Topology of the space of D-minimal metrics

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The Dirac operator

Let M be a (fixed) compact manifold with spin structure, $n = \dim M$.
For any metric g on M one defines

- the *spinor bundle* $\Sigma_g M$: a vector bundle with a metric, a connection and Clifford multiplication $TM \otimes \Sigma_g M \to \Sigma_g M$. Sections $M \to \Sigma_g M$ are called *spinors*.
- the *Dirac operator* $D\!\!\!/_g : \Gamma(\Sigma_g M) \to \Gamma(\Sigma_g M)$: a self-adjoint elliptic differential operator of first order.

$\Longrightarrow$ $\ker D\!\!\!/_g$ is finite-dimensional.

The elements of $\ker D\!\!\!/_g$ are called *harmonic spinors*.

Atiyah-Singer Index Theorem for $n = 4k$

Let $n = 4k$. $\Sigma_g M = \Sigma_g^+ M \oplus \Sigma_g^- M$. $D\!\!\!/_g = \begin{pmatrix} 0 & D\!\!\!/_g^- \\ D\!\!\!/_g^+ & 0 \end{pmatrix}$

$$\mathrm{ind}\ D\!\!\!/_g^+ = \dim\ker D\!\!\!/_g^+ - \mathrm{codim}\ \mathrm{im}\ D\!\!\!/_g^+ = \dim\ker D\!\!\!/_g^+ - \dim\ker D\!\!\!/_g^-$$

Theorem (Atiyah-Singer 1968)

$$\mathrm{ind}\ D\!\!\!/_g^+ = \int_M \widehat{A}(TM) =: \alpha(M)$$

Hence: $\dim\ker D\!\!\!/_g \geq |\int \widehat{A}(TM)|$

Index Theorem for $n = 8k + 1$ and $8k + 2$

$n = 8k + 1$:

$$\alpha(M) := \dim\ker \not{D}_g \mod 2$$

$n = 8k + 2$:

$$\alpha(M) := \frac{\dim\ker \not{D}_g}{2} \mod 2$$

$\alpha(M) \in \mathbb{Z}/2\mathbb{Z}$ is independent of g.
However, $\alpha(M)$ depends on the choice of spin structure.

Consequence

$$\dim\ker \not{D}^g \geq |\alpha(M)| := \begin{cases} |\int \widehat{A}(TM)|, & \text{if } n = 4k; \\ 1, & \text{if } n \equiv 1 \mod 8 \\ & \text{and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \\ & \text{and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

$D\!\!\!/$-minimal metrics

Definition

A metric g on a **connected** spin manifold is called $D\!\!\!/$-minimal if the bound given by Atiyah-Singer is attained, i.e.

$$\dim \ker D\!\!\!/^g = |\alpha(M)|$$

Theorem A (Ammann, Dahl, Humbert 2009)

Generic metrics on connected compact spin manifolds are $D\!\!\!/$-minimal.

Conjecture

On every closed spin manifold of dimension ≥ 3 *non-*$D\!\!\!/$*-minimal metrics exist.*

This conjecture was stated in the case $\alpha(M) = 0$ by Bär-Dahl 2002.
One might even expect:

Conjecture (Large kernel conjecture)

Let $\dim M \geq 3$*. For any* $k \in \mathbb{N}$ *there is a metric* g_k *with* $\dim \ker D\!\!\!/^{g_k} \geq k$*.*

Content of the talk

$$\mathcal{M}_{=|\alpha(M)|}(M) := \left\{ g \text{ Riem. metric on } M \,\middle|\, \dim\ker D\!\!\!/^{g} = |\alpha(M)| \right\}$$

- Proof of Theorem A.
  Collaboration with M. Dahl (Stockholm) and E. Humbert (Tours), ≈2007–2011
- Non-trivial topology of $\mathcal{M}_{=|\alpha(M)|}(M)$. Thus there are non-$D\!\!\!/$-minimal metrics. Work in progress with U. Bunke (Regensburg), M. Pilca (Regensburg) and N. Nowaczyk (London), ≈2015–??.

If M carries a psc metric, then $\alpha(M) = 0$ and

$$\mathcal{M}_{\mathrm{psc}}(M) \subsetneq \mathcal{M}_{=0}(M).$$

$\rightsquigarrow$ Talk of Boris Botvinnik

$D\!\!\!/$-minimality theorem

Theorem A ($D\!\!\!/$-minimality theorem, ADH, 2009)

Generic metrics on connected compact spin manifolds are $D\!\!\!/$-minimal.

Generic = dense in C^∞-topology and open in C^1-topology.

