平面凸ビリヤードの不変円の微分可能性について

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Abstract. Let C be a convex closed curve of class C^2 in the plane. We think the domain bounded by C as a billiard table. We state the following. If a convex billiard is integrable, the set of points with irrational slopes make invariant circles of class C^1 in the phase space. If the sets of points with rational slopes make invariant circles K, then the invariant circles K are of class C^1. Otherwise, we find closed curves of class C^1 in the union of invariant circles with the same slope.

1. Introduction

Let C be a simple closed and strictly convex curve of class C^k, $k \geq 2$, with length L in the Euclidean plane $\mathbf{E}$ and let $c : \mathbf{R} \longrightarrow \mathbf{E}$ be its representation with respect to the arclength, namely $|\dot{c}(s)| = 1$ for all $s \in \mathbf{R}$ where $\mathbf{R}$ is the set of all real numbers. Let $x = (x_j)_{j\in\mathbf{Z}}$ be a sequence of points in C where $\mathbf{Z}$ is the set of all integers. We say that x is a *billiard ball trajectory* if the angle between the tangent vector A to C at x_i and the oriented segment $T(x_{i-1}, x_i)$ from x_{i-1} to x_i is equal to the one between A and $T(x_i, x_{i+1})$ for all $i \in \mathbf{Z}$.

A billiard ball trajectory $x = (x_j)_{j\in\mathbf{Z}}$ in C is represented by a sequence $s = (s_j)_{j\in\mathbf{Z}}$ of real numbers such that $x_j = c(s_j)$ and $s_j < s_{j+1} < s_j + L$ for all $j \in \mathbf{Z}$ and the sequence $s = (s_j)_{j\in\mathbf{Z}}$ will be considered to be a configuration $\{(j, s_j)\}_{j\in\mathbf{Z}}$ in the configuration space $\mathbf{X} = \mathbf{Z} \times \mathbf{R} \subset \mathbf{R}^2$. A configuration $s = (s_j)_{j\in\mathbf{Z}}$ for x is determined uniquely up to the difference pL ($p \in \mathbf{Z}$).

Let $x_0, x_1 \in C$ and (x_0, x_1, x_2) the billiard ball trajectory. Let θ_0 (resp., θ_1) be the angle between the segment $T(x_0, x_1)$ from x_0 to x_1 (resp., $T(x_1, x_2)$) and the tangent vector to C at x_0 (resp., x_1). Set $u_0 = \cos\theta_0$ and $u_1 = \cos\theta_1$. We call $\Omega = C \times (-1, 1)$ the *phase space* which is the set of all pairs (x, u) for $x \in C$ and $u \in (-1, 1)$. Define a *billiard ball map* $\varphi : \Omega \longrightarrow \Omega$ as $\varphi(x_0, u_0) = (x_1, u_1)$. The billiard ball map is an example of a monotone twist map ([12]). Let $\bar{x} = (x_0, u_0) \in \Omega$ and $\varphi^j(\bar{x}) = (x_j, u_j)$ for all $j \in \mathbf{Z}$. Then, the sequence

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$x = (x_j)_{j\in\mathbf{Z}}$ is a billiard ball trajectory. Any billiard ball trajectory is given in this way.

A convex billiard is said to be *integrable* if a subset of full measure of the phase space is foliated by closed curves invariant under the billiard ball map φ. The billiards in circles and ellipses are integrable. G. Birkhoff's conjecture states that the only examples of integrable billiards are circular and elliptic billiards ([5]). M. Bialy ([4]) has given a partial answer of the conjecture, proving that C is a circle if Ω is foliated by φ-invariant continuous closed curves not null-homotopic in Ω. M. Wojtkowski ([13]) proved that C is a circle if the domain bounded by C is foliated by smooth caustics to which almost every billiard ball trajectories are tangent. E. Y. Amiran ([1]) proved that when C is a strictly convex bounded planar domain with a smooth boundary and is integrable near the boundary, its boundary is necessarily an ellipse. As was stated in [4] Bialy's theorem corresponds to a theorem of E. Hopf ([9]) concerning Riemannian metrics on tori without conjugate points. N. Innami ([10]) extended Bialy's theorem to the higher dimensional case and the nonpositive curvature case as L. Green ([7]) did.

We say that a φ-invariant continuous closed curve in Ω is an *invariant circle* if it is not null-homotopic. If the billiard table is of class C^2, then the map φ in Ω is an area preserving twist map of class C^1, and Birkhoff's theorem ensures only that the invariant circles are Lipshitz and any invariant circle is the graph of a Lipshitz function, $\{G(s) = (c(s), u(s)) : 0 \le s \le L\}$ ([8], [12]). N. Innami ([11]) discussed the differentiability of invariant circles by using the geometry of geodesics due to H. Busumann ([6]) which was reconstructed in the configuration space $\mathbf{X}$ by V. Bangert ([2], [3]). In this note his results applies to an integrable convex billiard and we note the differentiability of invariant circles.

