# Deformation and gluing constructions for scalar curvature, with applications

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Geometric Analysis in Geometry and Topology 2014
October 28-October 31, 2014

# Outline

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- Lecture I. Survey of scalar curvature
- Lecture II. Localized scalar curvature deformation, and gluing.
- Lecture III. Scalar curvature gluing and applications.

Applications may include a recent resolution of the remaining cases of the modified Kazdan-Warner problem by S. Matsuo, and constructions of interesting initial data in general relativity, as time permits.

# Curvature basics

Let $(M, g)$ be Riemannian, with compatible connection $\nabla$ ($\nabla g = 0$). The curvature tensor is defined by

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We use the index conventions (and the Einstein summation convention)

$$R = R_{ijk}{}^{\ell} \frac{\partial}{\partial x^{\ell}} \otimes dx^i \otimes dx^j \otimes dx^k$$

and with metric $g = \langle \cdot, \cdot \rangle$, $R_{ijk\ell} = \langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}), \frac{\partial}{\partial x^\ell} \rangle = R_{ijk}{}^{m} g_{m\ell}$.

## Gauss' Theorema Egregium

If $U \subset \mathbb{R}^2$, and $\mathbb{X} : U \to \Sigma = \mathbb{X}(U) \subset \mathbb{R}^3$ is an embedding, then if $\nabla^\Sigma$ is the compatible connection on $\Sigma$ in the induced metric,

$$\nabla^\Sigma_{\mathbb{X}_v} \nabla^\Sigma_{\mathbb{X}_u} \mathbb{X}_u - \nabla^\Sigma_{\mathbb{X}_u} \nabla^\Sigma_{\mathbb{X}_v} \mathbb{X}_u = K^\Sigma \left( \langle \mathbb{X}_u, \mathbb{X}_u \rangle \mathbb{X}_v - \langle \mathbb{X}_u, \mathbb{X}_v \rangle \mathbb{X}_u \right)$$

# Curvature basics

Sectional curvature: If $\Pi \subset T_pM$ is a two-plane, and if $\{e_1, e_2\}$ is any orthonormal basis of $\Pi$, $K(\Pi) := \langle R(e_1, e_2, e_2), e_1\rangle$ is independent of choice of orthonormal basis, and is the sectional curvature.
It is the Gauss curvature of a the geodesic two-surface tangent to $\Pi$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame for $T_pM$.

Ricci curvature: $\mathrm{Ric}_g(v, v) = \sum_{j=1}^{n} \langle R(v, e_j, e_j), v\rangle$,

or in component form for any basis of $T_pM$, $R_{jk} = R_{\ell jk}{}^{\ell}$.

Scalar curvature: $R(g) = g^{ij} R_{ij}$ is the trace of the Ricci tensor, which in an ON frame is just $R(g) = \sum_{i,j=1}^{n} \langle R(e_i, e_j, e_j), e_i\rangle$,
the total (average) of Gauss curvatures of geodesic two-surfaces through $p$.
The scalar curvature $R(g)$ is quasilinear, second-order in the components of $g$, second-order terms of the form $g * \partial^2 g$.

# Curvature basics

## Fundamental question

How to relate the curvatures to the geometry/topology of $M$.

Sectional and Ricci curvatures have stronger immediate ties to the geometry and topology of $M$.

## Sectional Curvature

Sectional curvature affects the behavior of geodesics. If $J$ is a variation field for a family of geodesics, say $\gamma_s$ is a geodesic for each $s \in I$, and $J = \partial_s \gamma_s$, then in case of constant sectional curvature $K$, the ODE $J'' + KJ = 0$ is satisfied.

This gives the relative spreading of geodesics in negative curvature (hyperbolic space), and the convergence of geodesics in positive curvature (sphere).

# Curvature basics

## Ricci curvature

The Ricci curvature then controls the geodesic deviation on average, nicely captured by the expansion of the volume element in normal coordinates: $dv_g = \left[1 - \frac{1}{6}R_{jk}x^j x^k + O(|x|^3)\right] \ dv_{\mathrm{Eucl}}$.

In fact recall

## Bishop's Theorem

For $\kappa$ constant, suppose $(M^n, g)$ has $\mathrm{Ric}(g) \geq (n-1)\kappa g$.
Then for any geodesic ball $B_r(p)$,

$$\mathrm{Vol}_g(B_r(p)) \leq \mathrm{Vol}(B_r^\kappa(O)),$$

where the volume of the right is taken in the simply connected space form of curvature $\kappa$.

# Curvature basics

We recall one last basic fact before moving to scalar curvature.

## Bonnet-Myers

$(M^n, g)$ complete Riemannian. Suppose $\mathrm{Ric}(g) \geq \frac{n-1}{r_0^2}\, g$.
Then $\mathrm{diam}(M, g) \leq \pi r_0$.
Moreover, $M$ is compact, and $\pi_1(M)$ is finite.

## Examples

- If $N$ closed, $\mathbb{S}^1 \times N$ admits no metric of $\mathrm{Ric}(g) > 0$.
- $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ $(n \geq 3)$ admits metrics with positive scalar curvature (PSC), e.g. product metric.

# Scalar curvature basics

So, what does scalar curvature measure?
We saw that the Ricci curvature appears in the expansion of the volume form, so integrating over geodesic balls $B_r(p)$, the scalar curvature appears in the expansion of the volume of geodesic balls:

**Volume expansion**

$$\mathrm{Vol}(B_p(r)) = \mathrm{Vol}_{\mathrm{Eucl}}(\mathbb{B}^n(1))r^n \left[1 - \frac{R(g)|_p}{6(n+2)} r^2 + O(r^3)\right].$$

Remark: This does give a $C^0$ way to define PSC (positive scalar curvature).
Flexibility: Scalar curvature should be reasonably flexible—degrees of freedom: single scalar equation for a metric. Before we focus on flexibility, let's note some obstructions.

# Surfaces

It's unfair to start with surfaces since on a closed, orientable two-surface $(\Sigma, g)$, $R(g) = 2K^{\Sigma}$, which is tightly controlled by the

## Gauss-Bonnet Theorem

$$\int_{\Sigma} K^{\Sigma} \, dA_{\Sigma} = 2\pi\chi(\Sigma).$$

In fact we have a complete solution to which functions are (scalar) curvatures of Riemannian metrics on $\Sigma$.

## Kazdan-Warner

$\Sigma^2$ closed, connected. $K \in C^{\infty}(\Sigma)$ is the Gauss curvature of some smooth metric $g$ on $\Sigma$ if and only if

- Case (i) $\chi(\Sigma) > 0$: $\{K > 0\} \neq \emptyset$.
- Case (ii) $\chi(\Sigma) = 0$: $K \equiv 0$; or $\{K > 0\} \neq \emptyset$ and $\{K < 0\} \neq \emptyset$
- Case (iii) $\chi(\Sigma) < 0$: $\{K < 0\} \neq \emptyset$.

# Scalar curvature in higher dimensions

As is expected, scalar curvature is more flexible in dimension three and higher. However, there are obstructions to manifolds admitting PSC, going back to Lichnerowicz, then Schoen and Yau (SY), and Gromov and Lawson (GL).

PSC and GR: There is an interesting connection with PSC and GR. The next theorem is intimately tied to the Positive Mass Theorem.

**Theorem (SY, GL)**

$\mathbb{T}^n$ does not admit any PSC metric. Any zero scalar curvature metric is flat.

# Schoen-Yau Theorem

## Theorem (SY)

$M^3$ closed, orientable. If $\pi_1(M)$ contains $\pi_1(\Sigma)$ for $\Sigma$ orientable surface of positive genus, then $M$ admits no PSC metric.

The proof involves an insightful rewriting of the second variation of area for closed oriented minimal hypersurfaces $\Sigma^2 \subset M^3$. We review that here.

