Gauge Invariant Perturbations and Covariance in Quantum Cosmology

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Introduction

- Our Universe is approximately homogeneous and isotropic: Background with cosmological **perturbations**.

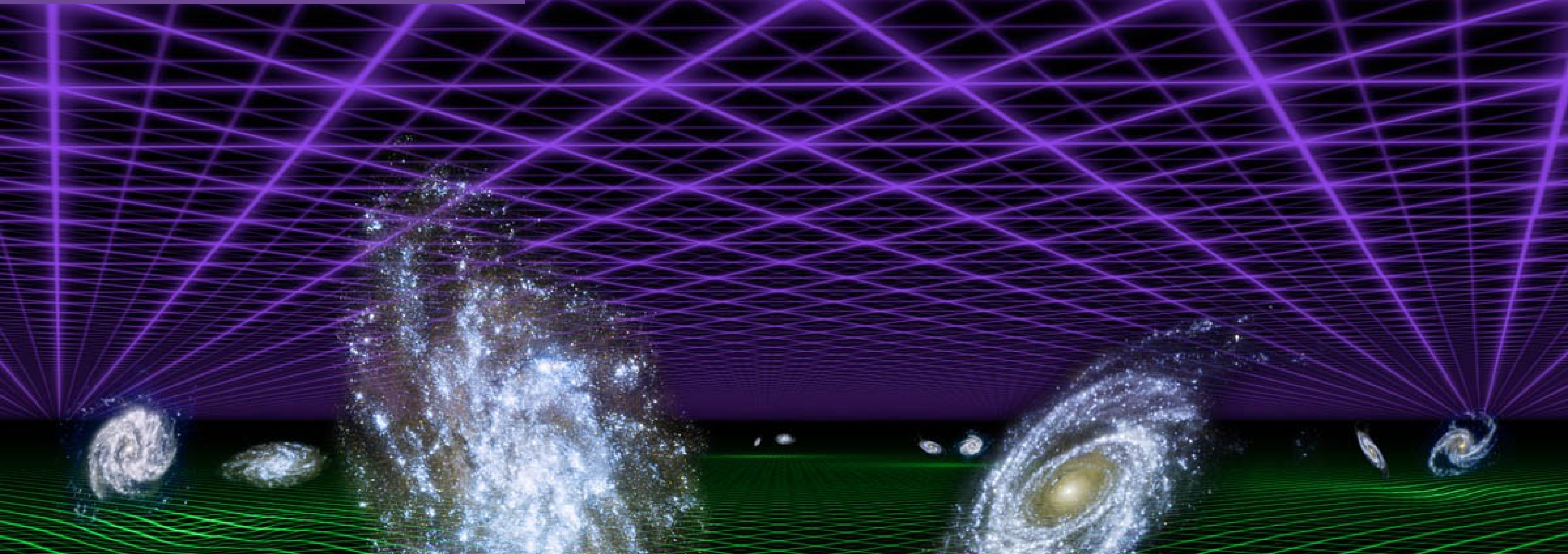
- Need of **gauge invariant** descriptions *(Bardeen, Mukhanov-Sasaki).*

- Perturbations: **Canonical formulation** with constraints *(Langlois, Pinto-Nieto)*.

- **Quantum** treatment including the background *(Halliwell-Hawking, Shirai-Wada)*.

- **Hybrid formalism** with a Born-Oppenheimer ansatz: Covariance.

Classical system



- We consider a **FLRW** universe with **compact flat** topology.
- We include a **scalar field** subject to a potential (e.g. a mass term).
- For simplicity, we analyze only **SCALAR pertubations**.

Classical system: Modes

- We expand the inhomogeneities in a **(real) Fourier** basis $\left(\vec{n}\in\mathbb{Z}^3\right)$:

$$Q_{\vec{n},+}=\sqrt{2}\cos\left(\vec{n}\cdot\vec{\theta}\right),\qquad Q_{\vec{n},-}=\sqrt{2}\sin\left(\vec{n}\cdot\vec{\theta}\right) \Leftarrow e^{\pm i\vec{n}\cdot\vec{\theta}}=\frac{\left(Q_{\vec{n},+}\pm i Q_{\vec{n},-}\right)}{\sqrt{2}}.$$

- We take $n_1\geq 0$. The eigenvalue of the Laplacian is $-\omega_n^2=-\vec{n}\cdot\vec{n}$.
- **Zero modes** are treated exactly (at linear perturbative order) in the expansions.

