N-TANGENTIAL OR N-NORMAL LINE THEOREMS FOR N-PARTICLES IN THE RIEMANNIAN SPACE FORMS AND DE SITTER SPACE.

TETSUYA TANIGUCHI AND SEIICHI UDAGAWA

1. Geometry of linear and angular momenta of N-particles

The N-body problem is

$$m_j \ddot{q}_j = -\frac{\partial U}{\partial q_j}, \qquad (j = 1, 2, , \cdots, N)$$

where $q_j = q_j(t) = (x_j(t), y_j(t), z_j(t)) \in \mathbb{R}^3$ and $U = -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}$ is the potential energy. Denote by $K(\dot{q})$ the total kinetic energy $\frac{1}{2}\sum_{j=1}^{n} m_j |\dot{q}_j|^2$. Consider the Lagrangian $L(q, \dot{q}) = K(\dot{q}) - U(q)$. Then, the Euler-Lagrange equation for L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}(q(t), \dot{q}(t))\right) = \frac{\partial L}{\partial q_j}(q(t), \dot{q}(t)), \qquad j = 1, 2, \cdots, N$$

is equivalent to the N-body problem. Thus, such $q = (q_1, q_2, \cdots, q_N)$ is a critical point of the action $\mathcal{A}(q) = \int_{t_0}^{t_1} L(q, \dot{q})dt$. Therefore, the variational method seems to be useful for solving N-body problem.

In [2], Chenciner and Montgomery considered the three body problem for the 3-particles with equal masses in the plane. In particular, they verified the existence of a figure-eight solution to the planer equal-masses three-body problem. They call this solution the *figure-eight solution*. However, the explicit solution to this problem has not been discovered yet. It is also unknown even whether the solutions are algebraic or not. On the other hand, by paying attention to the fact that the figure-eight solution has zero angular momentum, Fujiwara, Fukuda and Ozaki[3] found that three tangent lines at the three bodies meet a point at each instant. In fact, this fact holds in more general situation. Consider the motions of three particles with equal-masses 1 in the xy-plane. Denote by l_k (resp. by n_k) the tangential line (resp. normal line) of the curve of k-th particle at the point $(x_k(t), y_k(t))$ for any time t, where $k = 1, 2, 3$. Under these situations, Fujiwara et al. implemented a systematic study and proved the following theorem.

Theorem A([3], [4]) (1) *If both the linear momentum and the angular momentum of* 3*-particles vanish, then either the three tangential lines* l_1, l_2, l_3 *meet at a point or the three tangential lines are parallel each other.*
(2) *If the moment of inertia is constant and the linear momentum of* 3*-particles vanish, then either the three normal lines* n_1, n_2, n_3 *meet at a point or the three normal lines are parallel each other.*

In particular, the case where all the orbit curves for the motions of the three particles completely coincide and the three particles are on a closed plane curve is interesting. Such an example surely exists and the corresponding closed curve can be described by the Jacobi elliptic functions with some modulus. In fact, in [3] it is proved that the figure-eight solution which satisfies the assumption (1) of Theorem A exsists on the lemniscate.

In this paper, we develop the geometry of linear and angular momenta of the motions of N-particles in various ambient spaces, which generalizes the results of Fujiwara et al. stated above.

Let M be a Lagrangian submanifold of $(\mathbb{R}^{2N}, \omega = \sum_{i=1}^{N} dx_i \wedge dy_i)$. Let J be the almost complex structure on M, that is, J satisfies the relations $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$, $J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$. Let θ and ϕ be 1-forms on M defined by

$$\theta = \sum_{i=1}^{N}(x_i dy_i - y_i dx_i), \quad \phi = \sum_{i=1}^{N}(x_i dx_i + y_i dy_i),$$

respectively. Let L be a J-invariant 2-plane through the origin. For each point p on M, we define the dimensions $d_L(p)$ and $e_L(p)$ as follows:

$$\begin{aligned} d_L(p) &= \dim\{v \in T_pM \mid \theta_p(v) = 0, v \perp L\}, \\ e_L(p) &= \dim\{v \in T_pM \mid \phi_p(v) = 0, v \perp L\}. \end{aligned}$$

We regard T_pM as the affine space through p and denote it by $H(p)$. We then have the following theorem.

Theorem 1.1. *$H(p)$ and L (resp. $J(H(p))$ and L) are contained in some $(N+2)$-dimensional linear subspace if and only if $d_L(p) \geqq N-2$ (resp. $e_L(p) \geqq N-2$).*

Choose L in Theorem 1.1 as L_0 defined by

$$L_0 = \left\{(x_1, y_1, x_2, y_2, \cdots, x_N, y_N) \in \mathbb{R}^{2N} \,\middle|\, \begin{array}{l} x_1 = x_2 = \cdots = x_N, \\ y_1 = y_2 = \cdots = y_N \end{array}\right\}.$$

We have an N-tangential or N-normal lines theorem as a corollary of Theorem 1.1.

Corollary 1.2. *Consider the motions of N-particles in the xy-plane :*

$$c_1(t) = (x_1(t), y_1(t)), \quad c_2(t) = (x_2(t), y_2(t)), \quad \cdots, \quad c_N(t) = (x_N(t), y_N(t)).$$

Let M be a Lagrangian submanifold of $\mathbb{R}^{2N}$ defined by $M = c_1 \times c_2 \times \cdots \times c_N$. Let $p = (c_1(t_1), c_2(t_2), \cdots, c_N(t_N))$ be any point of M. Then, $d_L(p) \geqq N-2$ (resp. $e_L(p) \geqq N-2$) if and only if either the N-tangential lines (resp. N-normal lines) at p meet at a point or all the N-tangential lines (resp. N-normal lines) are parallel each other.