To prove the $D\!\!\!/$-minimality theorem it is sufficient to show that there is *one* $D\!\!\!/$-minimal metric, i.e.

$$\mathcal{M}_{=|\alpha(M)|}(M) \neq \emptyset.$$

History of $\mathcal{M}_{=|\alpha(M)|}(M) \neq \emptyset$

- Hitchin (1974): Some explicit examples, e.g. S^3 and surfaces.
- Maier (1996): $n = \dim M \leq 4$.
- Bär-Dahl (2002): $n \geq 5$ and $\pi_1(M) = \{e\}$.
- Ammann-Dahl-Humbert (2009): Version above.
- Ammann-Dahl-Humbert (2011): Stronger version: $D\!\!\!/$-minimality can be achievd via a perturbation on an arbitrarily small open set

Surgery

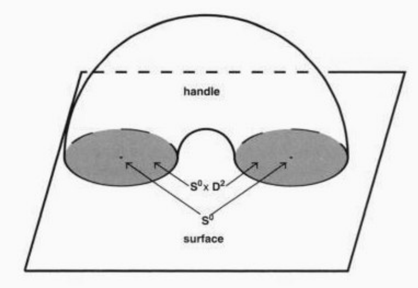
Let $f : S^k \times \overline{B^{n-k}} \hookrightarrow M$ be an embedding.
We define

$$M^{\#} := M \setminus f(S^k \times B^{n-k}) \cup (B^{k+1} \times S^{n-k-1}) / \sim$$

where $/ \sim$ means gluing the boundaries via

$$M \ni f(x, y) \sim (x, y) \in S^k \times S^{n-k-1}.$$

We say that $M^{\#}$ is obtained from M by surgery of dimension k.



Example: 0-dimensional surgery on a surface.

$D\!\!\!/$-minimality and surgery

Theorem ($D\!\!\!/$-Surgery Theorem, ADH 2009)

Let $k \leq n - 2$.
If M carries a $D\!\!\!/$-minimal metric, then $M^{\#}$ carries a $D\!\!\!/$-minimal metric as well.

We use a Gromov-Lawson type construction. In particular the new metric on $M^{\#}$ coincides with the old one away from the surgery sphere.
Bär-Dahl (2002) proved the theorem with other methods for $k \leq n - 3$.

Ammann-Dahl-Humbert, Math. Res. Lett. 2011

Theorem A_{loc} (Local $D\!\!\!/$-Minimality Theorem)

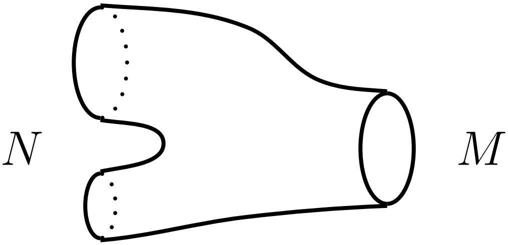
Let M be a compact connected spin manifold with a Riemannian metric g. Let U be a non-empty open subset of M. Then there is metric $\tilde{g}$ on M which is $D\!\!\!/$-minimal and which coincides with g on $M \setminus U$.

Theorem $A_{\mathrm{loc}} \Rightarrow$ Theorem A.

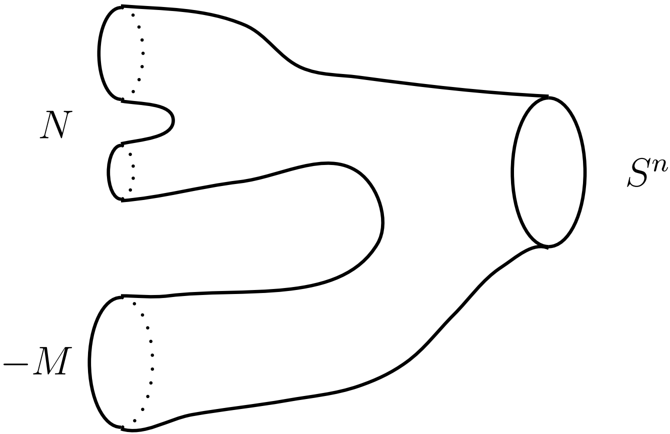
Proof of "$\not{D}$-surgery Thm $\Longrightarrow$ Local $\not{D}$-minimality Thm"

We use a theorem from Stolz 1992.
The given spin manifold M is spin bordant to $N = N_0 \cup P$, where

- P carries a metric of positive scalar curvature,
- N_0 is a disjoint union of products of S^1, a $K3$-surface and a Bott manifold, and carries a $\not{D}$-minimal metric.

