Gauge Invariant Perturbations and Covariance in Quantum Cosmology

Guillermo A. Mena Marugán
(IEM-CSIC)
With Laura Castelló Gomar, & Mercedes Martin-Benito

2nd APCTP-TUS Workshop, August 2015
Our Universe is approximately homogeneous and isotropic: Background with cosmological perturbations.

Need of gauge invariant descriptions (Bardeen, Mukhanov-Sasaki).

Perturbations: Canonical formulation with constraints (Langlois, Pinto-Nieto).

Quantum treatment including the background (Halliwell-Hawking, Shirai-Wada).

Hybrid formalism with a Born-Oppenheimer ansatz: Covariance.
We consider a FLRW universe with compact flat topology.

We include a scalar field subject to a potential (e.g. a mass term).

For simplicity, we analyze only SCALAR pertubations.
We expand the inhomogeneities in a (real) Fourier basis \((\vec{n} \in \mathbb{Z}^3)\):

\[
Q_{\vec{n},+} = \sqrt{2} \cos (\vec{n} \cdot \vec{\theta}), \quad Q_{\vec{n},-} = \sqrt{2} \sin (\vec{n} \cdot \vec{\theta})
\]

\[ e^{\pm i\vec{n} \cdot \vec{\theta}} = \frac{Q_{\vec{n},+} \pm i Q_{\vec{n},-}}{\sqrt{2}}. \]

We take \(n_1 \geq 0\). The eigenvalue of the Laplacian is 
\[ -\omega_n^2 = -\vec{n} \cdot \vec{n}. \]

Zero modes are treated exactly (at linear perturbative order) in the expansions.
Scalar perturbations: **metric and field.**

\[ h_{ij} = \sigma^2 e^{2\alpha} \left[ 0 h_{ij} + 2 \sum \left\{ a_{\vec{n}, \pm}(t) Q_{\vec{n}, \pm} + b_{\vec{n}, \pm}(t) \left( \frac{3}{\omega_n^2} Q_{\vec{n}, \pm, ij} + Q_{\vec{n}, \pm} + 0 h_{ij} \right) \right\} \right], \]

\[ N = \sigma \left[ N_0(t) + e^{3\alpha} \sum g_{\vec{n}, \pm}(t) Q_{\vec{n}, \pm} \right], \quad N_i = \sigma^2 e^{2\alpha} \sum \frac{k_{\vec{n}, \pm}(t)}{\omega_n^2} (Q_{\vec{n}, \pm, i}), \]

\[ \Phi = \frac{1}{\sigma (2\pi)^{3/2}} \left[ \varphi(t) + \sum f_{\vec{n}, \pm}(t) Q_{\vec{n}, \pm} \right]. \]

\[ \sigma^2 = \frac{G}{6\pi^2}, \quad \tilde{m} = m \sigma. \]

**Truncating at quadratic perturbative order** in the action:

\[ H = N_0 \left[ H_0 + \sum H_2^{\vec{n}, \pm} \right] + \sum g_{\vec{n}, \pm} H_1^{\vec{n}, \pm} + \sum k_{\vec{n}, \pm} \tilde{H}_1^{\vec{n}, \pm}. \]
Scalar constraint:

\[ H_0 = \frac{e^{-3\alpha}}{2} \left(-\pi_\alpha^2 + \pi_\phi^2 + e^{6\alpha} \tilde{m}^2 \phi^2\right), \]

Classical system: Inhomogeneities

\[ 2 e^{3\alpha} H_2^{\tilde{n},\pm} = -\pi_\alpha^2 a_{\tilde{n},\pm} + \pi_\alpha^2 b_{\tilde{n},\pm} + \pi_\phi^2 f_{\tilde{n},\pm} + 2 \pi_\alpha \left( a_{\tilde{n},\pm} \pi_{a_{\tilde{n},\pm}} + 4 b_{\tilde{n},\pm} \pi_{b_{\tilde{n},\pm}} \right) - 6 \pi_\phi a_{\tilde{n},\pm} \pi f_{\tilde{n},\pm} \]