This Corollary implies Theorem A of Fujiwara et al. because the linear momentum(denoted by $\overrightarrow{m}$), the angular momentum(denoted by ω) and the moment of inertia(denoted by I) are respectively given by

$$\overrightarrow{m} = \left(\sum_{i=1}^{3}\frac{dx_i}{dt}, \sum_{i=1}^{3}\frac{dy_i}{dt}\right), \quad \omega = \sum_{i=1}^{3}\left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right), \quad I = \sum_{i=1}^{3}(x_i^2 + y_i^2).$$

We choose a v in Theorem 1.1 by the following $v = \sum_{i=1}^{3} \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \sum_{i=1}^{3} \frac{dy_i}{dt}\frac{\partial}{\partial y_i}$. We then easily see that $v \perp L$ if and only if $\overrightarrow{m} = \overrightarrow{0}$, $\theta(v) = 0$ if and only if $\omega = 0$, and $\phi(v) = 0$ if and only if $dI(v) = 0$. Therefore, Corollary 1.2 implies the results of Fujiwara et al.

Let $\mathbb{R}^{3N}$ be a $3n$-dimensional Euclidean space. We endow it the metric

$$g_c = \sum_{j=1}^{N} \left((dx_j)^2 + (dy_j)^2 + c(dz_j)^2\right),$$

where $c = 1$ or $c = -1$. Let $(e_x^1, e_y^1, e_z^1, e_x^2, e_y^2, e_z^2, \cdots, e_x^N, e_y^N, e_z^N)$ be an orthonormal frame field for $c = 1$. Consider the $\mathbb{R}^{3N}$-valued 1-form $\widetilde{\theta}_c$ and $\widetilde{\phi}$ defined by

$$\widetilde{\theta}_c = \sum_{j=1}^{N} \left\{(y_j dz_j - z_j dy_j) \otimes e_x^j + (z_j dx_j - x_j dz_j) \otimes e_y^j + c\,(x_j dy_j - y_j dx_j) \otimes e_z^j\right\},$$

$$\widetilde{\phi} = \sum_{j=1}^{N} \left\{dx_j \otimes e_x^j + dy_j \otimes e_y^j + dz_j \otimes e_z^j\right\}.$$

Note that $d_{L_0}(p) = \dim\{v \in T_pM \mid \widetilde{\theta}_c\ |_{z=1}\ (v) = 0\}$ for plane $\{(x,y,z)|z = 1\}$. Let S^2 be a 2-sphere of constant curvature 1. Let $H^2 = K_1 \cup K_2$ be the union of the hyperboloid of two sheets and the hyperboloid of one sheet, where the former is 2-dimensional hyperbolic space and the latter is 2-dimensional de Sitter space with respect to the induced metric for $c = -1$. Denote by K^N the N-product $\underbrace{S^2 \times \cdots \times S^2}_{N\text{-times}}$ of spheres or $\underbrace{H^2 \times \cdots \times H^2}_{N\text{-times}}$ of the union of hyperboloids.

Let M be a N-dimensional submanifold of $K^N \subset \mathbb{R}^{3N}$. Let L be a 3-dimensional subspace of $\mathbb{R}^{3N}$. Define the indices $d_{L,c}(p)$ and $e_{L,c}(p)$ by

$$d_{L,c}(p) = \dim\left\{v \in T_pM \mid \widetilde{\theta}_c(v) \perp L\right\}, \quad e_{L,c}(p) = \dim\left\{v \in T_pM \mid \widetilde{\phi}(v) \perp L\right\},$$

where the orthogonality is taken with respect to g_c. When we choose a point $p = (a_1, b_1, c_1, a_2, b_2, c_2, \cdots, a_N, b_N, c_N)$, we denote by W_p an N-dimensional linear subspace of $\mathbb{R}^{3N}$ defined by

$$W_p = \mathbb{R}(a_1, b_1, c_1, 0, \cdots, 0) \oplus \mathbb{R}(0, 0, 0, a_2, b_2, c_2, 0, \cdots, 0) \oplus \mathbb{R}(0, \cdots, 0, a_N, b_N, c_N).$$

The first main result of the paper is then the following theorem.

Theorem 1.3. *Let M be an N-dimensional submanifold of $K^N \subset \mathbb{R}^{3N}$ which satisfy the condition $\widetilde{\theta}_c(T_pM) \perp T_pM$ for any point $p \in M$. Then, $d_{L,c}(p) \geqq N - 2$ (resp. $e_{L,c}(p) \geqq N - 2$) if and only if $T_pM \cup L \cup W_p$ (resp. $\widetilde{\theta}_c(T_pM) \cup L \cup W_p$) is contained in some $(2N+2)$-dimensional linear subspace of $\mathbb{R}^{3N}$.*

Choosing L in Theorem 1.3 as L_0 defined by

$$L_0 = \left\{(x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_N, y_N, z_N) \in \mathbb{R}^{3N} \mid x_1 = x_2 = \cdots = x_N, \right.$$
$$\left. y_1 = y_2 = \cdots = y_N, z_1 = z_2 = \cdots = z_N\right\},$$

we obtain N-tangential or N-normal line theorems for 2-dimensional non-flat Riemannian space form and de Sitter space as follows.

Corollary 1.4. *Consider the motions of N-particles in S^2: $c_1(t) = (x_1(t), y_1(t), z_1(t))$, $c_2(t) = (x_2(t), y_2(t), z_2(t)), \cdots, c_N(t) = (x_N(t), y_N(t), z_N(t))$. Let M be an N-dimensional submanifold of $\mathbb{R}^{3N}$ defined by $M = c_1 \times c_2 \times \cdots \times c_N$. Let $p = (c_1(t_1), c_2(t_2), \cdots, c_N(t_N))$ be any point of M. Then, $d_{L_0}(p) \geqq N-2$ (resp. $e_{L_0}(p) \geqq N-2$) if and only if either the N-geodesics generated by N-tangential lines (resp. N-normal lines) at p meet at a point each other.*

Corollary 1.5. *Three geodesics on S^2 meet at a point if and only if one of the following holds :*
(1) Three geodesics are generated by the three tangential lines of the orbit curves of the motions of three particles with the sum of angular momentum being zero.
(2) Three geodesics are generated by the three normal lines of the orbit curves of the motions of three particles with the sum of linear momentum being zero.