The notion of slope is usful to classify the invariant circles. Let $x = (x_j)_{j\in\mathbf{Z}}$ be a billiard ball trajectory and let $a(x_j, x_{j+1})$ be the arclength of the subarc of C from x_j to x_{j+1} measured with the positive orientation of C. We define the *slope* $\alpha(x)$ of x as

$$\alpha(x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} a(x_j, x_{j+1}) = \liminf_{n\to\infty} \frac{s_n}{n}.$$

where $s = (s_j)_{j\in\mathbf{Z}}$ is a configuration for x. Let $\alpha(\tilde{x})$ denote the slope of the billiard ball trajectory determined by $\tilde{x}$ for $\tilde{x} \in \Omega$. Set

$$\Omega(a) = \{\bar{x} \in \Omega \mid \alpha(\bar{x}) = aL\}.$$

If f is an invariant circle in Ω, then $\alpha(\bar{x})$ are constant for all $\bar{x} \in f$, and, therefore, $f \subset \Omega(a)$ for some a with $0 < a < 1$. We say that a closed curve f in Ω not null-homotoic is a *circle with constant slope* if $\alpha(\bar{x})$ are constant for all $\bar{x} \in f$.

Theorem 1. *Let C be a simple closed convex curve of class C^k, $k \geq 2$, with positive curvature κ and length L. Assume that its convex billiard is integrable. Let a be a real with $0 < a < 1$. Then the following are true.*

(1) *When a is irrational, $\Omega(a)$ is the invariant circle f with slope aL in Ω such that the graph $G_f(s)$ of $\Omega(a)$ is of class C^1.*

(2) *When a is rational, then the number of the invariant circles with slope aL in Ω is either one or two. If there exists just one invariant circle f with slope aL, then the graph $G_f(s)$ of f is of class C^1. Otherwise, there exist two closed curves $G_1(s) = (c(s), u_1(s))$ and $G_2(s) = (c(s), u_2(s))$, $0 \leq s \leq i(q)L$, of class C^1 with slope aL which are not null-homotopic where $i(q) = 1 + (1-(-1)^q)/2$ for $a = p/q$ reduced to its lowest terms. Moreover, $G_t(s) = (c(s), \max\{u_1(s), u_2(s)\})$ and $G_b(s) = (c(s), \min\{u_1(s), u_2(s)\})$, $0 \leq s \leq L$, are invariant circles with slope aL.*

If an invariant circle f is of class C^1, then the caustic K made from f is a continuous curve in the domain bounded by C. Here we say that a closed continuous curve K is a *caustic* if K has the following property. Let x_0 be an arbitrary point in C and let $T(x_0, x_1)$ be a segment tangent to K. If $x = (x_j)_{j \in \mathbf{Z}}$ is the billiard ball trajectory determined by $T(x_0, x_1)$, then $T(x_j, x_{j+1})$ is a segment tangent to K for all $j \in \mathbf{Z}$. Without C^2 differentiability condition on C the caustics are not continuous, in general.

2. Foliation by asymptotes and parallels

The contents in this section are based on the resuilts in [2], [3] and [11] and, therefore, we need not to prove the lemmas here again. We work in the configuration space $\mathbf{X}$. Let $s = (s_j)_{i \leq j \leq k}$ be the configuration of a billiard ball trajectory $x = (x_j)_{i \leq j \leq k}$. We say that $s = (s_j)_{i \leq j \leq k}$ is a *segment* from s_i to s_k in $\mathbf{X}$ if

$$\sum_{j=i}^{k-1} |c(s_{j+1}) - c(s_j)| = \max \left\{ \sum_{j=i}^{k-1} |c(t_{j+1}) - c(t_j)| \right\}$$

where $t = (t_j)_{j \in \mathbf{Z}}$ is any configuration such that $t_i = s_i$, $t_k = s_k$ and $t_j < t_{j+1} < t_j + L$. We say that $s = (s_j)_{j \in \mathbf{Z}}$ is a *straight line* in $\mathbf{X}$

if the restriction of s to the interval $i < j < k$ in $\mathbf{Z}$ is a segment for every $i < k \in \mathbf{Z}$. We say that a straight line s is (*positively*) *asymptotic* to a straight line t if the sequences of segments from s_i to t_k converges to the sub-ray $s = (s_j)_{j \geq i}$ of s as $k \longrightarrow \infty$ for every $i \in \mathbf{Z}$. We say that a straight line s is a *parallel* to a straight line t if the sequences of segments from s_i to t_k converge to the sub-ray $s = (s_j)_{j \geq i}$ and $s = (s_j)_{j \leq i}$ of s as $k \longrightarrow \infty$ and $k \longrightarrow -\infty$, respectively, for every $i \in \mathbf{Z}$. In general, the asymptote and parallel relation are not symmetric. A simple modification of the arguments in [11] proves the following.