If $\nu$ is the oriented unit normal along $\Sigma$, and if $A$ is the second fundamental form of $\Sigma$, then we consider the variation field $V = \varphi\nu$ along $\Sigma$. The Jacobi operator $\mathcal{L}_\Sigma$ is given by $\mathcal{L}_\Sigma\varphi = \Delta_\Sigma\varphi + (\|A\|^2 + \mathrm{Ric}(g)(\nu,\nu))\varphi$. Then if $A(t)$ is the area of $\Sigma_t$, where $V = \varphi\nu$ generates $\Sigma_t$, and if $\Sigma_0 = \Sigma$ is minimal, then

$$A''(0) = -\int_\Sigma \varphi\mathcal{L}_\Sigma\varphi \, dA_\Sigma = \int_\Sigma \left(|\nabla^\Sigma\varphi|^2 - (\|A\|^2 + \mathrm{Ric}(g)(\nu,\nu))\varphi^2\right) \, dA_\Sigma.$$

# Schoen-Yau Theorem

We want to work in the scalar curvature $R(g)$ into the second variation. To do this, let $E_1, E_2$ be a local orthonormal frame on $\Sigma$, let $E_3 = \nu$. Write $A = (h_{ij})$, where $h_{ij} = \langle \nabla^M_{e_i} e_j, \nu \rangle$, and let $K_{ij} = K(\text{span}(e_i, e_j))$ be a sectional curvature in $(M, g)$. We collect a few formulas:

- $\text{Ric}(g)(\nu, \nu) = K_{13} + K_{23}$
- $R(g) = 2(K_{12} + K_{13} + K_{23})$
- (Gauss Equation) $K^\Sigma = K_{12} + h_{11}h_{22} - h_{12}^2$.

Now recall $h_{12} = h_{21}$. Also, for $\Sigma$ minimal, $h_{11} + h_{22} = 0$, so that
$K^\Sigma = K_{12} - \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^2 = K_{12} - \frac{1}{2}\|A\|^2$.
Thus $\|A\|^2 + \text{Ric}(g)(\nu, \nu) = \|A\|^2 + K_{13} + K_{23} =$
$\frac{1}{2}\|A\|^2 - K^\Sigma + K_{12} + K_{13} + K_{23} = \frac{1}{2}\|A\|^2 + \frac{1}{2}R(g) - K^\Sigma$.

# Schoen-Yau Theorem

Inserting this into the Second Variation, we get

## Second Variation Formula

$A''(0) = \int_\Sigma \left[ |\nabla^\Sigma \varphi|^2 - \left( \frac{1}{2}\|A\|^2 + \frac{1}{2}R(g) - K^\Sigma \right) \varphi^2 \right] \, dA_\Sigma.$

## Definition

A minimal $\Sigma$ is stable if $A''(0) \geq 0$.

## Exercise

Let $\Sigma$ be an oriented, stable minimal two-surface in $(M^3, g)$. If $R(g) \geq 0$, then $\chi(\Sigma) \geq 0$, and $\chi(\Sigma) = 0$ implies $R(g) \equiv 0$ and $\Sigma$ is totally geodesic. In case $R(g) > 0$, then $\chi(\Sigma) > 0$ ($\Sigma$ is diffeomorphic to a sphere).

This is fun: use the stability inequality, with $\varphi \equiv 1$, and Gauss-Bonnet.

# Three-manifolds admitting PSC

## Theorem (SY, GL)

If $M_1^n$, $M_2^n$, $n \geq 3$, carry metrics of PSC, so does $M_1 \# M_2$.

Remark: The proof in GL is local to the surgical region.
Along with the resolution of the Poincaré and Spherical Space Form Conjectures, we have:

## Three-manifolds of PSC

$M^3$ closed, orientable. $M$ admits a PSC metric if and only if $M$ is a connected sum of factors of the form $\mathbb{S}^1 \times \mathbb{S}^2$, and $\mathbb{S}^3/\Gamma$ ($\Gamma \subset SO(4)$ finite subgroup).

# Kazdan-Warner classification

Goal: Determine which functions are scalar curvatures for some metric $g$ on $M$.

## Theorem(KW)

$M^n$, $n \geq 3$, closed and connected. Let $\mathcal{M}$ be the space of (smooth) metrics on $M$, $R : \mathcal{M} \to C^\infty(M)$ the scalar curvature map. There are three possibilities for the image $R(\mathcal{M})$:

- Case (i): Any $f \in C^\infty(M)$ can be realized as $R(g) = f$ for some $g \in \mathcal{M}$.
- Case (ii) $f \in C^\infty(M) \cap R(\mathcal{M})$ if and only if either $\{f < 0\} \neq \emptyset$, or $f \equiv 0$.
- Case (iii) $f \in C^\infty(M) \cap R(\mathcal{M})$ if and only if $\{f < 0\} \neq \emptyset$.

## Examples

(i) $\mathbb{S}^n$ (ii) $\mathbb{T}^n$ (iii) $\mathbb{T}^n \# \mathbb{T}^n$.

# Connection to Relativity

We recall briefly the Einstein constraint equations, which govern the space of allowable initial data for the Einstein equation in space-time.

Suppose $(M^3, g)$ is a totally geodesic space-like hypersurface in a Lorentzian manifold $(\mathcal{S}^4, \bar{g})$, with unit (time-like) unit normal $n$. The Einstein equation is $\text{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} + \Lambda\bar{g} = \kappa T$.

Vacuum case: $T = 0$. $\Lambda$ is the *cosmological constant*.

## Hamiltonian Constraint

The Gauss equation then implies the vacuum Hamiltonian constraint $R(g) = 2\Lambda$. ($g$ is CSC.)

Let $\Lambda = 0$. If there are matter fields, then $R(g) = \kappa\rho$, where $\kappa > 0$ is a constant, and $\rho = T(n, n)$ is the energy density of matter fields as measured by observer adapted to $M$. Thus $R(g) \geq 0$ is a natural assumption for the Positive Mass/Energy Theorem.

# Schwarzschild metric

The following metric is Lorentzian and Ricci-flat:

$$\bar{g}_S = -\left(\frac{1-\frac{m}{2|x|}}{1+\frac{m}{2|x|}}\right)^2 dt^2 + \left(1+\frac{m}{2|x|}\right)^4 g_{\mathrm{Eucl}}$$

$g_S := \left(1+\frac{m}{2|x|}\right)^4 g_{\mathrm{Eucl}}$ is the spatial Schwarzschild metric of mass $m$.

Since $\frac{1}{|x|}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$, we have $R(g_S) = 0$. It is asymptotically flat with two ends if $m > 0$.

$\Sigma = \{|x| = \frac{m}{2}\}$ is a totally geodesic surface. Easy to check (exercise)
$|\Sigma| = 16\pi m^2$, i.e. $m = \sqrt{\frac{|\Sigma|}{16\pi}}$ (equality case of Penrose inequality).

# Schwarzschild metric

Exercise: A simple computation shows

$$m = \frac{1}{16\pi} \lim_{r \to +\infty} \int\limits_{|x|=r} \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j}) \nu_e^j \, dA_e$$

where $\nu_e$ and $dA_e$ are the Euclidean outward unit normal and area element on the spheres $\{|x| = r\}$.

The flux integral limit provides the definition of mass/energy for more general asymptotically flat solutions of the vacuum ECE.

# Conformal Considerations

Conformal techniques give a classical vehicle for deforming/prescribing scalar curvature and solving the ECE.

Let $M^n$, $n \geq 3$, be smooth and connected. Let $\mathcal{M}$ be the space of (smooth) Riemannian metrics, and $\mathcal{M}_1 \subset \mathcal{M}$ the unit volume metrics.

For $g \in \mathcal{M}$, let $[g] = \{fg : f > 0, \;\; f \in C^\infty(M)\}$ be the *conformal class* of $g$.

Let $\mathcal{C}$ be the set of conformal classes.

# Conformal change of scalar curvature

For $n \geq 3$, $\tilde{g} = u^{\frac{4}{n-2}} g$ for $u > 0$. Then

$$R(\tilde{g}) = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \left[ \Delta_g u - \frac{n-2}{4(n-1)} R(g)\, u \right].$$

Let $\mathcal{L}_g u = \Delta_g u - \frac{n-2}{4(n-1)} R(g)\, u$ be the conformal Laplacian.
Remark: Finding $\tilde{g} \in [g]$ with CSC is a semi-linear elliptic equation, while $R(\tilde{g}) = 0$ is a linear elliptic PDE $\mathcal{L}_g u = 0$.

# Variational characterization of CSC

Einstein-Hilbert action: $\mathcal{R}(g) = \int_M R(g)\, dv_g$.

Volume-normalized action: $\overline{\mathcal{R}}(g) := \frac{\mathcal{R}(g)}{(\mathrm{Vol}(g))^{\frac{n-2}{n}}} = \mathcal{R}((\mathrm{Vol}(g))^{-\frac{2}{n}}\, g)$,

where $\bar{g} = (\mathrm{Vol}(g))^{-\frac{2}{n}}\, g \in \mathcal{M}_1$.