Corollary 1.6. *Consider the motions of N-particles in K: $c_1(t) = (x_1(t), y_1(t), z_1(t))$, $c_2(t) = (x_2(t), y_2(t), z_2(t)), \cdots, c_N(t) = (x_N(t), y_N(t), z_N(t))$. Let M be an N-dimensional submanifold of $\mathbb{R}^{3N}$ defined by $M = c_1 \times c_2 \times \cdots \times c_N$. Let p be any point of M. Then, $d_{L_0}(p) \geqq N-2$ (resp. $e_{L_0}(p) \geqq N-2$) if and only if either the N-geodesics generated by N-tangential lines (resp. N-normal lines) at p meet at a point in $K \cup \partial K$ each other, where $\partial K (\cong S^1 \cup S^1)$ is the ideal boundary of K.*

Corollary 1.7. *Three geodesics on $K \cup \partial K$ meet at a point if and only if one of the following holds :*
(1) Three geodesics are generated by the three tangent lines of the orbit curves of the motions of three particles with the sum of angular momentum being zero.
(2) Three geodesics are generated by the three normal lines of the orbit curves of the motions of three particles with the sum of linear momentum being zero.

2. **Further generalization**

Theorem 1.3 holds for more general case where the ambient space $\mathbb{R}^{3N}$ is given non-degenerate quadratic form.

Let $(\ ,\ )$ be a non-degenerate quadratic form on $\mathbb{R}^{3N}$. We use the musical isomorphism $\flat : T_p\mathbb{R}^{3N} \ni v \longrightarrow v^\flat \in T_p^*\mathbb{R}^{3N}$ defined by $v^\flat(w) = (v, w)$ for any $w \in T_p\mathbb{R}^{3N}$. For any subspace V in $T_p\mathbb{R}^{3N}$ we set

$$V^{\perp *} = \left\{ f \in T_p^*\mathbb{R}^{3N} \mid f(v) = 0 \text{ for any } v \in V \right\},$$
$$V^\flat = \left\{ v^\flat \in T_p^*\mathbb{R}^{3N} \mid v \in V \right\}.$$

We then have

Lemma 2.1. *For any subspaces V, W in $T_p\mathbb{R}^{3N}$ we have the following.*

$$V^{\perp *} = (V^\perp)^\flat, \tag{2.2}$$

where $V^\perp = \{ v \in T_p\mathbb{R}^{3N} \mid (v, w) = 0 \text{ for any } w \in V \}$.

$$V^\flat \cap W^\flat = (V \cap W)^\flat. \tag{2.3}$$

Let $\widetilde{\theta}$ be an element of the endomorphism bundle $\mathrm{End}(T\mathbb{R}^{3N})$ of $T\mathbb{R}^{3N}$. Let L and W_p be the linear subspaces in $\mathbb{R}^{3N}$. We regrad L and W_p as a subspace of

$T_p\mathbb{R}^{3N}$. Set

$$d = \dim\left\{\widetilde{\theta}(T_pM) \cap L^\perp\right\}, \quad e = \dim\left\{T_pM \cap L^\perp\right\}.$$

We have the following.

Theorem 2.4. *Let M be a submanifold of $\mathbb{R}^{3N}$ which satisfies the condition $\widetilde{\theta}(T_pM) = T_pM^\perp \cap W_p^\perp$ (resp. $\widetilde{\theta}(T_pM)^\perp \cap W_p^\perp = T_pM$). Then, $d \geqq N-2$ (resp. $e \geqq N-2$) if and only if $T_pM \cup W_p \cup L$ (resp. $\widetilde{\theta}(T_pM) \cup W_p \cup L$) is contained in some $(2N+2)$-dimensional linear subspace in $\mathbb{R}^{3N}$.*

We endow $\mathbb{R}^{3N}$ the metric $g_c = \displaystyle\sum_{j=1}^{N}\left((dx_j)^2 + (dy_j)^2 + c(dz_j)^2\right)$, where $c = 1$ or $c = -1$. Consider $\widetilde{\theta}$ in Theorem 2.4 as $\widetilde{\theta}_c$ in §1. In this case, we have

$$\ker\widetilde{\theta}_p = W_p := \mathbb{R}(x_1, y_1, z_1, 0, \cdots, 0) \oplus \cdots \oplus \mathbb{R}(0, \cdots, 0, x_N, y_N, z_N) \tag{2.5}$$

under the condtion $\dim W_p = N$. Moreover we have

$$\widetilde{\theta}(T_p\mathbb{R}^{3N}) \subset W_p^\perp, \tag{2.6}$$

where $\perp$ is taken with respect to g_c.

Corollary 2.7. *Let M be an N-dimensional submanifold of $\mathbb{R}^{3N}$ which satisfies the conditions $\widetilde{\theta}(T_pM) \subset T_pM^\perp$ and $\dim\widetilde{\theta}(T_pM) = \dim W_p = N$. Then, $d \geqq N-2$ if and only if $T_pM \cup W_p \cup L$ is contained in some $(2N+2)$-dimensional subspace in $\mathbb{R}^{3N}$.*

Next, consider $\widetilde{\phi}$ given in §1.

