# Positive Isotropic Curvature and Self-Duality in Dimension 4

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Let $\mathcal{R} : \wedge^2\mathbb{R}^n \longrightarrow \wedge^2\mathbb{R}^n$, $n \geq 3$, be an algebraic curvature operator. By definition $\mathcal{R}$ is a symmetric linear map satisfying the first Bianchi identity:

$$\langle \mathcal{R}(x \wedge y), z \wedge t \rangle + \langle \mathcal{R}(z \wedge x), y \wedge t \rangle + \langle \mathcal{R}(y \wedge z), x \wedge t \rangle = 0$$

for all $x, y, z, t \in \mathbb{R}^n$.

Let $S^2_B\Lambda^2\mathbb{R}^n$ denote the space of algebraic curvature operators on $\mathbb{R}^n$. There is a natural $O(n)$ action on $S^2_B\Lambda^2\mathbb{R}^n$ given by

$$\langle \gamma.\mathcal{R}(x \wedge y), z \wedge t \rangle = \langle \mathcal{R}(\gamma x \wedge \gamma y), \gamma z \wedge \gamma t \rangle \;\text{ for } \gamma \in O(n, \mathbb{R}).$$

A convenient way of describing a positive curvature condition is to regard it as an open $O(n)$-invariant convex cone $\mathcal{C}$ containing the identity operator $I$ and contained in the half-space

$$\mathcal{C}_{s>0} = \{\mathcal{R} \in S^2_B\Lambda^2\mathbb{R}^n \ : \ s = trace(\mathcal{R}) > 0\}$$

of operators with positive scalar curvature.

Let $\mathcal{C}$ be a cone as above and $(M, g)$ a Riemannian manifold. We say that $M$ has $\mathcal{C}$-positive curvature if, for any $p \in M$, a pullback of $\mathcal{R}_p$ to $S^2_B\Lambda^2\mathbb{R}^n$, via a linear isometry between $T_pM$ and $\mathbb{R}^n$, lies in $\mathcal{C}$. The $O(n)$ invariance of $\mathcal{C}$ ensures that this does not depend on the choice of the linear isometry.

Let $(M, g_0)$ be a compact Riemannian manifold and $\mathcal{C}$ be a positive curvature cone. Suppose that $M$ has $\mathcal{C}$-positive curvature.

Consider the Ricci flow starting at $g_0$:

$$\frac{\partial g}{\partial t} = -2Ric(g), \quad g(0) = g_0, \quad t \in [0, T].$$

One is interested in classifying cones $\mathcal{C}$ for which $\mathcal{R}_t \in \mathcal{C}$ for all $t \in (0, T]$.

A sufficient condition is given by Hamilton's maximum principle.

The evolution of the curvature operator is given by

$$\frac{\partial \mathcal{R}}{\partial t} = \Delta \mathcal{R} + Q(\mathcal{R})$$

where $Q : S^2_B \Lambda^2 \mathbb{R}^n \longrightarrow S^2_B \Lambda^2 \mathbb{R}^n$ is a certain universal homogeneous quadratic polynomial map.

We say that a curvature cone is *Ricci flow invariant* if it preserved by the ODE

$$\frac{d\mathcal{R}}{dt} = Q(\mathcal{R}),$$

i.e. if $\mathcal{R}(0) \in \mathcal{C}$ then $\mathcal{R}(t) \in \mathcal{C}$ for $t > 0$.

# Hamilton's Maximum Principle:

Let $\mathcal{C}$ be a Ricci flow invariant positive curvature cone and $(M, g_0)$ a compact Riemannian manifold with $\mathcal{C}$-positive curvature. Then the Ricci flow $g_t$, $t \in [0, T]$, starting at $g_0$ has $\mathcal{C}$-positive curvature for $t \in (0, T]$.

(1) $\mathcal{C}_{s>0}$

(2) $\mathcal{C} =$ the set of positive curvature operators

(3) $\mathcal{C} =$ the set of curvature operators with positive isotropic curvature (PIC)

# The PIC condition:

Let $\mathcal{R} \in S^2_B\Lambda^2\mathbb{R}^n$.