## Critical equations

- $g$ is critical for $\mathcal{R}$ iff $\mathrm{Ric}(g) = 0$ (Vacuum Einstein)
- $g$ is critical for $\overline{\mathcal{R}}$ iff $\mathrm{Ric}(g) = \frac{R(g)}{n}\, g$ (Einstein implies CSC), equivalently $g \in \mathcal{M}_1$ critical for $\mathcal{R}|_{\mathcal{M}_1}$ iff $g \in \mathcal{M}_1$ is Einstein
- $g$ critical for $\overline{\mathcal{R}}|_{[g]}$ iff $R(g)$ is constant, equivalently, $g \in \mathcal{M}_1$ critical for $\mathcal{R}|_{\mathcal{M}_1 \cap [g]}$ iff $R(g)$ is constant.

# Yamabe problem

With the latter condition in mind, let $\tilde{g} = u^{\frac{4}{n-2}} g \in [g]$, $u > 0$. Then $dv_{\tilde{g}} = u^{\frac{2n}{n-2}}\, dv_g$, so that using the formula $R(\tilde{g}) = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \left[\Delta_g u - \frac{n-2}{4(n-1)} R(g)\, u\right]$, with $\frac{2n}{n-2} - \frac{n+2}{n-2} = 1$, and integration by parts (easy check!), we get

$$\mathcal{R}(\tilde{g}) = \frac{4(n-1)}{n-2} \int_M \left[|\nabla u|^2 + \frac{n-2}{4(n-1)} u^2 R(g)\right]\, dv_g.$$

$$\overline{\mathcal{R}}(\tilde{g}) = \frac{\mathcal{R}(g)}{\|u\|^2_{L^{\frac{2n}{n-2}}(dv_g)}}.$$

# Yamabe problem

Goal: Look for CSC in $[g]$.

Let $Y(M,[g]) = \inf_{\tilde{g}\in[g]} \overline{\mathcal{R}}(\tilde{g}) = \inf_{u>0} \frac{\mathcal{R}(u^{\frac{4}{n-2}}g)}{\|u\|^2_{L^{\frac{2n}{n-2}}(dv_g)}}$.

Goal: Look for Einstein metrics: let $\sigma(M) = \sup_{\mathcal{C}} Y(M,\mathcal{C})$.

Fact: $\sigma(M) \le 0$ and $\overline{\mathcal{R}}(g) = \sigma(M)$, then $g$ is Einstein.

## Definition

A Yamabe metric is one which realizes the infimum, and thus is a CSC metric in $[g]$.

## Theorem (Yamabe, Trudinger, Aubin, Schoen)

For any $C \in \mathcal{C}$, there is $g \in \mathcal{C}$ with $R(g) = \frac{Y(M,\mathcal{C})}{(\mathrm{Vol}(g))^{\frac{2}{n}}}$.

# Yamabe problem

Before we move on, let's collect a few facts.

## Facts

- If $C \in \mathcal{C}$, and $g \in C$, then $Y(M,\mathcal{C})(\mathrm{Vol}(g))^{\frac{n-2}{n}} \leq \int_M R(g)\, dv_g$.
- For $r < \sigma(M)$, there is $C \in \mathcal{C}$, $Y(M,C) = r$, and thus there is $g \in \mathcal{M}_1$ with $R(g) = r < \sigma(M)$.
- $\sigma(M) > 0$ if and only if $M$ admits PSC.

# Modified Kazdan-Warner problem

Study image $R : \mathcal{M}_1 \to C^\infty(M)$. Again, $M^n$, $n \geq 3$, closed and connected.

**Theorem (O. Kobayashi)**

In case $\sigma(M) \leq 0$. Then
(i) If $Y(M, C) = \sigma(M)$ for some $C \in \mathcal{C}$, then $f \in R(\mathcal{M}_1)$ iff either $\min f < \sigma(M)$, or $f \equiv \sigma(M)$.
(ii) If for all $C \in \mathcal{C}$, $Y(M, C) < \sigma(M)$, then $f \in R(\mathcal{M}_1)$ iff $\min f < \sigma(M)$.

In case $\sigma(M) > 0$: $f \in R(\mathcal{M}_1)$ if $f$ is non-constant, or $f \equiv c < \sigma(M)$.

Remaining cases intriguing: In case $\sigma(M) > 0$, and $c \in [\sigma(M), +\infty)$. Is $c \in R(\mathcal{M}_1)$?
Yes! S. Matsuo ( 2013, Math. Ann., to appear).
Key idea: Any Yamabe metric $g$ isn't just CSC/critical, but also minimizing in $C$, so $Y(M, C) \leq \sigma(M)$. So a Yamabe metric in $\mathcal{M}_1$ has $R(g) \leq \sigma(M)$.
S. Matsuo: produce CSC non-Yamabe metrics in $\mathcal{M}_1$ by CSC surgery with volume control (C-Eichmair-Miao), and re-scaling.

# CSC gluing with volume constraint

**Theorem (C., Eichmair, Miao)**

If $(M_1, g_1)$ and $(M_2, g_2)$ are two closed $n$-manifolds ($n \geq 3$) of CSC $R(g_i) = c$ which are suitably non-degenerate, there is a CSC $c$ metric on $M_1 \# M_2$ with total volume $V(g_1) + V(g_2)$, and which equals the original metrics outside the gluing region.

You can also add a handle this way, preserving total volume.

Remarks: Conformal methods for CSC gluing (Mazzeo-Pollack-Uhlenbeck; Isenberg-Mazzeo-Pollack; Joyce; etc.) invoke global deformations.

Non-conformal techniques can be used in conjunction with well-established conformal methods to achieve controlled localized perturbations.

Goal: understand non-degeneracy condition and its role in scalar curvature deformation. We will explore this in Lecture 2

# Fischer-Marsden Analysis

Fischer and Marsden used Implicit Function Theorem arguments to study scalar curvature deformation, allowing the metric to move across conformal classes.

Consider the scalar curvature map $g \mapsto R(g)$, which has linearization $L_g$ , and formal adjoint $L_g^*$ given by

**Linearization of scalar curvature**

- $L_g(h) := \frac{d}{dt}\Big|_{t=0} R(g+th) = -\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g(\mathrm{div}_g(h)) - \langle h, \mathrm{Ric}(g)\rangle.$
- $L_g^*(f) = -(\Delta_g f)g + \mathrm{Hess}_g(f) - f\mathrm{Ric}(g).$

Convention: $\Delta_g f = \mathrm{tr}_g(\nabla^2 f) = \mathrm{tr}_g(\mathrm{Hess}_g f).$

Note that the principle part of $L_g L_g^*$ is $(n-1)\Delta_g^2$.

# Fischer-Marsden Analysis

$L_g^*(f) = -(\Delta_g f)g + \mathrm{Hess}_g(f) - f\mathrm{Ric}(g)$.

As $L_g^*$ is overdetermined-elliptic, it admits a Hodge decomposition/Fredholm alternative:

## Hodge-type Decomposition

On closed manifolds (or non-compact manifolds with asymptotic conditions, say) the appropriate function spaces (Sobolev, Hölder, possibly weighted) split as $\mathrm{Im}(L_g) \oplus \ker(L_g^*)$.

Thus by the IFT, we have

## Theorem (FM)

Suppose $(M^n, g)$ closed, and $\ker(L_g^*) = \{0\}$. Then there are $\epsilon > 0$ and $C > 0$ so that for all $S \in C^\infty(M)$ with $\|S\|_k < \epsilon$, there is a smooth $h$ with $\|h\|_{k+2} \leq C\|S\|_k$ so that $g + h$ is a Riemannian metric with $R(g + h) = R(g) + S$.

# Static metrics

From the preceding theorem, we can now focus on the linear obstruction to local deformation.

## Definition

A metric $g$ is called *static* (vacuum) if it admits a non-trivial solution to the system $L_g^*(f) = -(\Delta_g f)g + \mathrm{Hess}_g(f) - f\mathrm{Ric}(g) = 0$.

A solution $f$ is called a *static potential*.

## Remarks

- Static metrics are special. For example:
- A static metric has CSC, and in fact the associated $(n+1)$-space-time warped product metric $\bar{g} = -f^2\, dt^2 + g$ is Einstein: $\mathrm{Ric}(\bar{g}) = \frac{R(g)}{n-1}\, \bar{g}$.