Corollary 2.8. *Let M and G be the N-dimensional submanifolds of $\mathbb{R}^{3N}$ which satisfy the conditions $T_pM \subset T_pG^\perp, \widetilde{\theta}(T_pG) = T_pM$, $\dim W_p = N$. Then, $e \geqq N-2$ if and only if $T_pG \cup W_p \cup L$ is contained in some $(2N+2)$-dimensional subspace in $\mathbb{R}^{3N}$.*

3. Applications to Elementary Geometry

3.1. **Why do the circumcenter, inner center, centroid, orthocenter and excenter of triangles exist ?** Consider the triangle of the three vertices $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}$. Without loss of generality, we may assume that $\overrightarrow{x_3} = \overrightarrow{0}$, which is the origin of the plane. Let J_0 be the 2×2-matrix of rotation by $90°$ in the anti-clockwise direction.

Example 1. **Circumcenter**. The position vectors $\overrightarrow{p_1}, \overrightarrow{p_2}, \overrightarrow{p_3}$ of the middle points of the three edges are given by

$$\overrightarrow{p_1} = \frac{1}{2}\overrightarrow{x_1}, \quad \overrightarrow{p_2} = \frac{1}{2}\left(\overrightarrow{x_1} + \overrightarrow{x_2}\right), \quad \overrightarrow{p_3} = \frac{1}{2}\overrightarrow{x_2}.$$

We take $\overrightarrow{v_1} = J_0\overrightarrow{x_1}$, $\overrightarrow{v_2} = J_0\left(\overrightarrow{x_2} - \overrightarrow{x_1}\right)$, $\overrightarrow{v_3} = -J_0\overrightarrow{x_2}$. Set $\mathbb{P} = (\overrightarrow{p_1}\ \overrightarrow{p_2}\ \overrightarrow{p_3})$, $\mathbb{V}(\mathbb{P}) = (\overrightarrow{v_1}\ \overrightarrow{v_2}\ \overrightarrow{v_3})$ and regard them as vectors in $\mathbb{R}^6$. We then see that $\mathbb{V}(\mathbb{P}) \perp L$ because $\displaystyle\sum_{k=1}^{3}\overrightarrow{v_k} = \overrightarrow{0}$. The almost complex structure J on $\mathbb{R}^6$ is defined by $J(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}) =$

$(J_0\overrightarrow{x}, J_0\overrightarrow{y}, J_0\overrightarrow{z})$, where $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \in \mathbb{R}^2$. It follows that

$$\begin{aligned}\theta(\mathbb{V}(\mathbb{P})) &= \langle \mathbb{P}, J\mathbb{V}(\mathbb{P})\rangle \\ &= (\frac{1}{2}\overrightarrow{x_1}, -\overrightarrow{x_1}) + (\frac{1}{2}\overrightarrow{x_1} + \frac{1}{2}\overrightarrow{x_2}, -\overrightarrow{x_2} + \overrightarrow{x_1}) + (\frac{1}{2}\overrightarrow{x_2}, \overrightarrow{x_2}) = 0,\end{aligned}$$

where $(\ ,\ )$ is the standard inner product of $\mathbb{R}^2$. Thus, from Corollary 1.2 it follows that the three tangential lines in the directions $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ starting from the points $\overrightarrow{p_1}, \overrightarrow{p_2}, \overrightarrow{p_3}$, respectively, meet at a point, which is the circumcenter(the center of the circumcircle of the triangle).

Example 2. **Inner center**. In this case, we take $\overrightarrow{p_1} = \overrightarrow{0}, \overrightarrow{p_2} = \overrightarrow{x_1}, \overrightarrow{p_3} = \overrightarrow{x_2}$. Moreover, we take

$$\overrightarrow{v_1} = \frac{1}{\|\overrightarrow{x_1}\|}\overrightarrow{x_1} + \frac{1}{\|\overrightarrow{x_2}\|}\overrightarrow{x_2}, \quad \overrightarrow{v_2} = -\frac{1}{\|\overrightarrow{x_1}\|}\overrightarrow{x_1} + \frac{1}{\|\overrightarrow{x_2} - \overrightarrow{x_1}\|}\left(\overrightarrow{x_2} - \overrightarrow{x_1}\right),$$
$$\overrightarrow{v_3} = -\frac{1}{\|\overrightarrow{x_2}\|}\overrightarrow{x_2} + \frac{1}{\|\overrightarrow{x_1} - \overrightarrow{x_2}\|}\left(\overrightarrow{x_1} - \overrightarrow{x_2}\right).$$

We then see that $\displaystyle\sum_{k=1}^{3} \overrightarrow{v_k} = \overrightarrow{0}$, hence $\mathbb{V}(\mathbb{P}) \perp L$. Moreover, we have

$$\theta(\mathbb{V}(\mathbb{P})) =< \mathbb{P}, J\mathbb{V}(\mathbb{P}) >= (\overrightarrow{x_1}, J_0\overrightarrow{v_2}) + (\overrightarrow{x_2}, J_0\overrightarrow{v_3}) = \det(\overrightarrow{x_1}\ \overrightarrow{v_2}) + \det(\overrightarrow{x_2}\ \overrightarrow{v_3}) = 0.$$

Thus, from Corollary 1.2 it follows that the three tangential lines in the directions $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ starting from the points $\overrightarrow{p_1}, \overrightarrow{p_2}, \overrightarrow{p_3}$, respectively, meet at a point, which is the inner center(the center of inscribed circle of the triangle).

