Problem of definition: Jordan vs Einstein frame, beyond slow-roll

Godfrey Leung

godfrey.leung@apctp.org

Asia-Pacific Centre for Theoretical Physics

collaboration work with Jonathan White

[arXiv:1509.xxxxx]

3rd - 5th August 2015
APCTP-TUS joint workshop on Dark Energy
In Einstein gravity, it is assumed matter is minimally coupled to gravity

\[ S = \int d^4x \sqrt{-g} \left( M^2_p R/2 \right) + S_m[g_{\mu \nu}, \phi] \]

- \( R = \) Ricci Scalar, \( S_m = \) matter action
- For example, single scalar field \( \phi \)

\[ S_m = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] \]
Einstein’s General Relativity (con’t)

- Strong/weak equivalence principle
- Strong: The gravitational motion of a small test body depends only on its initial position in spacetime and velocity, and not on its constitution.
Can we violate the equivalence principle? Yes
Non-minimal Coupled Models

- Can we violate the equivalence principle? Yes
- Example, introducing non-minimal coupling to gravity

\[ S_J = \int d^4x \sqrt{-g}(f(\psi)M_P^2 R/2) + S_m[g_{\mu\nu}, \psi] \]

- generic in modified gravity and unified theories, such as string theory, f(R), Chameleons, TeVeS...
- conformally related to

\[ S_E = \int d^4x \sqrt{-g_E}(M_P^2 \tilde{R}/2) + \tilde{S}_m[(g_{\mu\nu})_E, \psi] \]

- by the conformal transformation \( g_{\mu\nu} \rightarrow (g_{\mu\nu})_E = f(\psi)g_{\mu\nu} \)
- They are mathematically equivalent
- Question: But are they physically equivalent?
Physics should be frame independent!

- Conformal transformation = field redefinition
- More precisely, conformal transformation = change of scale
- 1 meter is only meaningful with respect to a reference scale
In cosmology, density fluctuations are usually quantified in terms of $\zeta_q$

$$\zeta \equiv -\varphi + \frac{H\delta\rho}{\dot{\rho}}$$

can be defined in both conformal frames, where $\rho$ is the effective energy density from $G_{\mu\nu} = T_{\mu\nu}/M_P^2$.

For instance
But...

- In cosmology, density fluctuations are usually quantified in terms of $\zeta_q$

$$\zeta \equiv -\varphi + \frac{H\delta \rho}{\dot{\rho}}$$

- can be defined in both conformal frames, where $\rho$ is the effective energy density from $G_{\mu\nu} = T_{\mu\nu}/M_P^2$

- For instance

- dimensionless and gauge invariant, but *not frame independent* as we will see...
Aside: Isocurvature perturbation

- Perturbation is purely adiabatic if $\delta P = \frac{\dot{P}}{\rho} \delta \rho$. Not always true though...
Aside: Isocurvature perturbation

- Perturbation is purely adiabatic if $\delta P = \frac{\dot{P}}{\rho} \delta \rho$. Not always true though...
- Entropic/isocurvature perturbations
  - perturbations $\perp$ background trajectory, natural in multifield inflation models

Some hints in 2013 Planck results
In inequivalence of $\zeta$ in Einstein and Jordan frames:

- It was found that $\zeta$ is frame-dependent in the presence of isocurvature perturbation [White et al. 12, arXiv:1205.0656], [Chiba and Yamaguchi 13]
- reason: isocurvature perturbation is frame-dependent (artificial)
Inequivalence of $\zeta$ in Einstein and Jordan frames

- Examples, in multifield models

$$
\zeta - \tilde{\zeta} \approx A_{JK} \mathcal{K}^{JK} + B_{JK} \dot{\mathcal{K}}^{JK}, \quad \mathcal{K}^{JK} \equiv \delta^J_I \dot{\phi}^K - \delta^K_I \dot{\phi}^J
$$

where

$$
A_{JK} = \frac{1}{C} \left\{ \left[ \left( \frac{G_{PQ} \dot{\phi}^P \dot{\phi}^Q + 2 (\ddot{f} - H \dot{f})}{2f} \right) f_K G_{IJ} ight. \\
+ 2H \dot{\phi}^L f_{KL} G_{IJ} - 2H \dot{f} G_{IJ, K} \right] \dot{\phi}^I - 2H f_K G_{IJ} \ddot{\phi}^I \right\},
$$

$$
B_{JK} = \frac{2H f_K G_{IJ} \dot{\phi}^I}{C}
$$

and

$$
C = 2\kappa^2 f S_{MN} \dot{\phi}^M \dot{\phi}^N \left( G_{PQ} \dot{\phi}^P \dot{\phi}^Q + 2 (\ddot{f} - H \dot{f}) \right).
$$