Lemma 2. *Let f be a continuos curve in Ω with its graph $G_f(s) = (c(s), u(s))$, $s \in [t_0, t_1]$. Assume that the configurations $s(\bar{x})$ for all $\bar{x} \in f$ are straight lines and they are asymptotic to each other. Then, the graph $G_f(s)$ is of class C^1.*

The continuity of the curvature of C plays an important role in the proof of Lemma 2 as was seen in [11]. The situation in Lemma 2 appears in the case of irrational slopes.

Lemma 3. *Let a be an irrational number with $0 < a < 1$. Let f be a invariant circle in Ω with slope aL. Let $s(\bar{x})$ be the configuration corresponding to $\bar{x} \in f$. Then, all $s(\bar{x})$ are parallels to each other, and, therefore, f is of class C^1.*

Next we treat the case that a is rational. Let $a = p/q$ where p and q are mutually prime integers. In this case there exists a periodic straight line $s = (s_j)_{j \in \mathbf{Z}}$ with period (q, p), i.e., $s_{j+q} = s_j + pL$ for all $j \in \mathbf{Z}$. Let $A \subset \mathbf{R}$ be the set of those parameters s_0 such that $s = (s_j)_{j \in \mathbf{Z}}$ is a periodic straight line with period (q, p) and $B = \mathbf{R} \setminus A$. The set B is either an empty set or a union of open intervals (b^k, t^k), $k \in I$ where I ia an index set. If A is a discrete set, then we have $t^k = b^{k+1}$ for all $k \in I$. Let $u^k = ({u^k}_j)_{j \in \mathbf{Z}}$ and $v^k = ({v^k}_j)_{j \in \mathbf{Z}}$ be periodic straight lines with period (q, p) such that ${u^k}_0 = b^k$ and ${v^k}_0 = t^k$. For every $s_0 \in (b^k, t^k) \subset B$ there exists two straight lines with slope aL. One $\overline{s}$ is the positive asymptote to v^k through s_0 and the other $\underline{s}$ is the positive asymptote to u^k through s_0. Then, $\overline{s}$ and $\underline{s}$ is the negative asymptotes to u^k and v^k through s_0, respectively. Let $S(u^k, v^k) \subset \mathbf{X}$ be the strip bounded by two straight lines u^k and v^k. We have two foliations $\overline{F}_k = \{\overline{s} \,|\, s_0 \in (b^k, t^k)\}$ and $\underline{F}_k = \{\underline{s} \,|\, s_0 \in (b^k, t^k)\}$ of the interior of the strip $S(u^k, v^k)$ by parallels for each $k \in I$. Suppose that $S(u^{k+1}, v^{k+1})$ is next to $S(u^k, v^k)$. Let F_0 be the set of all periodic straight lines with period (q, p) through $s_0 \in A$. Then, each set of straight lines $F_1 = \cdots \cup \underline{F}_{k-1} \cup \overline{F}_k \cup \underline{F}_{k+1} \cup \cdots \cup F_0$ and $F_2 = \cdots \cup$

$\overline{F}_{k-1} \cup \underline{F}_k \cup \overline{F}_{k+1} \cup \cdots \cup F_0$ gives a foliation of $\mathbf{X}$ by paralleles to each other in the interior of each strip $S(u^k, v^k)$. Moreover, all straight lines in $\overline{F}_k \cup \underline{F}_{k+1}$ are asymptotic to the positive v^k and so to the negative v^k are all straight lines in $\underline{F}_k \cup \overline{F}_{k+1}$ for all $k \in I$. These foliations correspond to closed curves not null homotopic in Ω of class C^1. The curves cover C in Ω twice if q is odd.

Lemma 4. *Let $a = p/q$ be a rational number with $0 < a < 1$ where p and q are mutually prime integers. Let $s(\bar{x})$ be the configuration in $\mathbf{X}$ corresponing to $\bar{x} \in \Omega$. If $s(\bar{x})$ are periodic straight lines with period (q, p) for all $\bar{x} \in \Omega(a)$, then $\Omega(a)$ is an invariant circle of class C^1. Otherwise, there are two foliations of $\mathbf{X}$ which correspond to closed curves not null-homotopic and of class C^1.*

E. Gutkin and A. Katok ([8]) mentions some examples of invariant circles and caustics.

3. Proof of Theorem 1

In this section we prove Theorem 1. When $\Omega(a)$, $0 < a < 1$, is an invariant circle f, we have already proved that f is of class C^1. Suppose $\Omega(a)$ is not an invariant circle. Then, a is a rational number p/q where q and p are mutually prime integers. Let d_j (resp., e_j) be a sequence of irrational numbers with $d_j > a$ (resp., $e_j < a$) converging to a. Then, $\Omega(d_j)$ and $\Omega(e_j)$ converge to subsets G_b and G_t contained in $\Omega(a)$ which are invariant circles. More precisely, $G_b \cup G_t$ is the boundary of $\Omega(a)$ and the configurations $s(\bar{x})$ in $\mathbf{X}$ corresponding to $\bar{x} \in G_b \cap G_t$ are periodic straight lines with period (q, p). Lemma 4 completes the proof of Theorem 1.

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