

Analytic extensions of spacelike maximal surfaces in Minkowski 3-space to timelike surfaces

Geometric Analysis in Geometry and Topology, 2015
Tokyo University of Science

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This talk is based on the Joint works...



S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada

Zero mean curvature entire graphs of mixed type in Lorentz-Minkowski 3-space
preprint.



S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada

Analytic extension of Jorge-Meeks type maximal surfaces in Lorentz-Minkowski 3-space
preprint, [arXiv:1509.05853](https://arxiv.org/abs/1509.05853) (19 Sept. 2015)



S. Fujimori, Y. W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada and S.-D. Yang

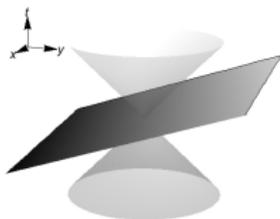
Zero mean curvature surfaces in Lorentz-Minkowski 3-space which change type across a light-like line
Osaka J. Math., vol. 52 (2015).

The Lorentz-Minkowski space

$$L^3 := \left((\mathbf{R}^3; t, x, y), \langle \cdot, \cdot \rangle = -dt^2 + dx^2 + dy^2 \right)$$

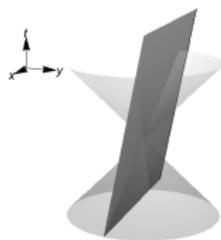
A plane Π in L^3 is said to be

Spacelike



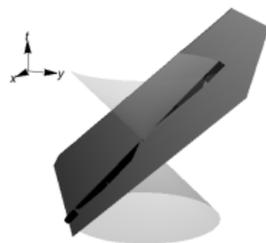
$$\langle \cdot, \cdot \rangle|_{\Pi} > 0$$

Timelike



$$\langle \cdot, \cdot \rangle|_{\Pi}: \text{indefinite}$$

Null



$$\langle \cdot, \cdot \rangle|_{\Pi}: \text{degenerates}$$

Surfaces in L^3

A surface in L^3 said to be

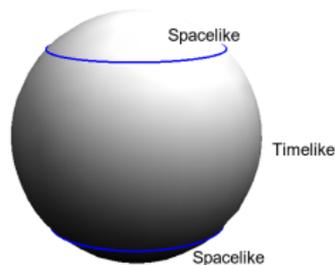
- **Spacelike** if all tangent planes are spacelike.
- **Timelike** if all tangent planes are timelike.



spacelike



timelike



type change

ZMC surfaces

Can type-changes occur for surfaces in a special class ?

ZMC surfaces

Can type-changes occur for **Zero Mean Curvature** (ZMC) surfaces?

ZMC surfaces

Can type-changes occur for **Zero Mean Curvature** (ZMC) surfaces?

This talk deals

- Type changes of ZMC surfaces along non-degenerate null curves;
- Type changes of ZMC surfaces along null lines;
- Examples of embedded ZMC surfaces.
- Examples of embedded ZMC graphs.

ZMC surfaces

Can type-changes occur for **Zero Mean Curvature** (ZMC) surfaces?

This talk deals

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- Examples of embedded ZMC graphs.

A ZMC surface:

- the mean curvature vanishes (if the surface is spacelike/timelike).
- a spacelike ZMC surface is called a (spacelike) maximal surface.
- a timelike ZMC surface is called a (timelike) minimal surface.

Maximal surfaces

A maximal surface is

- A spacelike surface in L^3 with vanishing mean curvature.
- A critical point of the area functional.
- A Weierstrass-type representation formula (Osamu Kobayashi, 1983)
Written in terms of holomorphic data on the surface.
cf. The Weierstrass representation for minimal surfaces in \mathbf{R}^3 .
- *A complete maximal surface in L^3 is a spacelike plane* (Calabi, 1970).
- It is natural to consider *maximal surfaces with singularities*.

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O. Kobayashi

Maximal surfaces in the 3-dimensional Minkowski space L^3
Tokyo J. Math., 6 (1983), 297–309.

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O. Kobayashi

Maximal surfaces with conelike singularities
J. Math. Soc. Japan, 36 (1984), 609–617.