Extend

- $\mathcal{R}$ to a $\mathbb{C}$-linear map $\mathcal{R} : \Lambda^2\mathbb{C}^n \longrightarrow \Lambda^2\mathbb{C}^n$.
- the inner product on $\mathbb{R}^n$ to a complex bilinear form $(.,.)$ on $\mathbb{C}^n$.
- the inner product on $\Lambda^2\mathbb{R}^n$ to a Hermitian inner product $\langle .,. \rangle$ on $\Lambda^2\mathbb{C}^n$.

We say that a subspace $V \subset \mathbb{C}^n$ is isotropic if $(v, v) = 0$ for all $v \in V$.

$\mathcal{R}$ is PIC if $\langle \mathcal{R}(v \wedge w), v \wedge w \rangle > 0$ whenever $v,\ w$ span an isotropic 2-plane.

This notion of curvature appears naturally in two contexts:

(1) As the curvature term in the complexified version of the second variation formula for the energy of harmonic maps of surfaces into Riemannian manifolds.

(2) As the curvature term in the Bochner formula for harmonic 2-forms.

Exploiting the Lie algebra structure of $\Lambda^2\mathbb{R}^n \sim \mathfrak{so}(n)$, B. Wilking gave a simple criterion for generating a large class of Ricci flow invariant curvature cones. It is known that all of these cones are contained in the PIC cone if $n \geq 5$. This latter fact uses the simplicity of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ for $n \geq 5$.

However it is not known if *any* Ricci flow invariant cone is always contained in the PIC cone when $n \geq 5$.

We begin by recalling the decomposition of $S^2_B\Lambda^2\mathbb{R}^n$, $n \geq 4$, into irreducible components under the $O(n)$ action:

$$S^2_B\Lambda^2\mathbb{R}^n = \mathbb{R}\,\mathrm{I} \oplus (S^2_0\mathbb{R}^n \wedge \mathrm{id}) \oplus \mathcal{W}.$$

Here the wedge product of two symmetric operators on $\mathbb{R}^n$ corresponds to the Kulkarni-Nomizu product of symmetric 2-tensors.

We will write $\mathcal{R} = \mathcal{R}_\mathrm{I} + \mathcal{R}_0 + \mathcal{R}_\mathcal{W}$ to denote the decomposition of a curvature operator along the three components.

We now consider $n = 4$: we have

$$\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4.$$

This has the following consequence: if we restrict the $O(4)$ action to an $SO(4)$ action, the decomposition is no longer irreducible. In fact, the Weyl part splits into self-dual and anti-self-dual parts:

$$S^2_B\Lambda^2\mathbb{R}^4 = \mathbb{R}\,\mathrm{I} \oplus (S^2_0\mathbb{R}^4 \wedge \mathrm{id}) \oplus \mathcal{W}_+ \oplus \mathcal{W}_-.$$

We write

$$\mathcal{R} = \mathcal{R}_\mathrm{I} + \mathcal{R}_0 + \mathcal{R}_{\mathcal{W}_+} + \mathcal{R}_{\mathcal{W}_-}.$$

Complexifying $\mathbb{R}^4$ we have

$$\Lambda^2\mathbb{C}^4 = \Lambda^2_+\mathbb{C}^4 \oplus \Lambda^2_-\mathbb{C}^4.$$

For $\mathcal{R} \in S^2_B\Lambda^2\mathbb{R}^4$ we say that $\mathcal{R}$ is $PIC_+$ if

$$(\mathcal{R}(v \wedge w), v \wedge w) > 0$$

whenever $v,\ w$ span an isotropic 2-plane and $v \wedge w \in \Lambda^2_+\mathbb{C}^4$.