Classical system: Inhomogeneities

- Scalar perturbations: **metric and field.**

$$h_{ij}=\sigma^2 e^{2\alpha}\left[{}^0h_{ij}+2\sum\left\{a_{\vec{n},\pm}(t)Q_{\vec{n},\pm}\,{}^0h_{ij}+b_{\vec{n},\pm}(t)\left(\frac{3}{\omega_n^2}(Q_{\vec{n};\pm})_{,ij}+Q_{\vec{n},\pm}\,{}^0h_{ij}\right)\right\}\right],$$

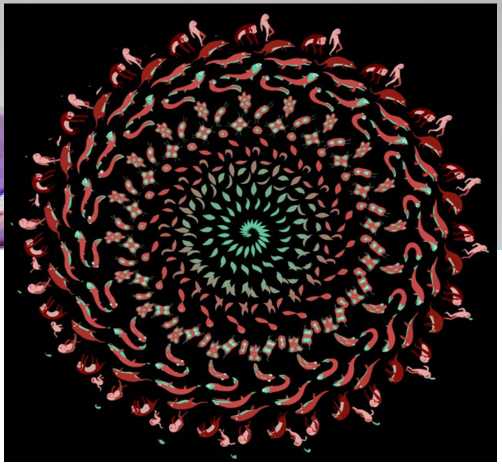
$$N=\sigma\left[N_0(t)+e^{3\alpha}\sum g_{\vec{n},\pm}(t)Q_{\vec{n},\pm}\right],\qquad N_i=\sigma^2 e^{2\alpha}\sum\frac{k_{\vec{n},\pm}(t)}{\omega_n^2}(Q_{\vec{n},\pm})_{;i},$$

$$\Phi=\frac{1}{\sigma(2\pi)^{3/2}}\left[\varphi(t)+\sum f_{\vec{n},\pm}(t)Q_{\vec{n},\pm}\right].\qquad \sigma^2=\frac{G}{6\pi^2},\qquad \tilde{m}=m\sigma.$$

- Truncating at **quadratic perturbative order** in the action:

$$H=N_0\left[H_0+\sum H_2^{\vec{n},\pm}\right]+\sum g_{\vec{n},\pm}H_1^{\vec{n},\pm}+\sum k_{\vec{n},\pm}\tilde{H}_{\uparrow 1}^{\vec{n},\pm}.$$

Classical system: Inhomogeneities



- **Scalar constraint**:

$$H_0=\frac{e^{-3\alpha}}{2}\left(-\pi_\alpha^2+\pi_\varphi^2+e^{6\alpha}\tilde{m}^2\varphi^2\right),$$

$$\begin{aligned}
2e^{3\alpha}\boldsymbol{H}_{\boldsymbol{2}}^{\vec{n},\pm}=&-\pi^2_{a_{\vec n,\pm}}+\pi^2_{b_{\vec n,\pm}}+\pi^2_{f_{\vec n,\pm}}+2\pi_\alpha\left(a_{\vec n,\pm}\pi_{a_{\vec n,\pm}}+4b_{\vec n,\pm}\pi_{b_{\vec n,\pm}}\right)-6\pi_\varphi a_{\vec n,\pm}\pi_{f_{\vec n,\pm}}\\
&+\pi_\alpha^2\left(\frac{1}{2}a^2_{\vec n,\pm}+10b^2_{\vec n,\pm}\right)+\pi_\varphi^2\left(\frac{15}{2}a^2_{\vec n,\pm}+6b^2_{\vec n,\pm}\right)-\frac{e^{4\alpha}}{3}\left(\omega_n^2a^2_{\vec n,\pm}+\omega_n^2b^2_{\vec n,\pm}-3\omega_n^2f^2_{\vec n,\pm}\right)\\
&-\frac{e^{4\alpha}}{3}\left(2\omega_n^2a_{\vec n,\pm}b_{\vec n,\pm}\right)+e^{6\alpha}\tilde{m}^2\left[3\varphi^2\left(\frac{1}{2}a^2_{\vec n,\pm}-2b^2_{\vec n,\pm}\right)+6\varphi a_{\vec n,\pm}f_{\vec n,\pm}+f^2_{\vec n,\pm}\right].
\end{aligned}$$