The Weierstrass-type representation

O. Kobayashi (1983):

A spacelike maximal immersion $f: M^2 \rightarrow L^3$ is expressed as

$$f = \operatorname{Re} F; \quad F = \int \left(-2g, 1 + g^2, \sqrt{-1}(1 - g^2) \right) \omega$$

where

$$(g; \omega) = \left(\text{a meromorphic fct.}, \text{a holomorphic 1-form} \right)$$

on M^2 (with complex structure induced by the induced metric).

The induced metric is

$$ds^2 = (1 - |g|^2)^2 |\omega|^2 \quad (|g| = 1: \text{singularity})$$

Remark:

$f^* = \operatorname{Im} F$ is also a maximal surface called the **conjugate** surface of f .

Maxfaces

Definition (Umehara-Y, 2006; [3])

M^2 : A Riemann surface

A map $f: M^2 \rightarrow L^3$ is a **maxface** \Leftrightarrow

$\exists W \subset M^2$: open dense, such that $f|_W$ is a conformal spacelike maximal immersion and $df(p) \neq 0$ ($\forall p \in M^2$).

$$f = \operatorname{Re} F; \quad F = \int \left(-2g, 1 + g^2, \sqrt{-1}(1 - g^2) \right) \omega$$

is a maxface $\Leftrightarrow ds_{\#}^2 := (1 + |g|^2)^2 |\omega|^2$ is positive definite. Then

- The singular set of f is $\{p \in M^2 \mid |g| = 1\}$.
- The conjugate surface $f^* = \operatorname{Im} F$ of f has the Weierstrass data $(g, \sqrt{-1}\omega)$.
- The singular set of f^* coincides with that of f .

Singularities of maxfaces

Theorem (Fujimori-Saji-Umehara-Y, 2008)

- *Generic singular points of maxfaces are cuspidal edges, swallowtails, and cuspidal crosscaps.*
- *The duality*

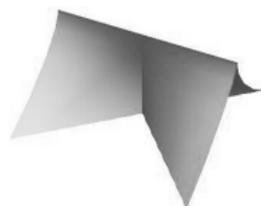
<i>Maxface f</i>	<i>The conjugate f^*</i>
<i>cuspidal edge</i>	<i>cuspidal edge</i>
<i>swallowtail</i>	<i>cuspidal crosscap</i>
<i>cuspidal crosscap</i>	<i>swallowtail</i>



Cuspidal edge



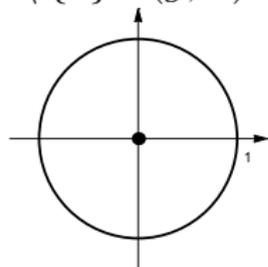
swallowtail



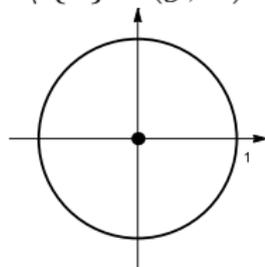
cuspidal crosscap

Example (the catenoid)

$$M^2 = \mathbf{C} \setminus \{0\}, \quad (g, \omega) = \left(z, \frac{dz}{z^2}\right) \quad M^2 = \mathbf{C} \setminus \{0\}, \quad (g, \omega) = \left(z, \frac{dz}{z^2}\right)$$



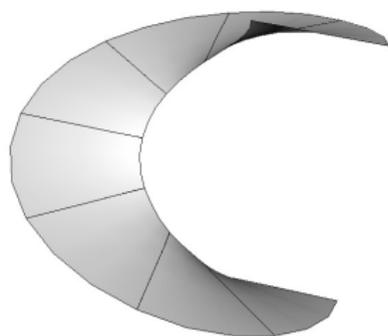
Lorentzian-Catenoid in L^3
 $|z| = 1$: cone-like singularities
non-generic



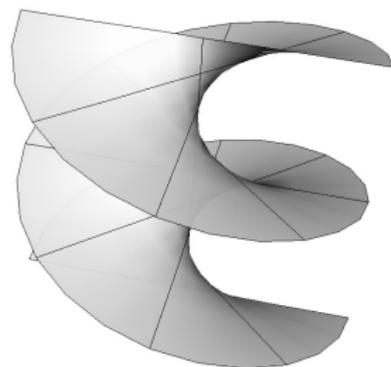
Euclidean-Catenoid in \mathbf{R}^3

Example (the helicoid)