Linear perturbative constraints:

\[ H_1^{\tilde{n},\pm} = -\pi_\alpha \pi_{a_{\tilde{n},\pm}} + \pi_\phi \pi_{f_{\tilde{n},\pm}} + \left( \pi_\alpha^2 - 3 \pi_\phi^2 + 3 e^{3\alpha} H_0 \right) a_{\tilde{n},\pm} - \frac{\omega_n^2}{3} e^{4\alpha} \left( a_{\tilde{n},\pm} + b_{\tilde{n},\pm} \right) \]

\[ + e^{6\alpha} \tilde{m}^2 \phi f_{\tilde{n},\pm}, \quad \overline{H}_1^{\tilde{n},\pm} = \frac{1}{3} \left[ -\pi_{a_{\tilde{n},\pm}} + \pi_{b_{\tilde{n},\pm}} + \pi_\alpha \left( a_{\tilde{n},\pm} + 4 b_{\tilde{n},\pm} \right) + 3 \pi_\phi f_{\tilde{n},\pm} \right]. \]
Consider the sector of zero modes as describing a fixed background.

Look for a transformation of the perturbations --canonical only with respect to their symplectic structure-- adapted to gauge invariance:

a) Find new variables that are an abelianization of the perturbative constraints.

\[ \tilde{H}_1^{\pm} = H_1^{\pm} - 3 e^{3\alpha} \ H_0 a,_{\pm}. \]

b) Include the Mukhanov-Sasaki variable, exploiting its gauge invariance.

\[ v_{\pm} = e^{\alpha} \left[ f,_{\pm} + \frac{\pi \phi}{\pi \alpha} (a,_{\pm} + b,_{\pm}) \right]. \]

c) Complete the transformation with suitable momenta.
Gauge invariant perturbations

- **Mukhanov-Sasaki** momentum:

\[
\pi_{v_{\tilde{n}}, \pm} = e^{-\alpha} \left[ \pi_{f_{\tilde{n}}, \pm} + \frac{1}{\pi_\Phi} \left( e^{6\alpha} \tilde{m}^2 \Phi f_{\tilde{n}}, \pm + 3 \pi_\Phi^2 b_{\tilde{n}}, \pm \right) \right] + F v_{\tilde{n}, \pm}.
\]

- There is an **ambiguity** in a function of the background variables, \( F \).

- The Mukhanov-Sasaki momentum is independent of \( (\pi_{a_{\tilde{n}}, \pm}, \pi_{b_{\tilde{n}}, \pm}) \).

  - The perturbative Hamiltonian constraint is independent of \( \pi_{b_{\tilde{n}}, \pm} \).
  - The perturbative momentum constraint depends through \( \pi_{a_{\tilde{n}}, \pm} - \pi_{b_{\tilde{n}}, \pm} \).

- It is straightforward to complete the transformation:

\[
\tilde{C}_{1}^{\tilde{n}, \pm} = 3 b_{\tilde{n}, \pm}, \quad \check{C}_{1}^{\tilde{n}, \pm} = - \frac{1}{\pi_\alpha} (a_{\tilde{n}, \pm} + b_{\tilde{n}, \pm}).
\]
The redefinition of the perturbative Hamiltonian constraint amounts to a **redefinition of the lapse** at our order of truncation in the action:

\[
H = \tilde{N}_0 \left[ H_0 + \sum_{\vec{n}, \pm} H_2^{\vec{n}, \pm} \right] + \sum_{\vec{n}, \pm} g_{\vec{n}, \pm} \tilde{H}_1^{\vec{n}, \pm} + \sum_{\vec{n}, \pm} k_{\vec{n}, \pm} \tilde{H}_{\uparrow 1}^{\vec{n}, \pm},
\]

\[
\tilde{N}_0 = N_0 + 3 e^{3\alpha} \sum_{\vec{n}, \pm} g_{\vec{n}, \pm} a_{\vec{n}, \pm}.
\]
We remove the ambiguity in the Mukhanov-Sasaki momentum by any of the following:

- It equals the time derivative of the Mukhanov-Sasaki variable.
- The scalar constraint is quadratic in this momentum (no linear terms).
- It is possible to adopt a Fock quantization with invariance under rigid rotations and unitary evolution (Cortez, Mena-Marugán, Velhinho).