- Linear **perturbative constraints**:

$$\begin{aligned}
\boldsymbol{H}_{\boldsymbol{1}}^{\vec{n},\pm}=&-\pi_\alpha\pi_{a_{\vec n,\pm}}+\pi_\varphi\pi_{f_{\vec n,\pm}}+\left(\pi_\alpha^2-3\pi_\varphi^2+3e^{3\alpha}H_0\right)a_{\vec n,\pm}-\frac{\omega_n^2}{3}e^{4\alpha}\left(a_{\vec n,\pm}+b_{\vec n,\pm}\right)\\
&+e^{6\alpha}\tilde{m}^2\varphi f_{\vec n,\pm},
\end{aligned}$$

$$\tilde{\boldsymbol{H}}_{\uparrow 1}^{\vec{n},\pm}=\frac{1}{3}\left[-\pi_{a_{\vec n,\pm}}+\pi_{b_{\vec n,\pm}}+\pi_\alpha\left(a_{\vec n,\pm}+4b_{\vec n,\pm}\right)+3\pi_\varphi f_{\vec n,\pm}\right].$$

Gauge invariant perturbations

- Consider the sector of zero modes as describing a fixed **background.**

- Look for a transformation of the **perturbations --canonical only** with respect to their symplectic structure-- adapted to gauge invariance:

a) Find new variables that are an **abelianization** of the perturbative constraints.

$$\breve{\boldsymbol{H}}_{\mathbf{1}}^{\vec{n},\pm} = H_1^{\vec{n},\pm} - 3e^{3\alpha} H_0 a_{\vec{n},\pm}.$$

b) Include the **Mukhanov-Sasaki** variable, exploiting its gauge invariance.

$$\boldsymbol{v}_{\vec{n},\pm} = e^{\alpha}\left[f_{\vec{n},\pm} + \frac{\pi_{\varphi}}{\pi_{\alpha}} \left(a_{\vec{n},\pm} + b_{\vec{n},\pm} \right) \right].$$

c) Complete the transformation with suitable **momenta**.

Gauge invariant perturbations

- **Mukhanov-Sasaki** momentum:

$$\pi_{v_{\vec{n},\pm}} = e^{-\alpha}\left[\pi_{f_{\vec{n},\pm}} + \frac{1}{\pi_\varphi}\left(e^{6\alpha}\tilde{m}^2\varphi f_{\vec{n},\pm} + 3\pi_\varphi^2 b_{\vec{n},\pm}\right)\right] + F v_{\vec{n},\pm}.$$

- There is an **ambiguity** in a function of the background variables, F.

- The Mukhanov-Sasaki momentum is independent of $\left(\pi_{a_{\vec{n},\pm}}, \pi_{b_{\vec{n},\pm}}\right)$.

  The perturbative Hamiltonian constraint is independent of $\pi_{b_{\vec{n},\pm}}$.
  The perturbative momentum constraint depends through $\pi_{a_{\vec{n},\pm}} - \pi_{b_{\vec{n},\pm}}$.

- It is straightforward to complete the transformation:

$$\tilde{C}_{\uparrow 1}^{\vec{n},\pm} = 3 b_{\vec{n},\pm}, \qquad \breve{C}_1^{\vec{n},\pm} = -\frac{1}{\pi_\alpha}\left(a_{\vec{n},\pm} + b_{\vec{n},\pm}\right).$$

Gauge invariant perturbations

- The redefinition of the perturbative Hamiltonian constraint amounts to a **redefinition of the lapse** at our order of truncation in the action**:**

$$H=\breve{N}_0\left[H_0+\sum_{\vec{n},\pm} H_2^{\vec{n},\pm}\right]+\sum_{\vec{n},\pm} g_{\vec{n},\pm}\,\breve{H}_1^{\vec{n},\pm}+\sum_{\vec{n},\pm} k_{\vec{n},\pm}\,\widetilde{H}_{\uparrow 1}^{\vec{n},\pm}\,,$$

$$\breve{\mathbf{N}}_{\mathbf{0}}=N_0+3\,e^{3\alpha}\sum_{\vec{n},\pm} g_{\vec{n},\pm}\,a_{\vec{n},\pm}\,.$$

Mukhanov-Sasaki momentum

- We remove the ambiguity in the Mukhanov-Sasaki **momentum** by any of the following:

- It equals the time derivative of the Mukhanov-Sasaki variable.

- The scalar constraint is quadratic in this momentum (no linear terms).