# Static metrics

## Examples

- Euclidean space with static potentials spanned by the functions 1, $x^1, \ldots, x^n$.
- The flat torus $\mathbb{T}^n$ with constant static potential.
- The round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, with static potentials given by the span of the restriction of the coordinate functions $x^j\Big|_{\mathbb{S}^n}$, $j = 1, \ldots, n+1$.
- $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ is static with static potential $f(t, \omega) = \sin(t)$ for the product metric, where the $S^1$ factor has metric $\frac{1}{n-2} dt^2$ with $t \in [0, 2\pi]$.

# Static metrics

## Example: Schwarzschild

Defined on $\mathbb{R}^3 \setminus \{0\}$:
$g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_{\mathrm{eucl}}$

This metric has zero scalar curvature and is static, with static potential $f(x) = \frac{1-\frac{m}{2|x|}}{1+\frac{m}{2|x|}}$.

The warped product $\bar{g}_S = -f^2 dt^2 + g_S$ gives the Schwarzschild space-time satisfying $\mathrm{Ric}(\bar{g}_S) = 0$.

The zero set $\{x : f(x) = 0\} = \{x : |x| = \frac{m}{2}\}$ is a minimal (in fact totally geodesic) sphere.

# A basic connection to the PMT

## Proposition (Simple case of PMT rigidity)

If $g$ is a metric on $\mathbb{R}^3$ which has $R(g) \geq 0$ and for which $g = g_{\mathrm{eucl}}$ outside a ball $B$, then $g$ is flat.

*Proof*: By forming the torus quotient outside the ball $B$, $g$ descends to a metric $\hat{g}$ on $\mathbb{T}^3$ with $R(\hat{g}) \geq 0$. By FM, $\hat{g}$ must be static, else one may perturb the scalar curvature up slightly everywhere, which is not allowed by SY, GL.

So in particular, then, $R(\hat{g}) \equiv 0$, and there must be a solution to $0 = L^*_{\hat{g}} f = -\Delta_{\hat{g}} f \hat{g} + \mathrm{Hess}_{\hat{g}}(f) - f \mathrm{Ric}(\hat{g})$. Taking the trace and using the scalar curvature condition we get $\Delta_{\hat{g}} f = 0$, so that $f$ is a non-zero constant. From $L^*_{\hat{g}} f = 0$, we have $\mathrm{Ric}(\hat{g}) = 0$.

# Variation of scalar curvature and volume

There are a number of remarks we want to make about static metrics and potentials. Since there are similar results to be made about the linear obstruction for specifying deformations of scalar curvature $R(g)$ and volume $V(g) = \mathrm{Vol}(g)$ jointly, we will economize and just make the comments for the joint problem.

## Variation formulae

- The linearization of the volume $g \mapsto V(g)$ is $\frac{d}{dt}\Big|_{t=0} V(g + th)) = \int_M \frac{1}{2}\mathrm{tr}_g(h)\, dv_g$.
- If we let $\Theta(g) = (R(g), V(g))$, and $\mathcal{S}_g = D\Theta_g$, then $\mathcal{S}_g^*(f, a) = L_g^*(f) + \frac{a}{2}\, g$. ($a$ is a constant.)

# $V$-static metrics

Recall $\Theta(g) = (R(g), V(g))$, $D\Theta_g =: \mathcal{S}_g$.

## $V$-static metrics

- A metric $g$ is called *$V$-static* if it admits a non-trivial solution to the system $\mathcal{S}_g^*(f, a) = -(\Delta_g f)g + \text{Hess}_g(f) - f\text{Ric}(g) + \frac{a}{2}\, g = 0$.
- A $V$-static metric has CSC: take the divergence of $\mathcal{S}_g^*(f, a) = 0$ and use the Bianchi identity (and that 0 is a regular value for a nontrivial potential, as we'll see).
- Static implies $V$-static: If $f$ were a static potential, then $\mathcal{S}_g^*(f, 0) = 0$.

## Examples

- If $g$ is Einstein of CSC $R(g) = c$, then $(1, \frac{2c}{n})$ is in the kernel of $\mathcal{S}_g^*$.
- The standard sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has the maximal-dimensional kernel. Indeed, the coordinate functions $x^j\Big|_{\mathbb{S}^n}$, $j = 1, \ldots, n+1$, restrict to static potentials, and $(1, 2(n-1))$ is in the kernel of $\mathcal{S}_g^*$.

# Scalar curvature and volume

## Volume functional on CSC metrics

For a closed manifold $M$, we let $\mathcal{M}^c$ be the space of metrics with constant scalar curvature (CSC) $R(g) = c$, and let volume function restricted to $\mathcal{M}^c$ be $V_c : \mathcal{M}^c \to (0, +\infty)$.

For a compact manifold-with-boundary $\overline{\Omega}$, if $\gamma$ is a metric on $\partial\Omega$, then we let $\mathcal{M}^c_\gamma$ denote the space of metrics $g$ on $\Omega$ with CSC $R(g) = c$, whose restriction agrees with $\gamma$: $g\Big|_{T(\partial\Omega)} = \gamma$.

For $c \neq 0$, critical points of $V_c$ correspond to critical points of the *Einstein-Hilbert action* $\mathcal{M}^c \ni g \mapsto \int_M R(g)\, dv_g$.

## Example

- $g$ is critical for $V_{-1}$ if and only if $g$ is Einstein of CSC negative.

# Volume criticality

The $V$-static condition is does not capture volume criticality on the nose.

## Example

- $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ is static with static potential $f(t,\omega) = \sin(t)$ for the product metric, where the $S^1$ factor has metric $\frac{1}{n-2}dt^2$ with $t \in [0, 2\pi]$.
- However, this is clearly not volume-critical, by scaling the $\mathbb{S}^1$-factor, which doesn't change the CSC $(n-1)(n-2)$.

We re-write the condition $\mathcal{S}_g^*(f, a) = 0$ as $L_g^*(f) = \kappa\, g$, where $\kappa = -\frac{a}{2}$.

## Proposition

Metrics in $\mathcal{M}^c$ or $\mathcal{M}_\gamma^c$ which are volume-critical are those which admit solutions to $L_g^*(f) = \kappa g$, with $\kappa \neq 0$, assuming the first eigenvalue of $(n-1)\Delta_g + c$ is positive. Dirichlet conditions are used in the boundary case (Miao-Tam).

# $V$-static metrics

The equation $\mathcal{S}_g^*(f,a)=0$ reduces to an ODE along geodesics. If $\gamma$ is a unit-speed geodesic and $h(t)=f(\gamma(t))$, then $h''(t)=\mathrm{Hess}_g f(\gamma'(t),\gamma'(t))$, and so

$$h''(t)=\left(\mathrm{Ric}(g)(\gamma'(t),\gamma'(t))-\frac{R(g)}{n-1}\right)h(t)+\frac{a/2}{n-1}.$$

Using this one can show that on a smooth connected domain, the kernel is at most $(n+2)$-dimensional, and the kernel elements extend smoothly to the boundary, and that zero is a regular value of a non-trivial element in the kernel.
We remark that *no boundary condition* has been imposed here.

Letting $a=0$, we have a similar statement for static metrics, where the associated maximal dimension is $n+1$.

Remark: One can interpret boundary values of static or $V$-static potentials, which by the finite-dimensionality of the space of potentials, must be special.

# Variational formulation for $V$-static potential

The next result generalizes work of P. Miao and L.-F. Tam, who considered the static case $\kappa = 0$.

## $V$-static action

For $\kappa$ a constant, and $\phi$ a function, we define

$$E_{\kappa,\phi} = \kappa V(g) - \int_{\partial\Omega} H \, \phi \, d\sigma.$$

## Proposition (C., Eichmair, Miao)

Let $\kappa$ be a constant, and let $\phi$ be a function so that $(\phi, \kappa)$ is not identically $(0, 0)$. Suppose $g$ has CSC $R(g) = c$, and suppose the first eigenvalue of $(n-1)\Delta_g + c$ is positive. Then $g$ is critical for $E_{\kappa,\phi}$ on $\mathcal{M}^c_\gamma$ if and only if there is a smooth function $f$ so that

$$L_g^* f = \kappa g \qquad \text{on } \Omega, \qquad f = \phi \qquad \text{on } \partial\Omega$$

# A volume comparison result

As a last result before we move on to the localized deformation theorem, we note a volume comparison result first observed by Miao-Tam; C-Eichmair-Miao removed the spin assumption in higher dimensions, by avoiding use of the Positive Mass Theorem.