Example 3. **Centroid**. As in Example 2, we take $\overrightarrow{p_1} = \overrightarrow{0}, \overrightarrow{p_2} = \overrightarrow{x_1}, \overrightarrow{p_3} = \overrightarrow{x_2}$. Moreover, we take

$$\overrightarrow{v_1} = \frac{1}{2}\left(\overrightarrow{x_1} + \overrightarrow{x_2}\right), \quad \overrightarrow{v_2} = \frac{1}{2}\overrightarrow{x_2} - \overrightarrow{x_1}, \quad \overrightarrow{v_3} = \frac{1}{2}\overrightarrow{x_1} - \overrightarrow{x_2}.$$

We then see that $\displaystyle\sum_{k=1}^{3} \overrightarrow{v_k} = \overrightarrow{0}$, hence $\mathbb{V}(\mathbb{P}) \perp L$. Moreover, we have

$$\theta(\mathbb{V}(\mathbb{P})) =< \mathbb{P}, J\mathbb{V}(\mathbb{P}) >= (\overrightarrow{x_1}, \frac{1}{2}J_0\overrightarrow{x_2} - J_0\overrightarrow{x_1}) + (\overrightarrow{x_2}, \frac{1}{2}J_0\overrightarrow{x_1} - J_0\overrightarrow{x_2}) = 0.$$

Thus, from Corollary 1.2 it follows that the three tangential lines in the directions $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ starting from the points $\overrightarrow{p_1}, \overrightarrow{p_2}, \overrightarrow{p_3}$, respectively, meet at a point, which is the centroid(the center of the gravity of triangle).

Example 4. **Orthocenter**. As in Example 2, we take $\overrightarrow{p_1} = \overrightarrow{0}, \overrightarrow{p_2} = \overrightarrow{x_1}, \overrightarrow{p_3} = \overrightarrow{x_2}$. Moreover, we take

$$\overrightarrow{v_1} = J_0\overrightarrow{x_1} - J_0\overrightarrow{x_2}, \quad \overrightarrow{v_2} = J_0\overrightarrow{x_2}, \quad \overrightarrow{v_3} = -J_0\overrightarrow{x_1}.$$

We then see that $\displaystyle\sum_{k=1}^{3} \overrightarrow{v_k} = \overrightarrow{0}$, hence $\mathbb{V}(\mathbb{P}) \perp L$. Moreover, we have

$$\theta(\mathbb{V}(\mathbb{P})) =< \mathbb{P}, J\mathbb{V}(\mathbb{P}) >= (\overrightarrow{x_1}, -\overrightarrow{x_2}) + (\overrightarrow{x_2}, \overrightarrow{x_1}) = 0.$$

Thus, from Corollary 1.2 it follows that the three tangential lines in the directions $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ starting from the points $\overrightarrow{p_1}, \overrightarrow{p_2}, \overrightarrow{p_3}$, respectively, meet at a point, which is the orthocenter of triangle.

Example 5. **Excenter**. As in Example 2, we take $\overrightarrow{p_1} = \overrightarrow{0}, \overrightarrow{p_2} = \overrightarrow{x_1}, \overrightarrow{p_3} = \overrightarrow{x_2}$. Moreover, we take

$$\overrightarrow{v_1} = \frac{1}{\|\overrightarrow{x_1}\|}\overrightarrow{x_1} + \frac{1}{\|\overrightarrow{x_2}\|}\overrightarrow{x_2}, \quad \overrightarrow{v_2} = -\frac{1}{\|\overrightarrow{x_1}\|}\overrightarrow{x_1} - \frac{1}{\|\overrightarrow{x_2} - \overrightarrow{x_1}\|}\left(\overrightarrow{x_2} - \overrightarrow{x_1}\right),$$
$$\overrightarrow{v_3} = -\frac{1}{\|\overrightarrow{x_2}\|}\overrightarrow{x_2} - \frac{1}{\|\overrightarrow{x_1} - \overrightarrow{x_2}\|}\left(\overrightarrow{x_1} - \overrightarrow{x_2}\right).$$

We then see that $\sum_{k=1}^{3} \overrightarrow{v_k} = \overrightarrow{0}$, hence $\mathbb{V}(\mathbb{P}) \perp L$. Moreover, we have

$$\begin{aligned}
&\theta(\mathbb{V}(\mathbb{P})) =< \mathbb{P}, J\mathbb{V}(\mathbb{P}) > \\
= \ & (\overrightarrow{x_1}, -\frac{J_0\overrightarrow{x_1}}{\|\overrightarrow{x_1}\|} - \frac{(J_0\overrightarrow{x_2} - J_0\overrightarrow{x_1})}{\|\overrightarrow{x_2} - \overrightarrow{x_1}\|}) + (\overrightarrow{x_2}, -\frac{J_0\overrightarrow{x_2}}{\|\overrightarrow{x_2}\|} - \frac{(J_0\overrightarrow{x_1} - J_0\overrightarrow{x_2})}{\|\overrightarrow{x_1} - \overrightarrow{x_2}\|}) \\
= \ & -\frac{1}{\|\overrightarrow{x_2} - \overrightarrow{x_1}\|}(\overrightarrow{x_1}, J_0\overrightarrow{x_2}) - \frac{1}{\|\overrightarrow{x_1} - \overrightarrow{x_2}\|}(\overrightarrow{x_2}, J_0\overrightarrow{x_1}) = 0.
\end{aligned}$$

Thus, from Corollary 1.2 it follows that the three tangential lines in the directions $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ starting from the points $\overrightarrow{p_1}, \overrightarrow{p_2}, \overrightarrow{p_3}$, respectively, meet at a point, which is the excenter of triangle.

These are famous properties of triangle, but Corollary 1.2 says more. In fact, although we didn't state here, many other examples of the three lines starting from some points on triangle which meet at a point can be understood from the point of view of Corollary 1.2.

3.2. **Desargues' Theorem, Pappus' Theorem and Pascal's Theorem.** For any point $p \in S^2$, we denote by $a(p) \in S^2$ the antipodal point of p. Take arbitrary six points $p_1, p_2, \cdots, p_6 \in S^2$ which are different from each other. For $i \neq j$, we denote by γ_{ij} the closed geodesic through p_i and p_j. We also denote by P_{ij} the linear subspace of $\mathbb{R}^3$ spanned by $\overrightarrow{p_i}$ and $\overrightarrow{p_j}$.