- It is possible to adopt a Fock quantization with invariance under rigid rotations and unitary evolution (*Cortez, Mena-Marugán, Velhinho*).

$$\bar{\pi}_{v_{\vec{n},\pm}} = e^{-\alpha}\left[\pi_{f_{\vec{n},\pm}} + \frac{1}{\pi_\varphi}\left(e^{6\alpha}\tilde{m}^2\varphi f_{\vec{n},\pm} + 3\pi_\varphi^2 b_{\vec{n},\pm}\right)\right] - e^{-2\alpha}\left(\frac{1}{\pi_\varphi}e^{6\alpha}\tilde{m}^2\varphi + \pi_\alpha + 3\frac{\pi_\varphi^2}{\pi_\alpha}\right)v_{\vec{n},\pm}.$$

Canonical transformation: Inverse

$$a_{\vec{n},\pm}=-\pi_\alpha \breve{C}_1^{\vec{n},\pm}-\frac{1}{3}\widetilde{C}_{\uparrow 1}^{\vec{n},\pm}, \qquad b_{\vec{n},\pm}=\frac{1}{3}\widetilde{C}_{\uparrow 1}^{\vec{n},\pm}, \qquad f_{\vec{n},\pm}=e^{-\alpha}v_{\vec{n},\pm}+\pi_\varphi \breve{C}_1^{\vec{n},\pm},$$

$$\pi_{a_{\vec{n},\pm}}=-\frac{1}{\pi_\alpha}\breve{H}_1^{\vec{n},\pm}+\frac{\pi_\varphi}{\pi_\alpha}e^{\alpha}\bar{\pi}_{v_{\vec{n},\pm}}+\frac{e^{-\alpha}}{\pi_\alpha}\left(e^{6\alpha}\tilde{m}^2\varphi+\pi_\varphi\pi_\alpha+3\frac{\pi_\varphi^3}{\pi_\alpha}\right)v_{\vec{n},\pm}$$
$$+\left(3\pi_\varphi^2+\frac{1}{3}e^{4\alpha}\omega_n^2-\pi_\alpha^2\right)\breve{C}_1^{\vec{n},\pm}-\frac{1}{3}\pi_\alpha\widetilde{C}_{\uparrow 1}^{\vec{n},\pm},$$

$$\pi_{b_{\vec{n},\pm}}=3\widetilde{H}_{\uparrow 1}^{\vec{n},\pm}-\frac{1}{\pi_\alpha}\breve{H}_1^{\vec{n},\pm}+\frac{\pi_\varphi}{\pi_\alpha}e^{\alpha}\bar{\pi}_{v_{\vec{n},\pm}}+\frac{e^{-\alpha}}{\pi_\alpha}\left(e^{6\alpha}\tilde{m}^2\varphi-2\pi_\varphi\pi_\alpha+3\frac{\pi_\varphi^3}{\pi_\alpha}\right)v_{\vec{n},\pm}$$
$$+\frac{1}{3}e^{4\alpha}\omega_n^2\breve{C}_1^{\vec{n},\pm}-\frac{4}{3}\pi_\alpha\widetilde{C}_{\uparrow 1}^{\vec{n},\pm},$$

$$\pi_{f_{\vec{n},\pm}}=e^{\alpha}\bar{\pi}_{v_{\vec{n},\pm}}+e^{-\alpha}\left(\pi_\alpha+3\frac{\pi_\varphi^2}{\pi_\alpha}\right)v_{\vec{n},\pm}-e^{6\alpha}\tilde{m}^2\varphi\breve{C}_1^{\vec{n},\pm}-\pi_\varphi\widetilde{C}_{\uparrow 1}^{\vec{n},\pm}.$$

Full system

- We now include the **zero modes** as variables of the system, and complete the transformation to a **canonical** one in their presence.

- We re-write the Legendre term of the action, keeping its canonical form at the considered **perturbative order**:

$$\int dt\left[\sum_a \dot{w}_q^a w_p^a + \sum_{l,\vec{n},\pm} \dot{X}_{q_l}^{\vec{n},\pm} X_{p_l}^{\vec{n},\pm}\right] \equiv \int dt\left[\sum_a \dot{\tilde{w}}_q^a \tilde{w}_p^a + \sum_{l,\vec{n},\pm} \dot{V}_{q_l}^{\vec{n},\pm} V_{p_l}^{\vec{n},\pm}\right].$$

- <u>Zero modes</u>: **Old** $\{w_q^a, w_p^a\} \rightarrow$ **New** $\{\tilde{w}_q^a, \tilde{w}_p^a\}$. $\left(\{w_q^a\} = \{\alpha, \varphi\}.\right)$