**Theorem**

Suppose $g$ is a smooth metric of zero scalar curvature on $\overline{\Omega}$, so that there is a function $f$ with $L_g^* f = g$ on $\Omega$, with $f = 0$ on $\partial\Omega$. Let $\gamma$ be the metric induced on $\Sigma = \partial\Omega$, which we assume is connected. We furthermore assume $(\Sigma, \gamma)$ can be isometrically embedded to a compact strictly convex hypersurface $\Sigma_0$ in $\mathbb{R}^n$. Then $V(\Omega, g) \geq V(\mathrm{int}(\Sigma_0), g_{\mathrm{eucl}})$, where equality holds if and only if $(\overline{\Omega}, g)$ is isometric to a standard ball in $\mathbb{R}^n$.

# Localized Deformation Theorem

Now that we've analyzed the linear obstruction to local deformations, and have noted that metrics which for which deformations are possibly obstructed are quite special. In the case when there is no obstruction, we have the following theorem.

**Theorem (C., Eichmair, Miao)**

Let $\Omega \subset (M, g)$ be a smooth bounded domain. Suppose the kernel of $\mathcal{S}_g^*$ is trivial on $\Omega$. For any $\Omega_0 \subset\subset \Omega$, there is an $\epsilon_0 > 0$ and a $C > 0$ so that for any $\sigma \in C_c^\infty(\Omega_0)$ and any $\tau \in \mathbb{R}$ with $|\tau| + \|\sigma\|_{C^{0,\alpha}} < \epsilon_0$, there is a smooth metric $g + h$ so that $h$ is supported in $\overline{\Omega}$, and $(R(g+h), V(g+h)) = (R(g) + \sigma, V(g) + \tau)$.

# Localized deformation theorem

**Theorem (Localized deformation of scalar curvature), C.**

Suppose $\Omega \subset (M, g)$ is a smooth bounded domain. Suppose that the kernel of $L_g^*$ on $\Omega$ is trivial (so $(\Omega, g)$ is not static). For $\Omega_0 \Subset \Omega$, there is an $\epsilon_0 > 0$ so that for any $S \in C^\infty(\Omega)$ with $\mathrm{spt}(S) \subset \Omega_0$ and $\|S\|_{C^{0,\alpha}} < \epsilon_0$, there is a smooth metric $g + h$ with $\mathrm{spt}(h) \subset \overline{\Omega}$ with
$R(g + h) = R(g) + S$ and $\|h\|_{C^{2,\alpha}} \leq C\|S\|_{C^{0,\alpha}}$.

- This is a localized analogue of the Fischer-Marsden theorem.
- This theorem doesn't help resolve the Min-Oo conjecture on the hemisphere, since the round sphere is static. Min-Oo conjecture: if $g$ is a Riemannian metric on $\mathbb{S}^n_+$ with $R(g) \geq n(n-1)$, so that $(\partial\mathbb{S}^n_+, g)$ is isometric to the standard unit sphere $\mathbb{S}^{n-1}$, and so that $\partial\mathbb{S}^n_+$ is totally geodesic, then $(\mathbb{S}^n_+, g)$ is isometric to the standard unit hemisphere. Brendle, Marques, Neves had to work harder to provide counterexamples.

# Localized deformations: scalar curvature melting

Lohkamp proved an interesting result in the late nineties.

**Lohkamp Theorem**

Let $(M, g)$ be Riemannian, and let $U \subset M$ be open. For any $\epsilon > 0$, and for any smooth function $f$ with $f < R(g)$ on $U$, and $f = R(g)$ on $M \setminus U$, there is a smooth metric $g_\epsilon$ with $g = g_\epsilon$ on $M \setminus U_\epsilon$ and $f - \epsilon \leq R(g_\epsilon) \leq f$ on $U_\epsilon$ (the $\epsilon$-neighborhood of $U$). Moreover, one may choose $g_\epsilon$ arbitrarily close to $g$ in $C^0$.

In general, one cannot keep strong control on the metrics $g_\epsilon$. One might consider whether one can estimate $g - g_\epsilon$ in terms of $f - R(g)$.
In terms of deforming scalar curvature down through a continuous path, one might try to specify a continuously varying family of smooth metrics $g_t$, $g_0 = g$, $t \geq 0$, with $R(g_t)$ strictly decreasing in $t$ inside some ball $B$, say, with $g_t = g$ outside $B$.
Of course, in case the metric $g$ is non-static this follows from work of C.

# Obstruction to locally bumping up scalar curvature

With Lohkamp's result holding for any metric, one might ask whether we can still so this *melting* of scalar curvature at static metrics. For Euclidean spaces, such metrics have been produced by J. Kim et. al.

Rigidity for the Positive Mass Theorem precludes locally bumping up the scalar curvature on Euclidean space and on hyperbolic space. It doesn't just preclude such deformations near the constant curvature metric, it precludes existence of other metrics which agree with those outside a compact set and have at least as much scalar curvature. A similar statement holds for regions of the Schwarzschild metric away from the minimal sphere (Penrose Inequality).

## Obstruction to bumping up

Forthcoming work of J. Qing and W. Yuan establishes some rigidity and non-rigidity results at static metrics, and, if I understand correctly, shows that at any static metric, and any point $p$, then for small enough balls around $p$, one cannot have a different metric *near* $g$ of scalar curvature at least as much as $R(g)$, which agrees with $g$ outside the ball.

# Proof of Local Deformation: Basic estimates

## Basic estimate

The key to obtaining $h$ compactly supported is the form of the operator $\mathcal{S}_g^*$: $\mathcal{S}_g^*(u,a) = -(\Delta_g u)g + \mathrm{Hess}_g u - u\,\mathrm{Ric}(g) + \frac{a}{2}\,g$. One can then get an immediate *a priori* estimate of the form

$$\|(u,a)\|_{H^2(U)\times\mathbb{R}} \leq C(n,g,\Omega)\left(\|\mathcal{S}_g^*(u,a)\|_{L^2(U)} + \|(u,a)\|_{H^1(U)\times\mathbb{R}}\right)$$

valid on any $U \subset \Omega$.

No boundary condition has been imposed.

*Proof*: $tr_g(\mathcal{S}_g^*(u,a)) = -(n-1)\Delta_g u - uR(g) + \frac{na}{2}$. Thus the full Hessian of $u$ can be expressed in terms of $\mathcal{S}_g^*(u,a)$ and lower order terms in $(u,a)$.

# Proof of Local Deformation: Basic estimates

## Coercivity estimate

In case $\Omega$ is not $V$-static, then one can use compactness arguments to prove the following estimate for small $\epsilon > 0$, where $\Omega_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\}$:

$$\|(u,a)\|_{H^2(\Omega_\epsilon)\times\mathbb{R}} \leq C\|\mathcal{S}_g^*(u,a)\|_{L^2(\Omega_\epsilon)}.$$

## Weighted estimates

Let $\rho$ be a smooth, positive function on $\Omega$, which near the boundary is $e^{-1/d}$, $d(x) = \mathrm{dist}(x, \partial\Omega)$.

The preceding estimate can be integrated to obtain the weighted $L^2$ estimates:

$$\|(u,a)\|_{H^2_\rho(\Omega)\times\mathbb{R}} \leq C\|\mathcal{S}_g^*(u,a)\|_{L^2_\rho(\Omega)}.$$

# Proof of Local Deformation: Linearized problem

Under the assumption that $\mathcal{S}_g^*$ has trivial kernel, solutions to $\mathcal{S}_g(h) = (\sigma, \tau)$ for $(\sigma, \tau) \in L^2_{\rho^{-1}}(\Omega) \times \mathbb{R}$ can be obtained from standard variational arguments. The solution $h$ will be of the form $h = \rho \mathcal{S}_g^*(u, a)$. The decaying weight $\rho$ will ensure $h$ extends smoothly by zero across $\partial\Omega$.