- $\mathcal{K}^{JK}$ is a measure of the isocurvature perturbation
- $\zeta - \tilde{\zeta} \to 0$ only if isocurvature vanishes in general
Relation between $\zeta$ and $\tilde{\zeta}$

- linear and non-linear order
- using the separate universe assumption, we can write $\zeta$ and $\tilde{\zeta}$ in terms of the $\delta N$ formalism [White, Minamitsuji and Sasaki et al.]

$$
\zeta = N_I \delta \phi^I + N_{IJ} \delta \phi^I_R \delta \phi^J_R + \\
\tilde{\zeta} = \tilde{N}_I \delta \phi^I + \tilde{N}_{IJ} \delta \phi^I_R \delta \phi^J_R + ...
$$

- $\delta \phi^I_R = $ flat-gauge field perturbations in Einstein frame
- observables can be expressed in terms of $\delta N$ coefficients, e.g. in Jordan frame

$$
n_s - 1 = -2(\tilde{\epsilon}_H)_* - \frac{2}{N_I N^I} + \frac{N^K N^L}{3 \tilde{H}_*^2 N_I N_J S^{|I|}} \left[ \tilde{\nabla}_K \tilde{\nabla}_L \tilde{\nabla} - \tilde{\nabla}_{KLPQ} \frac{d\phi^P}{d\tilde{t}} \frac{d\phi^Q}{d\tilde{t}} \right]_*
$$

$$
\mathcal{P}_\zeta = N^I N_I \left( \frac{\dot{H}}{2\pi} \right)_*^2 \quad \text{and} \quad r = \frac{8}{N^I N_I}
$$
Relation between $\zeta$ and $\tilde{\zeta}$ (con’t)

- general relation between $N$ and $\tilde{N}$
  \[ N = \int_{\mathcal{R}}^{\omega} \mathcal{H} d\eta = \int_{\mathcal{R}}^{\omega} \left( \tilde{\mathcal{H}} - \frac{f'}{2f} \right) d\eta = \tilde{N}(\omega, \mathcal{R}) - \frac{1}{2} \ln \left( \frac{f_\omega}{f_\mathcal{R}} \right) \]

- consider a simplified case where $f' \approx 0$ at the time of interest (late time)

- using the $\delta N$ formalism, the first and second order $\delta N$ coefficients are related by

  \[ \mathcal{N}_I \approx \tilde{\mathcal{N}}_I - \left( \frac{1}{2} + c \right) \left( \frac{f_J}{f} \right) \left( \frac{\partial \phi^K}{\partial \phi^*_J} \right) \omega \]

  \[ \mathcal{N}_{IJ} \approx \tilde{\mathcal{N}}_{IJ} - \left( \frac{1}{2} + c \right) \left[ \left( \frac{f_K f_L}{f^2} - \frac{f_K f_L}{f^2} \right) \left( \frac{\partial \phi^K}{\partial \phi^*_J} \right) \left( \frac{\partial \phi^L}{\partial \phi^*_I} \right) \omega \left( \frac{\partial \phi^*}{\partial \phi^*_I} \right) \omega \right] \]

- we have assumed $\epsilon_f \equiv |f'/\mathcal{H} f| \ll 1$

- $c \equiv \frac{\tilde{\mathcal{H}}}{p'}$, equals to $-1/3$ (matter era) and $-1/4$ (radiation era)