- <u>Inhomogeneities</u>: **Old** $\{X_{q_l}^{\vec{n},\pm}, X_{p_l}^{\vec{n},\pm}\} \rightarrow$ **New:**

$$\{V_{q_l}^{\vec{n},\pm}, V_{p_l}^{\vec{n},\pm}\} = \left\{\left(v_{\vec{n},\pm}, \breve{C}_1^{\vec{n},\pm}, \widetilde{C}_{\uparrow 1}^{\vec{n},\pm}\right), \left(\pi_{v_{\vec{n},\pm}}, \breve{H}_1^{\vec{n},\pm}, \widetilde{H}_{\uparrow 1}^{\vec{n},\pm}\right)\right\}.$$

Full system

- Using that the change of perturbative variables is linear, it is not difficult to find the **new zero modes**, which include modifications **quadratic in the perturbations**.

- Expressions:

$$w_q^a = \tilde{w}_q^a - \frac{1}{2}\sum_{l,\vec{n},\pm}\left[X_{q_l}^{\vec{n},\pm}\frac{\partial X_{p_l}^{\vec{n},\pm}}{\partial \tilde{w}_p^a} - \frac{\partial X_{q_l}^{\vec{n},\pm}}{\partial \tilde{w}_p^a}X_{p_l}^{\vec{n},\pm}\right],$$

$$w_p^a = \tilde{w}_p^a + \frac{1}{2}\sum_{l,\vec{n},\pm}\left[X_{q_l}^{\vec{n},\pm}\frac{\partial X_{p_l}^{\vec{n},\pm}}{\partial \tilde{w}_q^a} - \frac{\partial X_{q_l}^{\vec{n},\pm}}{\partial \tilde{w}_q^a}X_{p_l}^{\vec{n},\pm}\right].$$

$\left\{X_{q_l}^{\vec{n},\pm}, X_{p_l}^{\vec{n},\pm}\right\} \rightarrow$ Old perturbative variables in terms of the new.

New Hamiltonian

- Since the change of the zero modes is **quadratic in the perturbations**, the new scalar constraint at our **truncation order** is

$$H_0(w^a)+\sum_{\vec{n},\pm} H_2^{\vec{n},\pm}\left(w^a, X_l^{\vec{n},\pm}\right)\Rightarrow$$

$$H_0(\tilde{w}^a)+\sum_b\left(w^b-\tilde{w}^b\right)\frac{\partial H_0}{\partial \tilde{w}^b}(\tilde{w}^a)+\sum_{\vec{n},\pm} H_2^{\vec{n},\pm}\left[\tilde{w}^a, X_l^{\vec{n},\pm}(\tilde{w}^a, V_l^{\vec{n},\pm})\right],$$

$$w^a-\tilde{w}^a=\sum_{\vec{n},\pm}\Delta\tilde{w}^a_{\vec{n},\pm}.$$

- The perturbative contribution to the new scalar constraint is:

$$\bar{H}_2^{\vec{n},\pm}=H_2^{\vec{n},\pm}+\sum_a\Delta\tilde{w}^a_{\vec{n},\pm}\frac{\partial H_0}{\partial\tilde{w}^a}.$$

This is the change expected for zero modes treated as time dependent *external* variables with dynamics generated by H_0.

New Hamiltonian

- Carrying out the calculation explicitly, one obtains:

$$\bar{H}_2^{\vec{n},\pm} = \breve{H}_2^{\vec{n},\pm} + F_2^{\vec{n},\pm} H_0 + \breve{F}_1^{\vec{n},\pm} \breve{H}_1^{\vec{n},\pm} + \left(F_{\uparrow 1}^{\vec{n},\pm} - 3 \frac{e^{-3\tilde{\alpha}}}{\pi_{\tilde{\alpha}}} \breve{H}_1^{\vec{n},\pm} + \frac{9}{2} e^{-3\tilde{\alpha}} \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} \right) \widetilde{H}_{\uparrow 1}^{\vec{n},\pm},$$

$$\breve{H}_2^{\vec{n},\pm} = \frac{e^{-\tilde{\alpha}}}{2} \left\{ \left[\omega_n^2 + e^{-4\tilde{\alpha}} \pi_{\tilde{\alpha}}^2 + \tilde{m}^2 e^{2\tilde{\alpha}} \left(1 + 15\tilde{\varphi}^2 - 12\tilde{\varphi} \frac{\pi_{\tilde{\varphi}}}{\pi_{\tilde{\alpha}}} - 18 e^{6\tilde{\alpha}} \tilde{m}^2 \frac{\tilde{\varphi}^4}{\pi_{\tilde{\alpha}}^2} \right) \right] (v_{\vec{n},\pm})^2 + (\bar{\pi}_{v_{\vec{n},\pm}})^2 \right\}.$$

- The F 's are well determined functions.