## Proposition

Let $\Omega \subset M$ be a smooth, bounded domain and assume that $\mathcal{S}_g^*$ has trivial kernel in $H^2_{\mathrm{loc}}(\Omega) \times R$. Let $(\sigma, \tau) \in L^2_{\rho^{-1}}(\Omega) \times \mathbb{R}$. There is a unique minimizer $(u, a) \in H^2_\rho(\Omega) \times R$ of the functional $\mathcal{F}$, defined on $H^2_\rho(\Omega) \times \mathbb{R}$ given by

$$\mathcal{F}(u, a) = \int_\Omega \left( \frac{1}{2} \mid \mathcal{S}_g^*(u, a) \mid^2 \rho - \sigma u \right) d\mu_g - a\tau.$$

The minimizer is a weak solution of the equation $\mathcal{S}_g(\rho \mathcal{S}_g^*(u, a)) = (\sigma, \tau)$ and satisfies $\|(u, a)\|_{H^2_\rho(\Omega) \times \mathbb{R}} \leq C \|(\sigma, \tau)\|_{L^2_{\rho^{-1}}(\Omega) \times \mathbb{R}}$.

# Proof of the Local Deformation: Linearized problem

*Proof*: We use the injectivity estimate $\|(u,a)\|_{H^2_\rho(\Omega)\times\mathbb{R}} \leq C\|\mathcal{S}^*_g(u,a)\|_{L^2_\rho(\Omega)}$ to infer that the infimum $\mu \leq \mathcal{F}((0,0)) = 0$ of the functional

$$\mathcal{F}(u,a) = \int_\Omega \left(\frac{1}{2} \mid \mathcal{S}^*_g(u,a) \mid^2 \rho - \sigma u\right) d\mu_g - a\tau$$
$$\geq C'\|(u,a)\|^2_{H^2_\rho(\Omega)\times\mathbb{R}} - \|(u,a)\|_{H^2_\rho(\Omega)\times\mathbb{R}}\|(\sigma,\tau)\|_{L^2_{\rho^{-1}}(\Omega)\times\mathbb{R}}$$

is finite. Standard functional analysis then gives a minimizer $(u,a)$, which is the unique global minimizer, since for $(\hat{u},\hat{a}) \neq (u,a)$, $t \mapsto \mathcal{F}((1-t)(u,a) + t(\hat{u},\hat{a}))$ is strictly convex. For the minimizer $(u,a)$, from $\mu \leq 0$, we get an estimate

$$\frac{1}{2C^2}\|(u,a)\|^2_{H^2_\rho(\Omega)\times\mathbb{R}} \leq \int_\Omega \frac{1}{2} \mid \mathcal{S}^*_g(u,a) \mid^2 \rho\, d\mu_g = \mu + \int_\Omega \sigma u \, d\mu_g + a\tau$$
$$\leq \|(\sigma,\tau)\|_{L^2_{\rho^{-1}}(\Omega)\times\mathbb{R}} \cdot \|(u,a)\|_{H^2_\rho(\Omega)\times\mathbb{R}}$$

This gives us $\|(u,a)\|_{H^2_\rho(\Omega)\times\mathbb{R}} \lesssim \|(\sigma,\tau)\|_{L^2_{\rho^{-1}}(\Omega)\times\mathbb{R}}$.

# Comments on pointwise estimates

The proof of the Deformation Theorem proceeds through iterated linear corrections. We need to keep pointwise control during the iteration.

## Weighted Schauder Estimates

The operator $\rho^{-1}L_g\rho L_g^*$ to which we apply these estimates can be expressed in local coordinates as $P = (n-1)\Delta^2 + \sum_{|\beta|\leq 3} b_\beta D^\beta$, and analogously for the quasilinear operator $\rho^{-1}(R(g+\rho\mathcal{S}_g^*(u,a)) - R(g))$. If we had chosen $\rho$ to decay like a power of the distance $d$ to the boundary, then the lower order coefficients would go like $|b_\beta| \sim d^{-(4-|\beta|)}$, and a scale-invariant form of the interior Schauder estimates is standard. In case the weight $\rho$ decays exponentially, the lower order coefficients can behave like $|b_\beta| \sim d^{-2(4-|\beta|)}$ near $\partial\Omega$. The best way to handle such behavior is to choose the right scaling, in order to write interior Schauder estimates on $\Omega$ in terms of interior Schauder estimates on fixed balls.

# CSC gluing with volume constraint

We start with a gluing result for manifolds with boundaries. There are two non-degeneracy conditions—one for the conformal method, and one for the localized deformation technique discussed in Lecture 2.

**Theorem (C., Eichmair, Miao)**

Let $(M_1, g_1)$ and $(M_2, g_2)$ be compact manifolds with nonempty boundaries, and let $p_i \in \mathrm{int}(M_i)$ be marked points, and let $U_i$ be neighborhoods of $p_i$ which are not $V$-static. Assume that each metric has CSC $R(g_i) = \sigma_n$, where $\sigma_n \in \{-n(n-1), 0, n(n-1)\}$. In case $\sigma_n > 0$, we assume that the first Dirichlet eigenvalue of $((n-1)\Delta_{g_i} + \sigma_n)$ is positive.

Then there are neighborhoods $V_i \subset\subset U_i$ of $p_i$, and there is a metric $\bar{g}$ on $M_1 \# M_2$ which agrees with $g_i$ on each of $M_i \setminus V_i$, so that $R(\bar{g}) = \sigma_n$ and $V(\bar{g}) = V(g_1) + V(g_2)$.

# CSC gluing with volume constraint

Sketch of proof: Let $g$ be $g_1$ or $g_2$. $R > 0$ will be the radius of a sufficiently small ball around the chosen points. We omit subscripts. Let $\mathring{g}$ be the round unit metric on $\mathbb{S}^{n-1}$.

- Conformally blow up neighborhood of each chosen point in quasi-normal coordinates where $g_{ij}(x) = \delta_{ij} + Q_{ij}(x)$, $Q_{ij}(0) = 0 = \partial Q_{ij}(0)$, to produce an asymptotically cylindrical end:

$$r^{-2}g = r^{-2}dr^2 + \mathring{g} + r^{-2}Q,$$

  where $(r, \theta)$ are spherical coordinates based on the $x$ coordinates.
- For $T \gg 1$, let $s = -\log r + \log R - \frac{T}{2}$. This rescales $r = Re^{-s-T/2}$, with $r^{-2}dr^2 = ds^2$. Note that $r = R$ corresponds to $s = -\frac{T}{2}$. Also, let $\Psi$ be defined so that $\Psi^{\frac{4}{n-2}}$ is $r^2$ near $p$, interpolates to 1 between $B_R(p)$ and $B_{2R}(p)$.
- Then $r^{-2}g = \Psi^{-\frac{4}{n-2}}g = ds^2 + \mathring{g} + e^{-T}e^{-2s}R^2\hat{h}$ inside $B_R(p)$, i.e. for $s > -\frac{T}{2}$. If $\tilde{g}_c = ds^2 + \mathring{g}$ is the cylindrical metric, then $\|\widetilde{\nabla}^\ell \hat{h}\|$ has $T$-independent bound.

# CSC gluing with volume constraint

Sketch of proof, continued

- Now transition smoothly from $\Psi^{-\frac{4}{n-2}}g = ds^2 + \mathring{g} + e^{-T}e^{-2s}R^2\hat{h}$ in $(-\frac{T}{2}, -1)$ to the exact cylindrical metric $\tilde{g}_c = ds^2 + \mathring{g}$ in $(-\frac{1}{2}, -1)$.
- Do the same for both ends. One one end, take $s$ to $-s$, glue together in the exactly cylindrical part. Get a metric $\gamma_T$ on $C_T = [-\frac{T}{2}, \frac{T}{2}] \times \mathbb{S}^{n-1}$ of the form $\gamma_T = ds^2 + \mathring{g} + e^{-T}\cosh(2s)\hat{h}_T$, where $\hat{h}_T = 0$ on $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{S}^{n-1}$. $\|\widetilde{\nabla}^\ell \hat{h}_T\|$ bounded independent of $T$.
- Build an approximate solution: Let $\psi(t)$ interpolate from $t^{\frac{n-2}{2}}$ for $0 < t < R$, to 1 for $t > 2R$. ($\psi$ is essentially $\Psi$.) Let $\chi_{1,T}$ cut off from 1 to 0 in the interval $[\frac{T}{2} - 1, \frac{T}{2}]$, while $\chi_{2,T}$ cuts off from 1 to 0 going from $-\frac{T}{2} + 1$ down to $-\frac{T}{2}$.
- Let $\Psi_T(s,\theta) = \chi_{1,T}(s)\psi(e^{-s-\frac{T}{2}}R) + \chi_{2,T}(s)\psi(e^{s-\frac{T}{2}}R)$.