Theorem 3.1. (*Spherical Desargues' Theorem*) *Assume that each of the set of points $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6\}$ does not lie on a single geodesic in S^2. Then, three geodesics $\gamma_{14}, \gamma_{25}, \gamma_{36}$ meet at a point q (and $a(q)$) if and only if the intersection points $\gamma_{12} \cap \gamma_{45}, \gamma_{23} \cap \gamma_{56}, \gamma_{13} \cap \gamma_{46}$ lie on a single geodesic.*

Proof. Three geodesics $\gamma_{14}, \gamma_{25}, \gamma_{36}$ meet at a point q (and $a(q)$) if and only if $\det\left(\overrightarrow{p_1} \times \overrightarrow{p_4} \mid \overrightarrow{p_2} \times \overrightarrow{p_5} \mid \overrightarrow{p_3} \times \overrightarrow{p_6}\right) = 0$, which is equivalent to

$$\det\left(\overrightarrow{p_1} \mid \overrightarrow{p_2} \mid \overrightarrow{p_5}\right) \cdot \det\left(\overrightarrow{p_3} \mid \overrightarrow{p_4} \mid \overrightarrow{p_6}\right) = \det\left(\overrightarrow{p_2} \mid \overrightarrow{p_4} \mid \overrightarrow{p_5}\right) \cdot \det\left(\overrightarrow{p_1} \mid \overrightarrow{p_3} \mid \overrightarrow{p_6}\right). \tag{3.2}$$

On the other hand, $\gamma_{12} \cap \gamma_{45}, \gamma_{23} \cap \gamma_{56}, \gamma_{13} \cap \gamma_{46}$ lie on a single geodesic if and only if $P_{12} \cap P_{45}, P_{23} \cap P_{56}, P_{13} \cap P_{46}$ lie on a certain 2-dimensional subspace if and only if

(3.3)
$$\det\left((\overrightarrow{p_1} \times \overrightarrow{p_2}) \times (\overrightarrow{p_4} \times \overrightarrow{p_5}) \mid (\overrightarrow{p_2} \times \overrightarrow{p_3}) \times (\overrightarrow{p_5} \times \overrightarrow{p_6}) \mid (\overrightarrow{p_1} \times \overrightarrow{p_3}) \times (\overrightarrow{p_4} \times \overrightarrow{p_6})\right) = 0.$$

By the assumption we have $\det(\overrightarrow{p_1} \mid \overrightarrow{p_2} \mid \overrightarrow{p_3}) \neq 0$, hence (3.3) is equivalent to

$$\begin{aligned} (3.4) \qquad & \det(\overrightarrow{p_2} \mid \overrightarrow{p_4} \mid \overrightarrow{p_5}) \cdot \det(\overrightarrow{p_3} \mid \overrightarrow{p_5} \mid \overrightarrow{p_6}) \cdot \det(\overrightarrow{p_1} \mid \overrightarrow{p_4} \mid \overrightarrow{p_6}) \\ = \; & \det(\overrightarrow{p_1} \mid \overrightarrow{p_4} \mid \overrightarrow{p_5}) \cdot \det(\overrightarrow{p_2} \mid \overrightarrow{p_5} \mid \overrightarrow{p_6}) \cdot \det(\overrightarrow{p_3} \mid \overrightarrow{p_4} \mid \overrightarrow{p_6}) . \end{aligned}$$

Finally, we see that (3.4) is equivalent to (3.2) under the assumption $\det(\overrightarrow{p_4} \mid \overrightarrow{p_5} \mid \overrightarrow{p_6}) \neq 0$.

□

Considering the case where all the intersection points concerned with are lying on certain half sphere, identifying each point with its antipodal point, we obtain the planer Desargues' Theorem as a corollary of Theorem 3.1.

Corollary 3.5. (*Planer Desargues' Theorem*) *Consider two triangles* $\Delta\mathrm{ABC}$ *and* $\Delta\mathrm{A'B'C'}$ *in the projective plane* P^2. *Then three lines* $\mathrm{AA'}, \mathrm{BB'}, \mathrm{CC'}$ *meet at a point if and only if three intersection points* $\mathrm{AB} \cap \mathrm{A'B'}, \mathrm{BC} \cap \mathrm{B'C'}, \mathrm{CA} \cap \mathrm{C'A'}$ *lie on a single line.*

Theorem 3.6. *(Spherical Pappus' Theorem) Assume that each of the set of points* $\{p_1, p_2, p_3\}$ *and* $\{p_4, p_5, p_6\}$ *lie on a single geodesic in* S^2. *Then, three intersection points* $\gamma_{15} \cap \gamma_{24}, \gamma_{16} \cap \gamma_{34}, \gamma_{26} \cap \gamma_{35}$ *lie on a single geodesic.*

Proof. We have

$$\begin{aligned} & \det((\overrightarrow{p_1} \times \overrightarrow{p_5}) \times (\overrightarrow{p_2} \times \overrightarrow{p_4}) \mid (\overrightarrow{p_2} \times \overrightarrow{p_6}) \times (\overrightarrow{p_3} \times \overrightarrow{p_5}) \mid (\overrightarrow{p_1} \times \overrightarrow{p_6}) \times (\overrightarrow{p_3} \times \overrightarrow{p_4})) \\ = \; & \det(\overrightarrow{p_4} \mid \overrightarrow{p_5} \mid \overrightarrow{p_6}) \cdot \det(\overrightarrow{p_1} \mid \overrightarrow{p_3} \mid \overrightarrow{p_5}) \cdot \det(\overrightarrow{p_1} \mid \overrightarrow{p_2} \mid \overrightarrow{p_4}) \cdot \det(\overrightarrow{p_2} \mid \overrightarrow{p_3} \mid \overrightarrow{p_6}) \\ & - \det(\overrightarrow{p_1} \mid \overrightarrow{p_2} \mid \overrightarrow{p_3}) \cdot \det(\overrightarrow{p_1} \mid \overrightarrow{p_4} \mid \overrightarrow{p_5}) \cdot \det(\overrightarrow{p_2} \mid \overrightarrow{p_4} \mid \overrightarrow{p_6}) \cdot \det(\overrightarrow{p_3} \mid \overrightarrow{p_5} \mid \overrightarrow{p_6}) \\ =: \; & F(p_1, p_2, p_3, p_4, p_5, p_6), \end{aligned}$$