- The term $\breve{H}_2^{\vec{n},\pm}$ is the Mukhanov-Sasaki Hamiltonian.

- It has **no linear contributions** of the Mukhanov-Sasaki momentum.

- It is **linear in the momentum** $\pi_{\tilde{\varphi}}$.

New Hamiltonian

- We re-write the **total Hamiltonian** of the system at our **truncation order**, redefining the Lagrange multipliers:

$$\bar{H}_2^{\vec{n},\pm} = \breve{H}_2^{\vec{n},\pm} + F_2^{\vec{n},\pm} H_0 + \breve{F}_1^{\vec{n},\pm} \breve{H}_1^{\vec{n},\pm} + \left(F_{\uparrow 1}^{\vec{n},\pm} - 3 \frac{e^{-3\tilde{\alpha}}}{\pi_{\tilde{\alpha}}} \breve{H}_1^{\vec{n},\pm} + \frac{9}{2} e^{-3\tilde{\alpha}} \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} \right) \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} \quad \Rightarrow$$

$$H = \bar{N}_0 \left[H_0 + \sum_{\vec{n},\pm} \breve{H}_2^{\vec{n},\pm} \right] + \sum_{\vec{n},\pm} \breve{G}_{\vec{n},\pm} \breve{H}_1^{\vec{n},\pm} + \sum_{\vec{n},\pm} \widetilde{K}_{\vec{n},\pm} \widetilde{H}_{\uparrow 1}^{\vec{n},\pm} .$$

- The new *lapse* multiplier is $\bar{N}_0 = N_0 + \sum_{\vec{n},\pm} \left(3 e^{3\tilde{\alpha}} g_{\vec{n},\pm} a_{\vec{n},\pm} + N_0 F_2^{\vec{n},\pm} \right)$.

Hybrid quantization

Approximation: Quantum geometry effects are especially relevant in the background

- Adopt a **quantum cosmology** scheme for the zero modes and a **Fock quantization** for the perturbations. The scalar constraint **couples** them.

- We assume:

a) The zero modes (derived above) continue to **commute** with the perturbations under quantization,

b) Functions of $\tilde{\varphi}$ act by multiplication.

Uniqueness of the Fock description

The **Fock representation** in QFT is fixed (up to unitary equivalence) by:

1) The *background* **isometries**; **2)** The demand of a **UNITARY** evolution.

- The introduced **scaling** of the field by the scale factor is essential for unitarity.
- The proposal selects a **UNIQUE canonical pair** for the Mukhanov-Sasaki field, precisely the one we chose to fix the ambiguity in the momentum.
- We can use the massless representation (due to compactness), with its creation and annihilation operators, and the corresponding basis of occupancy number states $|N\rangle$.

Representation of the constraints

- We admit that the operators that represent the linear constraints (or an integrated version of them) act as derivatives (or as translations).

- Then, physical states are independent of $\left(\breve{C}_{1}^{\vec{n},\pm}, \widetilde{C}_{\uparrow 1}^{\vec{n},\pm}\right).$

- We pass to a space of states $\boldsymbol{H}_{kin}^{grav}\otimes\boldsymbol{H}_{kin}^{matt}\otimes\boldsymbol{F}$ that depend on the **zero modes** and the **Mukhanov-Sasaki modes**, with **no gauge fixing**.