Let $\mathcal{R} \in S^2_B\Lambda^2\mathbb{R}^4$. The following conditions are equivalent:

1. $\mathcal{R}$ is $PIC_+$.
2. The symmetric operator on $\Lambda^2_+\mathbb{R}^4$ defined by the quadratic form $\langle \mathcal{R}\eta, \eta\rangle$, $\eta \in \Lambda^2_+\mathbb{R}^4$, is 2-positive.
3. The symmetric operator $\frac{s}{6}I - \mathcal{R}_{\mathcal{W}_+}$ on $\Lambda^2_+\mathbb{R}^4$ is positive.
4. For any *oriented* orthonormal basis $(e_1, e_2, e_3, e_4)$ of $\mathbb{R}^4$, we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$$

where $R_{ijkl} = \left\langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_l \right\rangle$.

# Examples:

(1) Oriented 4-manifolds with positive scalar curvature and $\mathcal{R}_{\mathcal{W}_+} = 0$. An example would be $\mathbb{C}P^2$ with the Fubini-Study metric and opposite orientation.

(2) Kähler surfaces with their natural orientation and positive scalar curvature are $NNIC_+$ but not $PIC_+$. This follows from a result of Derdzinski.

We have :

**Theorem**:

(1) The cone $\mathcal{C}_{PIC_+}$ of $PIC_+$ curvature operators is Ricci flow invariant.

(2) Let $\mathcal{C} \subset S^2_B \Lambda^2 \mathbb{R}^4$ be a Ricci flow invariant positive curvature cone. Assume that $\mathcal{C} \neq \mathcal{C}_{s>0}$.

If $\mathcal{C}$ is $SO(4)$-invariant then $\mathcal{C}$ is contained in $\mathcal{C}_{PIC_+}$ or $\mathcal{C}_{PIC_-}$.

If $\mathcal{C}$ is $O(4)$-invariant then $\mathcal{C}$ is contained in the cone of $PIC$ operators.

## Theorem

*The connected sum of two $PIC_+$ manifolds admits a $PIC_+$ metric.*

## Theorem

*Let $(M, g)$ be a compact oriented locally irreducible Riemannian 4-manifold.*

- *If $(M, g)$ is $PIC_+$ then $b_+(M) = 0$.*
- *If $(M, g)$ is $NNIC_+$ then $b_+(M) \leq 3$, moreover, if $b_+(M) \geq 2$ then $(M, g)$ is either flat or isometric to a $K3$ surface.*

We recall the classical rigidity theorem for half-conformally flat Einstein 4-manifolds:

**Theorem**

*(N. Hitchin [**?**], T. Friedrich - H.Kurke [**?**]) Let $(M, g)$ be a compact oriented half-conformally flat Einstein 4-manifold.*

1. *If $M$ has positive scalar curvature then it is isometric, up to scaling, to $S^4$ or $\mathbb{C}P^2$ with their canonical metrics.*
2. *If $M$ is scalar flat then it is either flat or its universal cover is isometric to a $K3$ metric with its Calabi-Yau metric.*

The following result can be regarded as a generalization of the above theorem. This result is not new and follows from the work of C. LeBrun - M. Gursky in [**?**].

## Theorem

*An Einstein $PIC_-$ 4-manifold is isometric, up to scaling, to $\mathbb{S}^4$ or $\mathbb{CP}^2$ with their standard metrics. Moreover, an Einstein $NNIC_-$ manifold is either $PIC_-$ or flat or a negatively oriented Kähler-Einstein surface with nonnegative scalar curvature.*

*PIC* 4-manifolds have been classified up to diffeomorphism via Ricci flow with surgery. This was initiated by R. Hamilton and completed by B. L. Chen and X. P. Zhu.

*Can one classify compact oriented $PIC_+ 4$ -manifolds up to diffeomorphism by Ricci flow with surgery?*

A seemingly simpler question is

*Let $M$ be homeomorphic to $S^4$. If $M$ admits a $PIC_+$ metric does it follow that $M$ is diffeomorphic to $S^4$?*

The starting point for Ricci flow with surgery is the classification of blow-up limits of the Ricci flow. In turn this depends on understanding Ricci solitons with the given curvature property.

*Classify Ricci solitons with* $PIC_+$