\[
\bar{\pi}_{v_{n,\pm}} = e^{-\alpha} \left[ \pi_{f_{\bar{n}, \pm}} + \frac{\pi_{\varphi}}{\varphi} \left( e^{6\alpha} \tilde{m}^2 \varphi f_{\bar{n}, \pm} + 3 \pi_{\varphi}^2 b_{\bar{n}, \pm} \right) 
- e^{-2\alpha} \left( \frac{1}{\pi_{\varphi}} e^{6\alpha} \tilde{m}^2 \varphi + \pi_{\alpha} \right) \right] v_{\bar{n}, \pm}.
\]
Canonical transformation: Inverse

\[
a_{\tilde{n}, \pm} = -\pi_\alpha \tilde{C}\n, \pm - \frac{1}{3} \tilde{C}_{\uparrow 1}, \\
b_{\tilde{n}, \pm} = \frac{1}{3} \tilde{C}_{\uparrow 1}, \\
f_{\tilde{n}, \pm} = e^{-\alpha} \nu_{\tilde{n}, \pm} + \pi_\varphi \tilde{C}_{\uparrow 1}.
\]

\[
\pi_{a_{\tilde{n}, \pm}} = - \frac{1}{\pi_\alpha} \tilde{H}\n, \pm + \frac{\pi_\varphi}{\pi_\alpha} e^\alpha \tilde{\nu}_{\tilde{n}, \pm} + \frac{e^{-\alpha}}{\pi_\alpha} \left( e^{6\alpha} \tilde{m}^2 \varphi + \pi_\varphi \pi_\alpha + 3 \frac{\pi_\varphi^3}{\pi_\alpha} \right) \nu_{\tilde{n}, \pm} \\
+ \left( 3 \pi_\varphi^2 + \frac{1}{3} e^{4\alpha} \omega_n^2 - \pi_\alpha^2 \right) \tilde{C}\n, \pm - \frac{1}{3} \pi_\alpha \tilde{C}_{\uparrow 1},
\]

\[
\pi_{b_{\tilde{n}, \pm}} = 3 \tilde{H}_{\uparrow 1} - \frac{1}{\pi_\alpha} \tilde{H}\n, \pm + \frac{\pi_\varphi}{\pi_\alpha} e^\alpha \tilde{\nu}_{\tilde{n}, \pm} + \frac{e^{-\alpha}}{\pi_\alpha} \left( e^{6\alpha} \tilde{m}^2 \varphi - 2 \pi_\varphi \pi_\alpha + 3 \frac{\pi_\varphi^3}{\pi_\alpha} \right) \nu_{\tilde{n}, \pm} \\
+ \frac{1}{3} e^{4\alpha} \omega_n^2 \tilde{C}\n, \pm - \frac{4}{3} \pi_\alpha \tilde{C}_{\uparrow 1},
\]

\[
\pi_{f_{\tilde{n}, \pm}} = e^\alpha \tilde{\nu}_{\tilde{n}, \pm} + e^{-\alpha} \left( \pi_\alpha + 3 \frac{\pi_\varphi^2}{\pi_\alpha} \right) \nu_{\tilde{n}, \pm} - e^{6\alpha} \tilde{m}^2 \varphi \tilde{C}_{\uparrow 1} - \pi_\varphi \tilde{C}_{\uparrow 1}.
\]
We now include the **zero modes** as variables of the system, and complete the transformation to a **canonical** one in their presence.

We re-write the Legendre term of the action, keeping its canonical form at the considered **perturbative order**:

\[
\int dt \left[ \sum_a \dot{w}_q^a w_p^a + \sum_{l, \bar{n}, \pm} \dot{X}_{q_i}^{\bar{n}, \pm} X_{p_i}^{\bar{n}, \pm} \right] = \int dt \left[ \sum_a \dot{\tilde{w}}_q^a \tilde{w}_p^a + \sum_{l, \bar{n}, \pm} \dot{V}_{q_i}^{\bar{n}, \pm} V_{p_i}^{\bar{n}, \pm} \right].
\]

**Zero modes:** Old \( \{ w_q^a, w_p^a \} \rightarrow \) New \( \{ \tilde{w}_q^a, \tilde{w}_p^a \} \). \( \{ w_q^a \} = \{ \alpha, \varphi \} \).

