{-3} M_p$ and $\phi_* = 15.0 M_p$.




- special case: potential $\sim$ ridge like, initial conditions close to top of the ridge

# Discussion and Conclusion

## Take home message

- conventional definition of curvature perturbation is a not frame-dependent quantity
- in theory, using the wrong definition can lead to very different results
- e.g. $\zeta - \tilde{\zeta}$ can be arbitrarily large
- however asymptotically the difference between observables are negligible in general after reheating
- possible to realise counter examples, but need fine-tuned initial conditions

## Ongoing and Future Directions

- study the correlation between large (local) non-Gaussianity and the fractional difference
- decay rates are generically modulated in non-minimal coupled models even in simple perturbative reheating

$$\Gamma \rightarrow \Gamma(\chi)$$

- At quantum level? see Steinwachs