- $\zeta - \tilde{\zeta}$ can be arbitrarily large depending on $f$, but how about observables?
Defining the fractional difference between the power spectra amplitudes and the spectral indices

\[ \Delta P_\zeta \equiv \frac{P_\zeta - \tilde{P}_\zeta}{\tilde{P}_\zeta} \]

and slightly different definition for \( n_s \) and \( f_{NL} \)

\[ \frac{n_s - 1 + 2(\bar{\epsilon}_H)^*}{\bar{n}_s - 1 + 2(\bar{\epsilon}_H)^*} = (1 + \Delta P_\zeta)^{-1} (1 + \Delta n_s) \]
\[ \frac{f_{NL}}{\tilde{f}_{NL}} = (1 + \Delta P_\zeta)^{-2} (1 + \Delta f_{NL}) \]

Using the asymptotic relation between the \( \delta N \) coefficients, we therefore have

\[ \Delta P_\zeta = -\frac{1}{\tilde{N}_P \tilde{N}_P} \left[ (1 + 2c) \left( \frac{f_K}{f} \right) \left( \frac{\partial \phi^K}{\partial \phi^I} \right)_\omega \tilde{N}^I \right. \]
\[ - \left( \frac{1}{2} + c \right)^2 \left( \frac{f_K f_L}{f^2} \right) \left( \frac{\partial \phi^K}{\partial \phi^L} \right)_\omega \left( \frac{\partial \phi^K}{\partial \phi^J} \right)_\omega S_{IJ}^* \]
Model considered

- we consider the multifield model with the following action in the Jordan frame

\[
S_J = \int d^4x \sqrt{-g} \left\{ f(\Phi) R - \frac{1}{2} G_{IJ}(\Phi) g^{\mu\nu} \partial_\phi^I \partial_\phi^J - V(\Phi) \right\}
\]

- or in the Einstein frame

\[
S_E = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{M_p^2 R}{2} - \frac{1}{2} S_{IJ}(\Phi) \tilde{g}^{\mu\nu} \partial_\phi^I \partial_\phi^J - \tilde{V}(\Phi) \right\}
\]

- note that the field space metric and scalar potential are related by

\[
S_{IJ} = \frac{M_p^2}{2f} \left( G_{IJ} + 3 \frac{f_I f_J}{f} \right) \quad \text{and} \quad \tilde{V} = \frac{VM_p^4}{4f^2}
\]

- after inflation ends, reheating is modeled by adding a friction to the field EOM in Einstein frame

\[
(\phi^I)'' + \tilde{\Gamma}_{JK}^I (\phi^J)' (\phi^K)' + 2(\tilde{H} + \tilde{a}^{(I)} \Gamma) (\phi^I)' + \tilde{a}^2 S^{IJ} \tilde{V}_J = 0
\]

\[
\rho'_\gamma + 4\mathcal{H} \rho_\gamma = \frac{(I) \Gamma}{\tilde{a}} S_{IJ} (\phi^I)' (\phi^J)'
\]
Simple example: Non-minimal coupling $f' = 0$ always

- to illustrate, we consider the class of two-field models
- simple explicit example: $f = f(\chi)$ and $\chi$ is a frozen spectator field
  
  \[ S_{\chi\chi} \equiv \frac{M_p^2}{2f} \left( G_{\chi\chi} + 3 \frac{\chi^2}{f} \right) \quad \text{and} \quad \tilde{V} = V(\phi)M_p^2 \]

- we further assume $G_{\phi\chi} = 0$ and $G_{\phi\phi} = f$ such that there is no mixing in the kinetic term

- Einstein frame results = simple chaotic inflation
  
  \[ \tilde{n}_s - 1 = -6\tilde{\epsilon}_* + 2\tilde{n}_* , \quad \tilde{r} = 16\tilde{\epsilon}_* \]
Simple example: Non-minimal coupling $f' = 0$ always (con’t)

- results during slow-roll inflationary regime, consistent with previous analytic work

- model choice: $\tilde{V}(\phi) = \frac{1}{2} m^2 \phi^2$, $2f/M_p^2 = e^{-\beta x/M_p}$ with $\phi_* = 15.0 M_p$
- observables seem coincide at the end of inflation
- however, $\zeta$ still evolve in Jordan frame...
how about beyond slow-roll, particularly after reheating?
how about beyond slow-roll, particularly after reheating?