**Inhomogeneities:** Old \( \{ X_{q_i}^{\bar{n}, \pm}, X_{p_i}^{\bar{n}, \pm} \} \rightarrow \) New:

\[
\{ V_{q_i}^{\bar{n}, \pm}, V_{p_i}^{\bar{n}, \pm} \} = \left\{ (v_{\bar{n}, \pm}, \tilde{C}_{1}^{\bar{n}, \pm}, \tilde{C}_{\uparrow 1}^{\bar{n}, \pm}), (\pi_{v_{\bar{n}, \pm}}, \tilde{H}_{1}^{\bar{n}, \pm}, \tilde{H}_{\uparrow 1}^{\bar{n}, \pm}) \right\}.
\]
Using that the change of perturbative variables is linear, it is not difficult to find the **new zero modes**, which include modifications quadratic in the perturbations.

**Expressions:**

\[
\begin{align*}
\tilde{w}^a_q &= \tilde{w}^a_q - \frac{1}{2} \sum_{l,\tilde{n},\pm} X_{\tilde{n},\pm} \left( \frac{\partial X_{p_i}}{\partial \tilde{w}_p^a} - \frac{\partial X_{q_i}}{\partial \tilde{w}_q^a} \right) \tilde{w}_p^a, \\
\tilde{w}^a_p &= \tilde{w}^a_p + \frac{1}{2} \sum_{l,\tilde{n},\pm} X_{\tilde{n},\pm} \left( \frac{\partial X_{p_i}}{\partial \tilde{w}_q^a} - \frac{\partial X_{q_i}}{\partial \tilde{w}_p^a} \right) \tilde{w}_q^a.
\end{align*}
\]

\[\left\{ X_{q_i}^{\tilde{n},\pm}, X_{p_i}^{\tilde{n},\pm} \right\} \rightarrow \text{Old perturbative variables in terms of the new.}\]
Since the change of the zero modes is quadratic in the perturbations, the new scalar constraint at our truncation order is

\[ H_0(w^a) + \sum \bar{\n}, \pm H_2^\bar{\n}, \pm (w^a, X_l^\bar{\n}, \pm) \Rightarrow \]

\[ H_0(\tilde{w}^a) + \sum_b (w^b - \tilde{w}^b) \frac{\partial H_0}{\partial \tilde{w}^b}(\tilde{w}^a) + \sum \bar{\n}, \pm H_2^\bar{\n}, \pm [\tilde{w}^a, X_l^\bar{\n}, \pm (\tilde{w}^a, V_l^\bar{\n}, \pm)], \]

\[ w^a - \tilde{w}^a = \sum \bar{\n}, \pm \Delta \tilde{w}^a_{\bar{\n}, \pm}. \]

The perturbative contribution to the new scalar constraint is:

\[ \bar{H}^\bar{\n}, \pm = H_2^\bar{\n}, \pm + \sum_a \Delta \tilde{w}^a_{\bar{\n}, \pm} \frac{\partial H_0}{\partial \tilde{w}^a}. \]

This is the change expected for zero modes treated as time dependent external variables with dynamics generated by \( H_0 \).
Carrying out the calculation explicitly, one obtains:

$$\begin{align*}
\ddot{H}_2^{\tilde{n}, \pm} = \dot{H}_2^{\tilde{n}, \pm} + & F_2^{\tilde{n}, \pm} H_0 + \ddot{F}_1^{\tilde{n}, \pm} \dot{H}_1^{\tilde{n}, \pm} + \\
& \left( \frac{F_{\uparrow 1}^{\tilde{n}, \pm}}{2} - 3 \frac{e^{-3 \tilde{\alpha}}}{\pi \tilde{\alpha}} \dot{H}_1^{\tilde{n}, \pm} + \frac{9}{2} e^{-3 \tilde{\alpha}} \ddot{H}_{\uparrow 1}^{\tilde{n}, \pm} \right) \ddot{H}_1^{\tilde{n}, \pm},
\end{align*}$$

$$\begin{align*}
\ddot{H}_2^{\tilde{n}, \pm} = & \frac{e^{-\tilde{\alpha}}}{2} \left[ \omega_n^2 + e^{-4 \tilde{\alpha}} \pi \tilde{\alpha}^2 + \tilde{m}^2 e^{2 \tilde{\alpha}} \left( 1 + 15 \tilde{\Phi}^2 - 12 \tilde{\Phi} \frac{\pi \tilde{\Phi}}{\pi \tilde{\alpha}} - 18 e^{6 \tilde{\alpha}} \tilde{m}^2 \frac{\tilde{\Phi}^4}{\pi^2 \tilde{\alpha}} \right) \right] \left( \nu_{\tilde{n}, \pm} \right)^2 + \left( \pi_{\nu_{\tilde{n}, \pm}} \right)^2.
\end{align*}$$

The $F$'s are well determined functions.