- In this covariant construction, physical states still must satisfy the scalar constraint given by the FLRW and the Mukhanov-Sasaki contributions.

$$H_S=e^{-3\alpha}\left(H_0+\sum_{\vec{n},\pm}\breve{H}_2^{\vec{n},\pm}\right)=0.$$

Representation of the constraints

- With $\tilde{\varphi}$ acting by multiplication, we introduce the following functions on the phase space of the **gravitational zero mode**:

$$H_0^{(2)}=\pi_{\tilde{\alpha}}^2-e^{6\tilde{\alpha}}\tilde{m}^2\tilde{\varphi}^2, \qquad \vartheta_o=-12\,e^{4\tilde{\alpha}}\tilde{m}^2\frac{\tilde{\varphi}}{\pi_{\tilde{\alpha}}}, \qquad \vartheta_e=e^{2\tilde{\alpha}},$$

$$\vartheta_e^q=e^{-2\tilde{\alpha}}H_0^{(2)}\left(19-18\frac{H_0^{(2)}}{\pi_{\tilde{\alpha}}^2}\right)+\tilde{m}^2e^{4\tilde{\alpha}}\left(1-2\tilde{\varphi}^2\right).$$

- Including the Mukhanov-Sasaki modes, we define:

$$\Theta_o^{\vec{n},\pm}=-\vartheta_o\left(v_{\vec{n},\pm}\right)^2, \qquad \Theta_e^{\vec{n},\pm}=-\left[\left(\vartheta_e\omega_n^2+\vartheta_e^q\right)\left(v_{\vec{n},\pm}\right)^2+\vartheta_e\left(\bar{\pi}_{v_{\vec{n},\pm}}\right)^2\right],$$

$$\Theta_o=\sum_{\vec{n},\pm}\Theta_o^{\vec{n},\pm}, \qquad \Theta_e=\sum_{\vec{n},\pm}\Theta_e^{\vec{n},\pm}.$$

- We get the constraint:

$$\hat{H}_S=\frac{1}{2}\left[\hat{\pi}_{\tilde{\varphi}}^2-\hat{H}_0^{(2)}-\hat{\Theta}_e-\frac{1}{2}\left(\hat{\Theta}_o\hat{\pi}_{\tilde{\varphi}}+\hat{\pi}_{\tilde{\varphi}}\hat{\Theta}_o\right)\right].$$

Representation of the constraints

- Quantum constraint:

$$\hat{H}_S = \frac{1}{2}\left[\hat{\pi}_{\tilde{\varphi}}^2 - \hat{H}_0^{(2)} - \hat{\Theta}_e - \frac{1}{2}\left(\hat{\Theta}_o \hat{\pi}_{\tilde{\varphi}} + \hat{\pi}_{\tilde{\varphi}} \hat{\Theta}_o\right)\right].$$

- This constraint is quadratic in the momentum of the zero mode of the scalar field.

- The linear contribution goes with the derivative of the field potential.

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Born-Oppenheimer ansatz

- Consider states whose dependence on the FLRW geometry and the inhomogeneities (N) **split:**

$$\Psi = \Gamma(\tilde{\alpha}, \tilde{\varphi}) \psi(N, \tilde{\varphi}),$$

- The FLRW state is **normalized, peaked** and **evolves unitarily**:

$$\Gamma(\tilde{\alpha}, \tilde{\varphi}) = \hat{U}(\tilde{\alpha}, \tilde{\varphi}) \chi(\tilde{\alpha}).$$

- $\hat{U}$ is a unitary evolution operator close to the **unperturbed** one:

  $[\hat{\pi}_{\tilde{\varphi}}, \hat{U}] = \hat{\tilde{H}}_0$ is a $\tilde{\varphi}$-dependent operator on the FLRW geometry and

  $(\hat{\tilde{H}}_0)^2 - \hat{H}_0^{(2)} + [\hat{\pi}_{\tilde{\varphi}}, \hat{\tilde{H}}_0]$ is *negligible* on Γ.

Born-Oppenheimer ansatz

- In the quantum constraint, we can disregard transitions from Γ to other FLRW states if the following operators have small relative dispersion:

i) $\hat{\tilde{H}}_0$, ii) $\hat{\vartheta}_e$, iii) $-i\,\boldsymbol{d}_{\tilde{\varphi}}\hat{\vartheta}_o+(\hat{\vartheta}_o\hat{\tilde{H}}_0+\hat{\tilde{H}}_0\hat{\vartheta}_o)+2\hat{\vartheta}_e^q$.

- We have defined the *total* derivative $\boldsymbol{d}_{\tilde{\varphi}}\hat{O}=i\,[\hat{\pi}_{\tilde{\varphi}}-\hat{\tilde{H}}_0,\hat{O}]$.

- Then, taking the inner product with Γ in the FLRW geometry, one gets a **quantum evolution** constraint for the Mukhanov-Sasaki field.