for this particular model, since the non-minimal coupled field $\chi$ is frozen, things simplify

the fractional difference

$$\Delta P_\zeta = \frac{1}{16} \left( \frac{1}{\tilde{N}_\phi^2} \right) \left( \frac{f_\chi}{f} \right)^2 S_{\chi\chi}$$

$$\frac{\tilde{n}_s - n_s}{n_s} = \Delta P_\zeta \left[ \frac{n_s - 1 + 2(\tilde{\epsilon}_H)_*}{n_s} \right]$$

difference can only be large if $\Delta P_\zeta \gg O(1)$
Non-minimal Coupled Models
Asymptotic relation beyond slow roll
Simple explicit example
Counter-example
Discussion and Conclusion

Simple example: Non-minimal coupling $f' = 0$ always (con’t)

- Beyond slow-roll regime

- with reheating parameter $\Gamma_\phi = 0.1(m/\sqrt{2})$
- evolution terminates when $\Omega_\gamma > 0.9999$
we see the fractional difference between observables are negligible even $\zeta - \tilde{\zeta}$ is large

why?
we see the fractional difference between observables are negligible even \( \zeta - \tilde{\zeta} \) is large

why? recall

\[
\Delta P_\zeta = -\frac{1}{\tilde{N}_P \tilde{N}^P} \left[ (1 + 2c) \left( \frac{f_K}{f} \right) \left( \frac{\partial \phi^K}{\partial \phi^*_I} \right) \omega \tilde{N}^I \right. \\
\left. - \left( \frac{1}{2} + c \right)^2 \left( \frac{f_K f_L}{f^2} \right) \left( \frac{\partial \phi^K}{\partial \phi^*_I} \right) \omega \left( \frac{\partial \phi^L}{\partial \phi^*_J} \right) \omega S^I_J \right]
\]

reason: Einstein frame field space metric \( S^I_J \) also depends on \( f \)

\[
S_{\chi \chi} \equiv \frac{M_P^2}{2f} \left( G_{\chi \chi} + 3 \frac{f^2}{\chi} \right)
\]
Caveat I: negative Jordan frame field space metric

- We may tune \( (G_{\chi\chi})^* \rightarrow (f_\chi^2/f)^* \), with \( S_{\chi\chi} \) remains positive
- Example: \( G_{\chi\chi} = -b_1(f_\chi^2/f) \), \( \tilde{V}(\phi) = \frac{1}{2} m^2 \phi^2 \) and \( 2f/M_p^2 = e^{-\beta \chi/M_p} \), with \( b_1 \leq 3 \)

- only in the very fine-tuned limit the difference becomes significant

\[ \begin{array}{ll}
\text{Spectral index} & \tilde{N} \\
\text{Jordan, } b_1 = 1 & \text{Jordan, } b_1 = 2.5 \\
\text{Jordan, } b_1 = 2.9 & \text{Einstein} \\
\end{array} \]

\[ \begin{array}{ll}
\text{Tensor-to-scalar ratio} & \tilde{N} \\
\text{Jordan, } b_1 = 1 & \text{Jordan, } b_1 = 2.5 \\
\text{Jordan, } b_1 = 2.9 & \text{Einstein} \\
\end{array} \]
Caveat II: non-frozen $f$

- more generic case: $f$ evolves
- the model choice: $\tilde{V}(\phi) = \frac{1}{2} m^2 \phi^2 \exp(-\lambda \chi^2/M_p^2)$, $S_{\chi\chi} = S_{\phi\phi} = 1$ and $2f/M_p^2 = \exp(-0.5\lambda \chi^2/M_p^2)$.
- $\lambda = \{0.05, 0.06\}$, initial conditions $\chi_* = 10^{-3} M_p$ and $\phi_* = 15.0 M_p$.

![Graphs showing spectral index and tensor-to-scalar ratio](image)

- special case: potential $\sim$ ridge like, initial conditions close to top of the ridge
Take home message

- conventional definition of curvature perturbation is a not frame-dependent quantity
- in theory, using the wrong definition can lead to very different results
- e.g. $\zeta - \tilde{\zeta}$ can be arbitrarily large
- however asymptotically the difference between observables are negligible in general after reheating
- possible to realise counter examples, but need fine-tuned initial conditions

Ongoing and Future Directions

- study the correlation between large (local) non-Gaussianity and the fractional difference
- decay rates are generically modulated in non-minimal coupled models even in simple perturbative reheating

$$\Gamma \rightarrow \Gamma(\chi)$$

- At quantum level? see Steinwachs