The term $\ddot{H}_2^{\tilde{n}, \pm}$ is the Mukhanov-Sasaki Hamiltonian.

It has no linear contributions of the Mukhanov-Sasaki momentum.

It is linear in the momentum $\pi_{\tilde{\Phi}}$. 

We re-write the **total Hamiltonian** of the system at our **truncation order**, redefining the Lagrange multipliers:

\[
\begin{align*}
\bar{\mathcal{H}}_{2}^{\vec{n},\pm} &= \bar{\mathcal{H}}_{2}^{\vec{n},\pm} + F_{2}^{\vec{n},\pm} H_{0} + \bar{F}_{1}^{\vec{n},\pm} \bar{H}_{1}^{\vec{n},\pm} + \left(F_{\uparrow 1}^{\vec{n},\pm} - 3 \frac{e^{-3\bar{\alpha}}}{\pi \bar{\alpha}} \bar{H}_{1}^{\vec{n},\pm} + \frac{9}{2} e^{-3\bar{\alpha}} \bar{H}_{\uparrow 1}^{\vec{n},\pm} \right) \bar{H}_{\uparrow 1}^{\vec{n},\pm} \Rightarrow \\
H &= \bar{N}_{0} \left[H_{0} + \sum_{\vec{n},\pm} \bar{H}_{2}^{\vec{n},\pm} \right] + \sum_{\vec{n},\pm} \bar{G}_{\vec{n},\pm} \bar{H}_{1}^{\vec{n},\pm} + \sum_{\vec{n},\pm} \bar{K}_{\vec{n},\pm} \bar{H}_{\uparrow 1}^{\vec{n},\pm}.
\end{align*}
\]

The new **lapse** multiplier is \(\bar{N}_{0} = N_{0} + \sum_{\vec{n},\pm} \left(3 e^{3\bar{\alpha}} g_{\vec{n},\pm} a_{\vec{n},\pm} + N_{0} F_{2}^{\vec{n},\pm} \right)\).
Approximation: Quantum geometry effects are especially relevant in the background.

- Adopt a quantum cosmology scheme for the zero modes and a Fock quantization for the perturbations. The scalar constraint couples them.

- We assume:
  a) The zero modes (derived above) continue to commute with the perturbations under quantization,
  b) Functions of $\tilde{\phi}$ act by multiplication.
The **Fock representation** in QFT is fixed (up to unitary equivalence) by:

1) The *background isometries*; 2) The demand of a **UNITARY** evolution.

- The introduced **scaling** of the field by the scale factor is essential for unitarity.

- The proposal selects a **UNIQUE canonical pair** for the Mukhanov-Sasaki field, precisely the one we chose to fix the ambiguity in the momentum.

- We can use the massless representation (due to compactness), with its creation and annihilation operators, and the corresponding basis of occupancy number states $|N\rangle$. 
We admit that the operators that represent the linear constraints (or an integrated version of them) act as derivatives (or as translations).

Then, physical states are independent of \((\tilde{C}_{1}^{\vec{n}, \pm}, \tilde{C}_{\uparrow \downarrow}^{\vec{n}, \pm})\).

We pass to a space of states \(H_{\text{kin}}^{\text{grav}} \otimes H_{\text{kin}}^{\text{matt}} \otimes F\) that depend on the zero modes and the Mukhanov-Sasaki modes, with no gauge fixing.

In this covariant construction, physical states still must satisfy the scalar constraint given by the FLRW and the Mukhanov-Sasaki contributions.