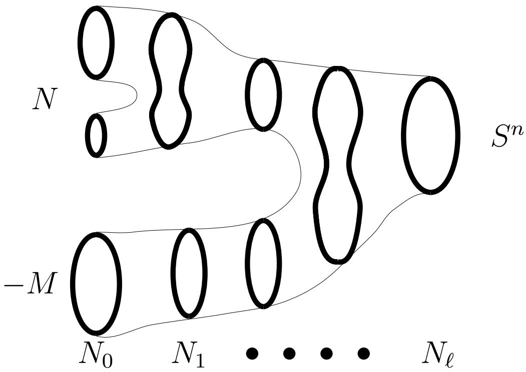


Assume now $n \geq 5$. Remove a ball and move M to the other side.



Modify the bordism W such that W is connected and $\pi_1(W) = \pi_2(W) = 0$.
As $\pi_1(S^n) = 0$, the bordism W can be decomposed into pieces corresponding to surgeries of dimension

$$k \in \{0, 1, \ldots, n-3\}$$

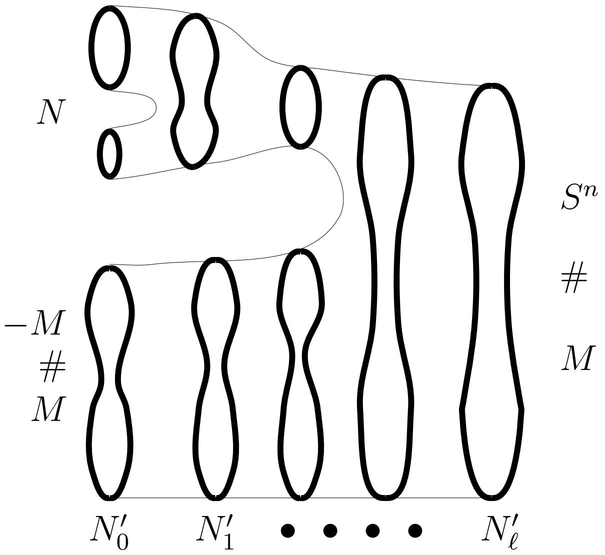


Invertible Double

Proposition

Let M be compact, connected and spin. Then there is a metric g on $M\#(-M)$ with invertible $\not{D}_g$.

See e.g. the book by Booß-Bavnbek and Wojchiekowski.
Uses the unique continuation property of $\not{D}^g$.



N
$-M$
$\#$
M
S^n
$\#$
M
N'_0
N'_1
N'_ℓ

Status of the non-$\not{D}$-minimality conjecture

Conjecture (Non-$\not{D}$-minimality)

On every closed spin manifold of dimension ≥ 3 non-$\not{D}$-minimal metrics exist.

This conjecture has been proved by

- Hitchin (1974): on $M = S^3$, and surfaces on genus ≥ 3, resp. ≥ 5
- Hitchin (1974): in dimensions $n \equiv 0, 1, 7 \mod 8$, $\alpha(M) = 0$
- Bär (1996): in dimensions $n \equiv 3, 7 \mod 8$,
- Seeger (2000): on S^{2m}, $m \geq 2$,
- Dahl (2008): on S^n, $n \geq 5$, for $k = 1$,
- Ammann, Bunke, Nowazcyk, Pilca: See below.

New impact from psc

Three recent techniques to get non-trivial elements of $\pi_k(\mathcal{M}_{\mathrm{psc}}(M))$.

[CS] D. Crowley, T. Schick, ArXiv April 2012, using $KO_{2+8k} = \mathbb{Z}/2$.

[HSS] B. Hanke, W. Steimle, T. Schick, ArXiv Dec 2012, using $KO_{8k} = \mathbb{Z}$.

[BER] B. Botvinnik, J. Ebert, O. Randal-Williams, ArXiv Nov 2014, using homotopy theory.

Trivial Conclusion

If $\alpha(M) = 0$, then non-trivial elements in $\pi_k(\mathcal{M}_{psc}(M))$ detected in these approaches are also non-trivial in $\pi_k(\mathcal{M}_{=0}(M))$.

A more detailed analysis yields

Conclusion

If $\alpha(M) = 0$*, and* $n = \dim M \geq 6$*, then the techniques above yield non-trivial elements of* $\pi_k(\mathcal{M}_{=0}(M))$ *for appropriate* k*.*

Corollary

Let M *be a closed connected spin manifolds of dimension* $\dim M = 3$ *or* $\dim M \geq 6$*. Assume* $\alpha(M) = 0$*, then non-*$D\!\!\!/$*-minimal metrics exist.*

Recent work by Ammann, Bunke, Pilca, Nowaczyk

Now $\alpha(M) \neq 0$, in particular $n := \dim M \equiv 0, 1, 2, 4 \mod 8$.