# CSC gluing with volume constraint

Sketch of proof, continued: Recall that $\gamma_T$ on $[-\frac{T}{2}, \frac{T}{2}]$ transitions from $\Psi^{-\frac{4}{n-2}} g = r^{-2} g$ to the cylindrical metric from one manifold, then back the other way.

- Approximate solution: $\Psi_T^{\frac{4}{n-2}} \gamma_T$.
- $R(\Psi_T^{\frac{4}{n-2}} \gamma_T) = \frac{4(n-1)}{n-2} \Psi_T^{-\frac{n+2}{n-2}} \left[ -\Delta_{\gamma_T} \Psi_T + \frac{n-2}{4(n-1)} R(\gamma_T) \Psi_T \right]$.
- On $[-\frac{T}{2}+1, \frac{T}{2}-1] \times \mathbb{S}^{n-1}$, $\Psi_T(s,\theta) = 2R^{\frac{n-2}{2}} e^{-\frac{n-2}{4}T} \cosh\left(\frac{n-2}{2}\, s\right)$. On $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{S}^{n-1}$, the metric $\gamma_T = ds^2 + \mathring{g}$, so that $R(\gamma_T) = (n-1)(n-2)$, while $\Delta_{\gamma_T} \Psi_T = \left(\frac{n-2}{2}\right)^2 \Psi_T$. Thus $R(\Psi_T^{\frac{4}{n-2}} \gamma_T) = 0$ on $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{S}^{n-1}$.
- The approximate solution is thus the two given manifolds joined by Schwarzschild neck of mass $m_T = 2R^2 e^{-\frac{(n-2)T}{2}}$. The volume is approximately $V(g_1) + V(g_2)$.

# CSC gluing with volume constraint

## Sketch of proof, continued

- Goal: Solve for conformal factor to return to CSC: solve $\mathcal{N}_T(\Psi_T + \eta_T) = 0$ for small $\eta_T$, where

$$\mathcal{N}_T(f) = -\Delta_{\gamma_T} f + \frac{n-2}{4(n-1)} R(\gamma_T) f - \frac{n-2}{4(n-1)} \sigma_n f^{\frac{n+2}{n-2}}.$$

- $\mathcal{N}_T(\Psi_T)$ is exponentially small in $T$. A contraction mapping argument will give the solution, once the linearized operator $\mathcal{L}_T = D\mathcal{N}_T\Big|_{\Psi_T}$ is analyzed.
-

$$\mathcal{L}_T(f) = -\Delta_{\gamma_T} f + \frac{n-2}{4(n-1)} R(\gamma_T) f - \frac{n-2}{4(n-1)} \sigma_n \frac{n+2}{n-2} \Psi_T^{\frac{4}{(n-2)}} f.$$

- The operators $\mathcal{L}_T$ converge locally on punctured $M_1$ or $M_2$ to $\mathcal{L}$, obtained by replacing $\gamma_T$ by $\tilde{g} = \Psi^{-\frac{4}{n-2}} g$ and $\Psi_T$ by $\Psi$.

# CSC gluing with volume constraint

## Sketch of proof, continued

- $\mathcal{L} = -\Delta_{\tilde{g}} + \frac{n-2}{4(n-1)} R(\tilde{g}) - \frac{n-2}{4(n-1)} \sigma_n \frac{n+2}{n-2} \Psi^{\frac{4}{n-2}}$.
- By conformal invariance, since $R(g) = \sigma_n$, we have

$$(-\Delta_{\tilde{g}} + \tfrac{n-2}{4(n-1)} R(\tilde{g}))\eta = \Psi^{\frac{n+2}{n-2}} (-\Delta_g + \tfrac{n-2}{4(n-1)} \sigma_n)(\Psi^{-1}\eta).$$

- $\mathcal{L}\eta = 0$ implies, with a little arithmetic, $\left(\Delta_g + \frac{\sigma_n}{n-1}\right)(\Psi^{-1}\eta) = 0$.
- The linearized operator has no kernel for large $T$, by contradiction. If there were elements $\eta_j$ of maximum value 1 in the kernel of $\mathcal{L}_{T_j}$ for $T_j \to +\infty$, then there are two cases to consider.
- Case (i): For infinitely many $j$, $\max |\eta_j|$ is uniformly bounded away from zero outside the neck, so that a subsequence will determine a nontrivial solution $\hat{\eta} = \Psi^{-1}\eta$ of $(\Delta_g + \frac{\sigma_n}{n-1})\hat{\eta} = 0$ (Dirichlet), which cannot exist (by assumption in case $\sigma_n > 0$).

# CSC gluing with volume constraint

## Sketch of proof, continued

- $\mathcal{L}_T = -\Delta_{\gamma_T} + \frac{n-2}{4(n-1)} R(\gamma_T) - \frac{n-2}{4(n-1)} \sigma_n \frac{n+2}{n-2} \Psi_T^{\frac{4}{(n-2)}}$ converges locally uniformly on the neck to $-\Delta_{\tilde{g}_c} + \frac{n-2}{4(n-1)} R(\tilde{g}_c) = -\Delta_{\tilde{g}_c} + \frac{(n-2)^2}{4}$.
- Case (ii): If $\eta_j$ instead go to zero locally uniformly on the punctured manifolds, then a subsequence converges to a bounded solution of $(\Delta_{\tilde{g}_c} - \frac{(n-2)^2}{4})\eta = 0$ on the cylinder, which cannot exist.
- A similar argument shows that the inverse $\mathcal{G}_T$ is bounded independent of $T$.
- We can estimate the solution, in order to estimate the resulting volume, close to $V(g_1) + V(g_2)$.
- We have a family of metrics parametrized by $T$, with CSC $\sigma_n$, with volume approaching $V(g_1) + V(g_2)$ and which outside the gluing region approach $g_1 \sqcup g_2$.
- We now want a localized deformation procedure to get the original metrics back outside the gluing region, and to correct the volume.

# CSC gluing with volume constraint

- Now, find annuli around each $p_i$ which are not $V$-static. If there were no such annulus, get a sequence $A_i$ of annuli, increasing to the punctured manifolds, each with non-trivial $V$-static potentials. The kernels of $\mathcal{L}^*_{g_i}$ decrease—but by finite dimensionality, then, they must stabilize. From this, we get a nontrivial solution with an isolated singularity. The ODE argument recalled in Lecture 2 shows that such a potential extends smoothly to $p_i$, which contradicts the assumption that the original manifolds are not $V$-static.
- For large $T$, the CSC metric obtained above is close to $g_i$ on each respective annulus, and the volume approaches $V(g_1) + V(g_2)$.
- Now glue back in the original metrics $g_i$ in the non-static annuli. Use the localized perturbation result to adjust scalar curvature and volume.

# CSC gluing with volume constraint

As a corollary we obtain the following.

**Corollary (C., Eichmair, Miao)**

If $(M_1, g_1)$ and $(M_2, g_2)$ are two $n$-manifolds of CSC $\sigma_n$ which are not $V$-static, there is a family of CSC $\sigma_n$ metrics $\gamma_T$ on $M_1 \# M_2$ which have total volume $V(g_1) + V(g_2)$, and which are equal to the original metrics outside the gluing region.

To see this: pick small neighborhoods for which $((n-1)\Delta_{g_i} + \sigma_n)$ has positive first Dirichlet eigenvalue.

You can also add a handle this way, preserving total volume.

# Application: Matsuo's resolution of Modified KW problem

- We now return to a problem introduced the first lecture, the modified Kazdan-Warner problem, to determine the range of the scalar curvature mapping on unit volume metrics, as studied by O. Kobayashi.
- We had discussed that the remaining cases were the cases $R(g) = c \geq \sigma(M) > 0$.
- S. Matsuo recently resolved these cases, by finding CPSC non-Yamabe metrics through gluing. We outline the proof here.
- We first note the following corollary of the gluing result.

# Application: Matsuo's resolution of Modified KW problem

**Corollary**

If $(M_1, g_1)$ and $(M_2, g_2)$ are two $n$-manifolds of CPSC, with unit volume, and which are not $V$-static, there is a CPSC metric $g$ on $M_1 \# M_2$ which has $(R(g))^{\frac{n}{2}} = (R(g_1))^{\frac{n}{2}} + (R(g_2))^{\frac{n}{2}}$ and total volume $V(g) = 1$, and which is equal to the original metrics outside the gluing region.