Quantum constraint on the perturbations

- With the Born-Oppenheimer ansatz and our assumptions, we can write the scalar constraint as:

$$\hat{\pi}_{\tilde{\varphi}}^2 \psi + 2\langle \hat{\tilde{H}}_0 \rangle_\Gamma \hat{\pi}_{\tilde{\varphi}} \psi = \left[\langle \hat{\Theta}_e + \frac{1}{2}(\hat{\Theta}_o \hat{\tilde{H}}_0 + \hat{\tilde{H}}_0 \hat{\Theta}_o) \rangle_\Gamma - \frac{i}{2} \langle \boldsymbol{d}_{\tilde{\varphi}} \hat{\Theta}_o \rangle_\Gamma \right] \psi .$$

- Besides, if we can **neglect**: a) The term $\hat{\pi}_{\tilde{\varphi}}^2 \psi$.
  b) The total $\tilde{\varphi}$-derivative of $\hat{\Theta}_o$.

$$\hat{\pi}_{\tilde{\varphi}} \psi = \frac{\langle 2\hat{\Theta}_e + (\hat{\Theta}_o \hat{\tilde{H}}_0 + \hat{\tilde{H}}_0 \hat{\Theta}_o) \rangle_\Gamma}{4\langle \hat{\tilde{H}}_0 \rangle_\Gamma} \psi .$$

Schrödinger-like equation for the gauge invariant perturbations

Effective Mukhanov-Sasaki equations

- Starting from the **Born-Oppenheimer** form of the constraint and assuming a direct **effective** counterpart for the **inhomogeneities**:

$$d^2_{\eta_\Gamma} v_{\vec{n},\pm} = -v_{\vec{n},\pm}\left[4\pi^2\omega_n^2 + \langle \hat{\theta}_e + \hat{\theta}_o \rangle_\Gamma\right],$$

$$\langle \hat{\theta}_e \rangle_\Gamma = 4\pi^2 \frac{\langle \hat{\vartheta}_e^q \rangle_\Gamma}{\langle \hat{\vartheta}_e \rangle_\Gamma}, \qquad \langle \hat{\theta}_o \rangle_\Gamma = 2\pi^2 \frac{\langle (\hat{\vartheta}_o \hat{\tilde{H}}_0 + \hat{\tilde{H}}_0 \hat{\vartheta}_o) - i\, \boldsymbol{d}_{\tilde{\varphi}} \hat{\vartheta}_o \rangle_\Gamma}{\langle \hat{\vartheta}_e \rangle_\Gamma}.$$

where we have defined the **state-dependent conformal time**

$$2\pi\, d\eta_\Gamma = \langle \hat{\vartheta}_e \rangle_\Gamma dT, \qquad \text{with} \qquad dt = \sigma e^{3\tilde{\alpha}} dT.$$

Effective Mukhanov-Sasaki equations

- For all modes:

$$d^2_{\eta_\Gamma} v_{\vec{n},\pm} = -v_{\vec{n},\pm}\left[4\pi^2\omega_n^2 + \langle \hat{\theta}_e + \hat{\theta}_o \rangle_\Gamma\right].$$

- The expectation value depends (only) on the conformal time, through $\tilde{\varphi}$. It is the time dependent part of the frequency, but it is mode independent.

- The effective equations are of harmonic **oscillator** type, with no dissipative term, and **hyperbolic in the ultraviolet** regime.

Conclusions

- We have considered a FLRW universe with a massive scalar field perturbed at **quadratic** order in the action.
- We have found a canonical transformation for the **full system** that respects *covariance* at the perturbative level of truncation.
- The system is described by the **Mukhanov-Sasaki** gauge invariants, linear perturbative constraints and their momenta, and zero modes.
- The resulting system is a **constrained symplectic manifold,** where zero modes incorporate quadratic contributions of the perturbations.
- Lagrange multipliers are redefined as well. In particular, this affects the **lapse** function.

Conclusions

- **Backreaction** is included at the considered perturbative order in the scalar constraint, by means of the **Mukhanov-Sasaki Hamiltonian**.
- We have discussed the **hybrid quantization** of the system. This can be applied to a variety of quantum FLRW cosmology approaches.
- Under general assumptions, physical states depend only on the **zero modes** (*quantum background*) and the **Mukhanov-Sasaki field**.
- A **Born-Oppenheimer** ansatz leads to a quantum evolution equation for the inhomogeneities, without recurring to **any semiclassical approximation**.
- We have derived effective **Mukhanov-Sasaki equations**, which include quantum corrections. The ultraviolet regime is **hyperbolic**.