from which the result follows. □

Corollary 3.7. *Planer Pappus' Theorem*

Consider the curve defined by $F(p_1, p_2, p_3, p_4, p_5, p) = 0$, where p is the parameter. Then, this defines a quadratic curve $\mathcal{C}$. Clearly, $p_1, p_2, p_3, p_4, p_5 \in \mathcal{C}$. Hence, we may take p_6 as arbitrary point on $\mathcal{C}$ which is different from p_1, p_2, p_3, p_4, p_5. Therefore we obtain

Corollary 3.8. *Planer Pascal's Theorem*

Remark. Using the idea by Arnold([1]), Tomihisa([5]) gave a certain criterion in terms of the Poisson bracket for those theorems of Desargues', Pappus', Pascal's and Brianchon's. This is because the Poisson bracket of two qudratic forms on a $\mathbf{R}^2$ corresponds to the vector cross product of the two vectors made by the coefficients of the quadratic forms in a certain way.

4. Geodesics on Riemann surfaces

4.1. **Geodesics on torus.** Consider flat torus $T = \Lambda \backslash \mathbb{R}^2$, where Λ is a lattice of rank 2 in $\mathbb{R}^2$. Let $l_1, l_2, \cdots, l_N$ be geodesics on the torus T. Denote by $\widetilde{l_1}, \widetilde{l_2}, \cdots, \widetilde{l_N}$ be arbitrary lifts of $l_1, l_2, \cdots, l_N$, respectively. We may define an action of Λ^N on $\mathbb{R}^{2N} \times \Lambda^N$ by $(p, \tau)^\eta = (\eta + p, \eta + \tau)$ for $\eta \in \Lambda^N, (p, \tau) \in \mathbb{R}^{2N} \times \Lambda^N$. The equivalence class of (p, τ) is denoted by $[[p, \tau]]$. The set of all equivalence classes is denoted by

$I^T = \Gamma^N \backslash (\mathbb{R}^{2N} \times \Gamma^N)$. We have a natural projection map $pr : I^T \ni [[p,\tau]] \longrightarrow [p] \in T^N$. Note that for any $[p] \in T^N$ the fiber $I^T_{[p]} := pr^{-1}([p])$ is nothing but $[[p, \Lambda^N]]$. we introduce the equivalence relation $(p,u) \sim (q,v)$ by $p = q + \eta, u = v$ for some $\eta \in \Lambda^N$. We denote by $[p,u]$ the equivalence class of (p,u). The set of all equivalence classes is the tangent space $T_{[p]}T$.

Definition 4.1. Consider submanifold $M = c_1 \times c_2 \times \cdots \times c_N$ in T^N. For any $p = (p_1, p_2, \cdots, p_N) \in T^N$ and $\tau \in \Lambda^N$, we define an index $d_{[[p,\tau]]}$ by

$$d_{[[p,\tau]]} = \dim \left\{ [p,v] \in T_{[p]}M \mid \sum_{j=1}^{N} v_j = 0,\ \sum_{j=1}^{N} (p_j - \tau_j) \times v_j = 0 \right\}.$$

From Corollary 1.2 we have the following.

Corollary 4.2. *N-geodesics $l_1, l_2, \cdots, l_N$ on the torus $T = \Lambda \backslash \mathbb{R}^2$ meet at a point or all l_j^s are parallel if and only if $d_{[[p,\tau]]} \geqq N-2$ for some $[[p,\tau]] \in I^T_{[p]}$.*

4.2. **Geodesics on Rieman surfaces of hyperbolic type.** Let K_1^o denotes the connected component of K_1, where K_1 is the hyperboloid of two sheets as that in section 1. Note that, for any Riemann surface R, K_1^o becomes the universal cover $K_1^o \xrightarrow{pr} R$ of R and R is expressed as $R = \Gamma \backslash K_1^o$, where $\Gamma = \pi_1(R)$ is the fundamental group of R.

Let $E_{K_1^o}$ be the tangent bundle $T\mathbb{R}^3$ restricted to K_1^o. We introduce the equivalence relation on $E_{K_1^o}$ by Γ. Two elements $(p,u), (q,v) \in E_{K_1^o}$ are equivalent if and only if $q = \eta p, v = \eta u$ for some $\eta \in \Gamma$. We denote by $[p,u]$ the equivalence class of (p,u). The set of all equivalence classes is denoted by $E_R = \Gamma \backslash E_{K_1^o}$. Note that the tangent bundle TR of Riemann surface R is a subbundle of E_R. Set

$$E^N_{K_1^o} = \underbrace{E_{K_1^o} \times \cdots \times E_{K_1^o}}_{N-\text{times}}, \qquad E^N_R = \underbrace{E_R \times \cdots \times E_R}_{N-\text{times}},$$

$$TR^N = \underbrace{TR \times \cdots \times TR}_{N-\text{times}}, \qquad \Gamma^N = \underbrace{\Gamma \times \cdots \times \Gamma}_{N-\text{times}}.$$

For $p = (p_1, p_2 \cdots, p_N) \in K_1^{oN}$ and $u = (u_1, u_2, \cdots, u_N) \in E^N_{K_1^o}$, we write $[p,u] = ([p_1,u_1], [p_2,u_2], \cdots, [p_N,u_N]) \in E^N_R$. For $\tau = (\tau_1, \cdots, \tau_N) \in \Gamma^N$, define L_τ by

$$L_\tau := \tau L_0 = \mathbb{R} \sum_{j=1}^{N} \tau_j \mathbf{e}_x^j \oplus \mathbb{R} \sum_{j=1}^{N} \tau_j \mathbf{e}_y^j \oplus \mathbb{R} \sum_{j=1}^{N} \tau_j \mathbf{e}_z^j.$$