\[
H_{S} = e^{-3\alpha} \left( H_{0} + \sum_{\vec{n}, \pm} \tilde{H}_{2}^{\vec{n}, \pm} \right) = 0.
\]
With \( \tilde{\Phi} \) acting by multiplication, we introduce the following functions on the phase space of the gravitational zero mode:

\[
H_0^{(2)} = \pi_\alpha^2 - e^{6\tilde{\alpha}} \tilde{m}^2 \tilde{\Phi}^2, \quad \Theta_0 = -12 e^{4\tilde{\alpha}} \tilde{m}^2 \frac{\tilde{\Phi}}{\pi_\alpha}, \quad \Theta_e = e^{2\tilde{\alpha}},
\]

\[
\Theta_0^{\tilde{n}, \pm} = -\Theta_0 (v_{\tilde{n}, \pm})^2, \quad \Theta_e^{\tilde{n}, \pm} = -\left[ (\Theta_e \omega_{n}^2 + \Theta_e^{q}) (v_{\tilde{n}, \pm})^2 + \Theta_e (\pi_{v_{n}, \pm})^2 \right],
\]

\[
\Theta_0 = \sum_{\tilde{n}, \pm} \Theta_0^{\tilde{n}, \pm}, \quad \Theta_e = \sum_{\tilde{n}, \pm} \Theta_e^{\tilde{n}, \pm}.
\]

Including the Mukhanov-Sasaki modes, we define:

\[
\Theta_0^{\tilde{n}, \pm} = -\Theta_0 (v_{\tilde{n}, \pm})^2, \quad \Theta_e^{\tilde{n}, \pm} = -\left[ (\Theta_e \omega_{n}^2 + \Theta_e^{q}) (v_{\tilde{n}, \pm})^2 + \Theta_e (\pi_{v_{n}, \pm})^2 \right],
\]

\[
\Theta_0 = \sum_{\tilde{n}, \pm} \Theta_0^{\tilde{n}, \pm}, \quad \Theta_e = \sum_{\tilde{n}, \pm} \Theta_e^{\tilde{n}, \pm}.
\]

We get the constraint:

\[
\hat{H}_S = \frac{1}{2} \left[ \hat{\pi}_{\tilde{\Phi}}^2 - \hat{H}_0^{(2)} - \hat{\Theta} - \frac{1}{2} \left( \hat{\Theta}_o \hat{\pi}_{\tilde{\Phi}} + \hat{\pi}_{\tilde{\Phi}} \hat{\Theta}_o \right) \right].
\]
Quantum constraint: 

\[ \hat{H}_S = \frac{1}{2} \left[ \hat{\pi}_{\Phi}^2 - \hat{H}_0^{(2)} - \hat{\Theta} - \frac{1}{2} \left( \hat{\Theta}_o \hat{\pi}_\Phi + \hat{\pi}_\Phi \hat{\Theta}_o \right) \right]. \]

This constraint is quadratic in the momentum of the zero mode of the scalar field.

The linear contribution goes with the derivative of the field potential.
Consider states whose dependence on the FLRW geometry and the inhomogeneities \( (N) \) split:

\[
\Psi = \Gamma (\tilde{\alpha}, \tilde{\phi}) \psi (N, \tilde{\phi}),
\]

The FLRW state is \textbf{normalized}, \textbf{peaked} and \textbf{evolves unitarily}:

\[
\Gamma (\tilde{\alpha}, \tilde{\phi}) = \hat{U} (\tilde{\alpha}, \tilde{\phi}) \chi (\tilde{\alpha}).
\]

\( \hat{U} \) is a unitary evolution operator close to the \textbf{unperturbed} one:

\[
[\hat{\pi}_{\tilde{\phi}}, \hat{U}] = \hat{H}_0 \quad \text{is a } \tilde{\phi} \text{-dependent operator on the FLRW geometry and }
\]

\[
(\hat{H}_0)^2 - \hat{H}_0^{(2)} + [\hat{\pi}_{\tilde{\phi}}, \hat{H}_0] \quad \text{is negligible on } \Gamma .
\]
• In the quantum constraint, we can disregard transitions from $\Gamma$ to other FLRW states if the following operators have small relative dispersion:

  i) $\hat{H}_0$,  
  ii) $\hat{\varphi}$,  
  iii) $-i d_\varphi \hat{\varphi} + (\hat{\varphi} \hat{H}_0 + \hat{H}_0 \hat{\varphi}) + 2\hat{\varphi}^q$.

• We have defined the total derivative $d_\varphi \hat{O} = i [\pi_\varphi - \hat{H}_0, \hat{O}]$.