Proof: Let $\rho_j = R(g_j) > 0$, and $\hat{g}_j = \rho_j g_j$. Then $R(\hat{g}_j) = 1$, $V(g_j) = \rho_j^{\frac{n}{2}}$. By C-E-M, there is a a metric $\hat{g}$ with $R(\hat{g}) = 1$, $V(\hat{g}) = \rho_1^{\frac{n}{2}} + \rho_2^{\frac{n}{2}}$. Then $g = (V(\hat{g}))^{-\frac{2}{n}} \hat{g}$ satisfies the requirements.

# Application: Matsuo's resolution of Modified KW problem

Matsuo notes the following criterion for non-degeneracy in his paper. The goal then is to arrange so that the condition holds.

**Lemma**

Suppose $(M, g)$ has CSC. If $g$ is not Einstein and if $\frac{R(g)}{n-1} \notin \mathrm{spec}(\Delta_g)$, then $g$ is not $V$-static.

Proof: If $0 = \mathcal{S}_g^*(f, a) = -(\Delta_g f)g + \mathrm{Hess}_g f - f\,\mathrm{Ric}(g) + \frac{a}{2}g$, then tracing we get

$$\Delta_g f = -\frac{R(g)}{n-1}\left(f - \frac{na}{2}\right).$$

Thus $\left(f - \frac{na}{2}\right)$ must be zero, else it would be an eigenfunction. Thus $f$ must be constant, and by the equation $\mathcal{S}_g^*(f, a) = 0$, we see $g$ is Einstein, unless $a = 0$, in which case $f = 0$ too.

We now turn to the Main Existence Result, giving building blocks for the final construction.

**Proposition**

$M^n$ closed, $n \geq 3$, $\sigma(M) > 0$. There exists $\rho > 0$ so that for all $r \in (0, \rho]$, there is a $g \in \mathcal{M}_1$, $R(g) = r$, and $g$ is not $V$-static.

Proof: Since $\sigma(M) > 0$, there is a metric of PSC on $M$. There are always metrics of negative scalar curvature, and consider a path $\tilde{g}_t$ between one of each of these metrics. The first eigenvalue of the conformal Laplacian $\mathcal{L}_t = \Delta_{\tilde{g}_t} - \frac{n-2}{4(n-1)} R(\tilde{g}_t)$ is continuous in $t$, and goes from positive to negative. There is thus a $t_0$ and $u > 0$ so that $\mathcal{L}_{t_0} u = 0$. Thus $g_{t_0} := u^{\frac{4}{n-2}} \tilde{g}_{t_0}$ has zero scalar curvature.

Proof, continued: Apply Koiso's Local Yamabe Theorem (1979) to obtain $g_t \in [\tilde{g}_t] \cap \mathcal{M}_1$, each with CSC. We arrange parameter so that $R(g_0) = 0$ and $R(g_t) > 0$ for $t > 0$.

Since the first positive eigenvalue of the Laplacian is also continuous in $t$, we may assume for small $t > 0$, $\frac{R(g_t)}{n-1}$ is not in the spectrum of $\Delta_{g_t}$, one of the conditions required.

If $g_0$ were not Einstein, then $g_t$ is not Einstein for small $t$. From here we can conclude.

If $g_0$ is Einstein, we apply another result of Koiso (1983): there is an infinite-dimensional connected slice $\Sigma_{g_0}$ in the space of unit volume metrics around $g_0$, transverse to the action of $\mathrm{Diff}(M)$, and a finite-dimensional real analytic submanifold of $\Sigma_{g_0}$ that contains all the Einstein metrics in $\Sigma_{g_0}$. As there are non-Einstein PSC and NSC metrics in $\Sigma_{g_0}$, there is a path of non-Einstein metrics a PSC to a NSC metric. From here the above applies.

# Application: Matsuo's resolution of Modified KW problem

Now that the gluing construction can be applied, we arrive at the final step in the proof.

- Let $r_1 > 0$ be small enough that for $M$ and $\mathbb{S}^n$, there are non-Einstein metrics of CSC $\rho \in (0, r_1]$. Given $r > 0$, we write $r^{\frac{n}{2}} = r_0^{\frac{n}{2}} + (k^{\frac{2}{n}} r_1)^{\frac{n}{2}}$, for $r_0 \in (0, r_1)$ and $k \in \mathbb{Z}_+ \cup \{0\}$. Let $g_{r_0}$ be a metric on $M$ with $R(g_{r_0}) = r_0$ and $V(g_{r_0}) = 1$. Let $g_{r_1}$ be a metric on $\mathbb{S}^n$ with $R(g_{r_1}) = r_1$ and $V(g_{r_1}) = 1$.
- In case $k \in \mathbb{Z}_+$ (else done), let $(M_2, g_2)$ be the C-E-M connect sum of the disjoint union of $k$ spheres, each with metric $k^{-\frac{2}{n}} g_{r_1}$. Then $(R(g_2))^{\frac{n}{2}} = k r_1^{\frac{n}{2}}$ and $V(g_2) = 1$. Apply the Corollary of C-E-M to $M \# M_2 \cong M$ to get a metric $g$ with $(R(g))^{\frac{n}{2}} = (R(g_{r_0}))^{\frac{n}{2}} + (R(g_2))^{\frac{n}{2}} = r_0^{\frac{n}{2}} + k r_1^{\frac{n}{2}} = r^{\frac{n}{2}}$ and $V(g) = 1$.
- So $R(g) = r$, $V(g) = 1$. But $r > 0$ was arbitrary!

# Other applications

Other applications of the gluing include applications to constructing interesting solutions of the Einstein constraint equations, and constructing more complicated counterexamples to the Min-Oo conjecture. Given the time, let's focus on the Min-Oo examples.

Recall the Min-Oo conjecture: if $g$ is a Riemannian metric on $\mathbb{S}^n_+$, or more generally on $M$ with $R(g) \geq n(n-1)$, so that $(\partial M, g)$ is isometric to the standard unit sphere $\mathbb{S}^{n-1}$, and so that $\partial M$ is totally geodesic, then $(M, g)$ is isometric to the standard unit hemisphere. Brendle, Marques, Neves provide counterexamples on $\mathbb{S}^n_+$ that are identical to the standard sphere metric in a neighborhood of the equator, and for which $\{R(g) > n(n-1)\} \neq \emptyset$.

We remark that a number of rigidity results hold, such as on $\mathbb{S}^n_+$ in case the scalar curvature bound is replaced by Ricci curvature (Hang-Wang), and in case $g$ is near the round metric $\mathring{g}$ and $V(g) \geq V(\mathring{g})$ (Miao-Tam).

# Other applications

In light of the previous comment, the following corollary might be interesting.

**Corollary**

There exists counterexamples to the Min-Oo conjecture with arbitrarily large volume.

Proof: Any counterexample which is the round sphere metric near the equation cannot be $V$-static (e.g. analyticity). One can then choose a small geodesic ball $B(p, \rho) \subset \mathbb{S}^n_+$ on which the scalar curvature is $n(n-1)$, and for which the exterior of $B(p, \frac{\rho}{2})$ is not $V$-static, and for which the first Dirichlet eigenvalue of $\Delta_g + n$ is positive. One can glue two copies together, using the C-E-M CSC gluing resulting in a metric with scalar curvature at least $n(n-1)$, with double the total volume of the single counterexample. Since the metrics agree with the round metric near the equator, one end can be capped off smoothly. This construction can be repeated.

# Other applications

Remark: One could also apply Gromov-Lawson technique to Brendle-Marques-Neves examples, where the scalar curvature is strictly bigger than $n(n-1)$. Remark: K. Akutagawa notes that results of O. Kobayashi can also be used.

**Corollary**

There exists counterexamples to the Min-Oo conjecture with non-trivial topology.

Proof: Take a counterexample of Min-Oo as above, and add a CSC handle. Or, consider metrics on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ or $\mathbb{S}^n/\Gamma$, with scalar curvature at least $n(n-1)$, with constant scalar curvature $n(n-1)$ in an open subset. Such metrics exists by Kazdan-Warner, and can be chosen to be non $V$-static (e.g. non-CSC).

# Open problems

A couple open problems came up in discussions with Eichmair and Miao:

- Are there any CSC counterexamples to the Min-Oo conjecture?

- Are there static counterexamples to the Min-Oo conjecture?

THANK YOU: A big thank you to the organizers, for the invitation and gracious hospitality in Tokyo.