Theorem B

Let $n := \dim M \equiv 0, 1, 2 \mod 8$, $\ell \geq 1$, $n + \ell + 1 \equiv 2 \mod 8$. In the case $n \equiv 0$ we additionally assume $|\alpha(M)| \leq 5$ and $\ell = 9$. Then $\pi_\ell(\mathcal{M}_{=|\alpha(M)|}(M))$ contains a non-trivial element of order 2.

Proof based on [CS].

"Theorem C"

For each $A \in \mathbb{N}$ and each $\ell \equiv 3 \mod 4$ with $\ell > 2A$ there is a $k_0 = k_0(A) \in \mathbb{N}$ such for any closed connected spin manifold M of dimension $4k$, $k \geq k_0$ with $|\alpha(M)| \leq A$ there is a non-trivial element of $\pi_\ell(\mathcal{M}_{=|\alpha(M)|}(M))$.

Proof based on [HSS, Thm 1.4], but different way to conclude.

We wrote "Theorem C" to indicate, that this theorem is not yet written up, and some unexpected difficulties might arise.

Corollary

Non-$\not{D}$-minimal metrics exist on the closed connected spin manifold M, $n = \dim M$ if

- $n = 3$
- $n \equiv 1, 2, 3, 5, 6, 7 \mod 8$ *and* $n \geq 6$
- $n \equiv 0 \mod 8$, $n \geq 8$, *and* $|\alpha(M)| \leq 5$
- $n \equiv 0 \mod 8$, $n \geq 4k_0(\alpha(M))$
- $n \equiv 4 \mod 8$, $n \geq 12$, $\alpha(M) = 0$
- $n \equiv 4 \mod 8$, $n \geq 4k_0(\alpha(M))$

About the proofs of Theorem B and C

The articles [CS] and [HSS] define maps

$$\phi : S^\ell \to \mathrm{Diff}_{\mathrm{spin}}(M'), \quad y \mapsto \phi_y,$$

and $M = M'$ for [CS] and M' spin bordant to M for [HSS]. Define

$$\Phi : M' \times S^\ell \to M' \times S^\ell, \quad (x, y) \mapsto (\phi_y(x), y).$$

Then

$$\pi : \underbrace{(M' \times D^{\ell+1}) \cup_\Phi (M' \times D^{\ell+1})}_{W:=} \to \underbrace{D^{\ell+1} \cup_\partial D^{\ell+1}}_{S^{\ell+1}=}$$

is a fiber bundle with fiber M'.

$$\alpha(W) \neq 0 \in \begin{cases} KO_{2+8k} & \text{in Theorem B, using [CS]} \\ KO_{8k} & \text{in Theorem C, using [HSS]} \end{cases}$$

Fix a $\not{D}$-minimal metric g_0 on M'.

Claim

$$S^\ell \to \mathcal{M}_{=|\alpha(M)|}(M'), \quad y \mapsto \phi_y^* g_0$$

is a non-trivial element in $\pi_\ell(\mathcal{M}_{=|\alpha(M)|}(M'))$.

Proof of the claim:
If this sphere of metrics were contractible, then we would get a family of $\not{D}$-minimal metrics on the fibers of W.
$\rightsquigarrow$ $\ker \not{D} \to S^{\ell+1}$ is a $\mathbb{K}$-vector bundle, where

$$\mathbb{K} = \begin{cases} \mathbb{R} & \text{if } n \equiv 0 \\ \mathbb{C} & \text{if } n \equiv 1 \\ \mathbb{H} & \text{if } n \equiv 2, 4. \end{cases}$$

Question

Is $\ker \not{D} \to S^{\ell+1}$ *a trivial* $\mathbb{K}$*-vector bundle?*

No example known where the answer is "No".
"Yes" under the conditions of the theorems.

We prove and use a family index theorem

$$0 \neq \alpha(W) = \alpha(M') \cdot \alpha(S^{\ell+1}) = 0.$$

We have obtained a non-trivial element in $\pi_\ell(\mathcal{M}_{=|\alpha(M)|}(M'))$.
This yields Theorem B.
To get Theorem C, note that a suitable bordism from M to M' yields a homotopy equivalence

$$\pi_\ell(\mathcal{M}_{=|\alpha(M)|}(M')) \to \pi_\ell(\mathcal{M}_{=|\alpha(M)|}(M)).$$

Thanks for the attention

My publications:

`http://www.berndammann.de/publications`