• Then, taking the inner product with $\Gamma$ in the FLRW geometry, one gets a quantum evolution constraint for the Mukhanov-Sasaki field.
With the Born-Oppenheimer ansatz and our assumptions, we can write the scalar constraint as:

\[ \hat{\pi}_\phi^2 \psi + 2 \langle \hat{H}_0 \rangle_\Gamma \hat{\pi}_\phi \psi = \left[ \langle \hat{\Theta}_e + \frac{1}{2} (\hat{\Theta}_o \hat{H}_0 + \hat{H}_0 \hat{\Theta}_o) \rangle_\Gamma - \frac{i}{2} \langle d_{\hat{\phi}} \hat{\Theta}_o \rangle_\Gamma \right] \psi. \]

Besides, if we can neglect:

a) The term \( \hat{\pi}_\phi^2 \psi \).

b) The total \( \hat{\phi} \)-derivative of \( \hat{\Theta}_o \).

\[ \hat{\pi}_\phi \psi = \frac{2 \hat{\Theta}_e + (\hat{\Theta}_o \hat{H}_0 + \hat{H}_0 \hat{\Theta}_o) \rangle_\Gamma \psi}{4 \langle \hat{H}_0 \rangle_\Gamma}, \]

Schrödinger-like equation for the gauge invariant perturbations.
Starting from the Born-Oppenheimer form of the constraint and assuming a direct effective counterpart for the inhomogeneities:

\[
d_{\eta_{\Gamma}} v_{\tilde{n}, \pm} = - v_{\tilde{n}, \pm} \left[ 4 \pi^2 \omega_n^2 + \langle \hat{\theta}_e + \hat{\theta}_o \rangle_{\Gamma} \right],
\]

\[
\langle \hat{\theta}_e \rangle_{\Gamma} = 4 \pi^2 \frac{\langle \hat{\mathcal{S}}_{\tilde{e}}^q \rangle_{\Gamma}}{\langle \hat{\mathcal{S}}_{\tilde{e}} \rangle_{\Gamma}}, \quad \langle \hat{\theta}_o \rangle_{\Gamma} = 2 \pi^2 \frac{\langle (\hat{\mathcal{S}}_o \hat{H}_0 + \hat{H}_0 \hat{\mathcal{S}}_o) - i d_{\phi} \hat{\mathcal{S}}_o \rangle_{\Gamma}}{\langle \hat{\mathcal{S}}_{\tilde{e}} \rangle_{\Gamma}}.
\]

where we have defined the state-dependent conformal time

\[
2 \pi \ d \ \eta_{\Gamma} = \langle \hat{\mathcal{S}}_{\tilde{e}} \rangle_{\Gamma} dT,
\]

with

\[
dt = \sigma e^{3\alpha} dT.
\]
Effective Mukhanov-Sasaki equations

- For all modes:

\[
\frac{d^2}{d\eta^2} v_{n,\pm} = - v_{n,\pm} \left[ 4\pi^2 \omega_n^2 + \langle \hat{\theta}_e + \hat{\theta}_o \rangle \Gamma \right].
\]

- The expectation value depends (only) on the conformal time, through \( \tilde{\phi} \). It is the time dependent part of the frequency, but it is mode independent.

- The effective equations are of harmonic **oscillator** type, with no dissipative term, and **hyperbolic in the ultraviolet** regime.
Conclusions

- We have considered a FLRW universe with a massive scalar field perturbed at **quadratic** order in the action.

- We have found a canonical transformation for the **full system** that respects **covariance** at the perturbative level of truncation.

- The system is described by the **Mukhanov-Sasaki** gauge invariants, linear perturbative constraints and their momenta, and zero modes.

- The resulting system is a **constrained symplectic manifold**, where zero modes incorporate quadratic contributions of the perturbations.

- Lagrange multipliers are redefined as well. In particular, this affects the **lapse** function.
• **Backreaction** is included at the considered perturbative order in the scalar constraint, by means of the Mukhanov-Sasaki Hamiltonian.

• We have discussed the **hybrid quantization** of the system. This can be applied to a variety of quantum FLRW cosmology approaches.

• Under general assumptions, physical states depend only on the **zero modes** (*quantum background*) and the Mukhanov-Sasaki field.

• A **Born-Oppenheimer** ansatz leads to a quantum evolution equation for the inhomogeneities, without recurring to any **semiclassical approximation**.

• We have derived effective **Mukhanov-Sasaki equations**, which include quantum corrections. The ultraviolet regime is **hyperbolic**.