We regard L_τ as a 3-plane of E^N_p, which is the fiber of $E^N_{K_1^o}$ at $p \in K_1^o$. Then, we may regard $[p, L_\tau]$ as a 3-plane of $E^N_{[p]}$, which is the fiber of E^N_R at $[p] \in R$. Recall $\widetilde{\theta}_{-1}$ used in section 1. Since $\left(\widetilde{\theta}_{-1}\right)_{\tau p} (\tau v) = \tau \left(\widetilde{\theta}_{-1}\right)_p (v)$ for $\tau \in \Gamma^N, v \in E^N_{K_1^o}$, we give the following definition.

Definition 4.3. For $p = (p_1, p_2, \cdots, p_N) \in K_1^{oN}$, we define $\widetilde{\theta}^R$ by

$$\widetilde{\theta}^R([p,v]) = \left[p, \left(\widetilde{\theta}_{-1}\right)_p (v)\right], \tag{4.4}$$

where $[p,u] \in TR^N$. We call $\widetilde{\theta}^R$ the *angular momentum for R^N*.

We say that $[p, u]$ is perpendicular to $[p, v]$ if $g_{\mathcal{L}}(u, v) = 0$. In this case, we also write $[p, u] \perp [p, v]$.

Definition 4.5. Let $M = c_1 \times c_2 \times \cdots \times c_N$ be an N-dimensional submanifold of R^N, where each c_j is a smooth curve in R. We define the index $d_{L_\tau}([p])$ by

$$d_{L_\tau}([p]) = \dim \left\{ [p, v] \in T_{[p]}M \mid \widetilde{\theta}^R([p, v]) \perp [p, L_\tau] \right\}.$$

We have easily the following.

$$d_{L_\tau}([p]) = \dim \left\{ [p, v] \in T_{[p]}M \mid \sum_{j=1}^{N} \tau_j^{-1} \theta_{p_j}(v_j) = 0 \right\}, \tag{4.6}$$

where θ is given by

$$\theta = \sum_{j=1}^{N} (y_j dz_j - z_j dy_j, z_j dx_j - x_j dz_j, x_j dy_j - y_j dx_j).$$

Let $l_1, l_2, \cdots, l_N$ be geodesics on R given by the initial condition $l(0) = c_j(t_j), \dot{l}(0) = \dot{c}_j(t_j)$ for $j = 1, 2, \cdots, N$, where $p_j = c_j(t_j)$. Denote by $\widetilde{l}_1, \widetilde{l}_2, \cdots, \widetilde{l}_N$ arbitrary lifts of $l_1, l_2, \cdots, l_N$ to K_1^o, respectively. We also denote by $P_1, P_2, \cdots, P_N \in G_2(\mathbb{R}^3)$ the 2-dimensional linear subspaces corresponding to $\widetilde{l}_1, \widetilde{l}_2, \cdots, \widetilde{l}_N$, respectively.

Lemma 4.7. *Set $H = P_1 \times P_2 \times \cdots \times P_N$. Choose some element $\tau = (\tau_1, \tau_2, \cdots, \tau_N) \in \Gamma^N$. Then, $\tau_1^{-1}\widetilde{l}_1, \tau_2^{-1}\widetilde{l}_2, \cdots, \tau_N^{-1}\widetilde{l}_N$ meet at a point in $K \cup \partial K$ if and only if $\dim\{H^\perp \cap L_\tau^\perp\} \geqq N - 2$.*

We here give the following definition.

Definition 4.8. We say that $l_1, l_2, \cdots, l_N$ are *parallel* if the intersection point x of the above lifts lies in $\partial K \cup K_2$.

Now, we have the following.

Theorem 4.9. *Let $l_1, l_2, \cdots, l_N$ be N-geodesics on Riemann surface R. Then, $d_{L_\tau}([p]) \geqq N - 2$ for some $[p, L_\tau] \in G_3(E_{[p]}^N)$ if and only if $l_1, l_2, \cdots, l_N$ meet at a point in R each other or all the l_j's are parallel.*

Next, we define another index t_r. For any point $r = (r_1, r_2, \cdots, r_N)$ of $l_1 \times \cdots \times l_N$, we define the index $t_{r,\tau}$ by

$$t_r = \operatorname*{Max}_{\tau \in \Gamma^N} \dim \left\{ S \mid\ S \text{ is a subspace of } < T_p K_1^{oN} \cup W_p > \cap L_\tau,\ (s, s) < 0, \forall s (\neq 0) \in S \right\}.$$

We easily see that t_r does not depend on the choice of the lifts.

Theorem 4.10. *Let $l_1, l_2, \cdots, l_N$ be N-geodesics on R and $r = (r_1, r_2, \cdots, r_N)$ any point of $l_1 \times \cdots \times l_N$. Then, $d_{L_\tau}([p]) \geqq N - 2$ for some $[p, L_\tau] \in G_3(E_{[p]}^N)$ and $t_r \geqq 1$ if and only if $l_1, l_2, \cdots, l_N$ meet at a point in R each other.*

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Tetsuya TANIGUCHI : Kitasato University, College of Liberal Arts and Sciences, 1-15-1 Kitasato, Minami, Sagamihara, Kanagawa, 252-0373
E-mail address: tetsuya@kitasato-u.ac.jp

Seiichi UDAGAWA : Nihon University, School of Medicine, Department of Mathematics, Itabashi, Tokyo, 173-0032
E-mail address: sudagawa@med.nihon-u.ac.jp