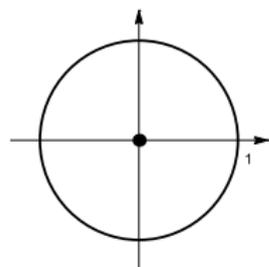
The conjugate surface of the catenoid:



Lorentz-Helicoid in L^3

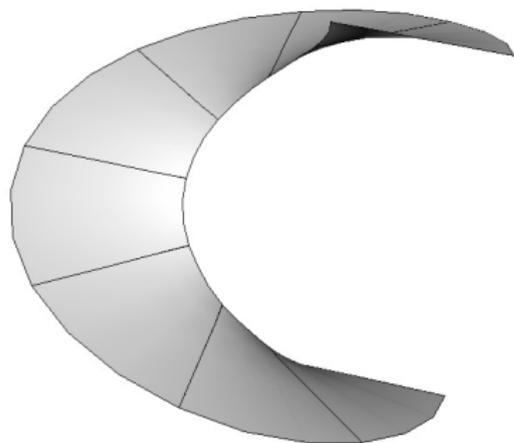
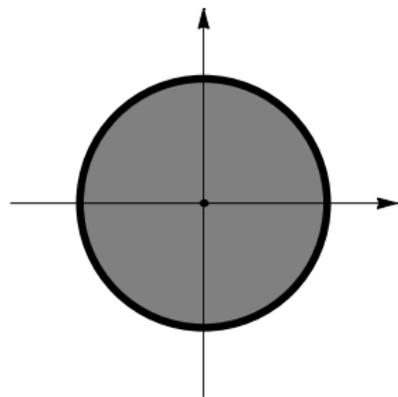


Helicoid in R^3

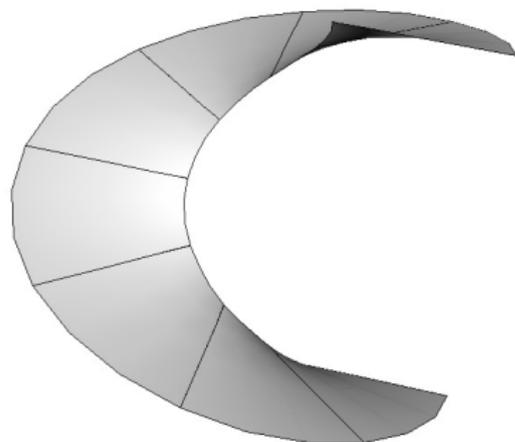
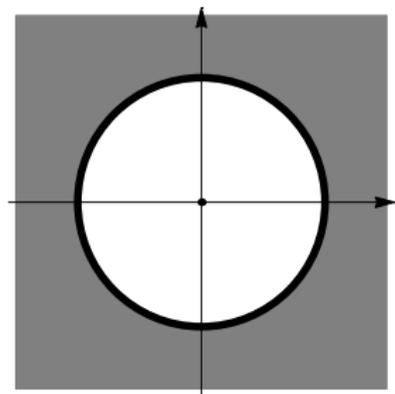


- $|z| = 1$ corresponds to the **fold** singularities.
- the image of fold singularities consists of a **null (light-like) curve**.

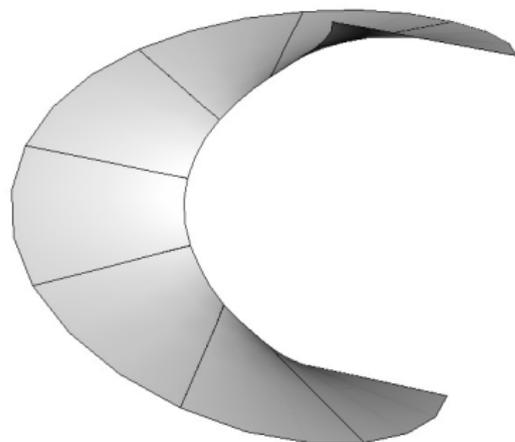
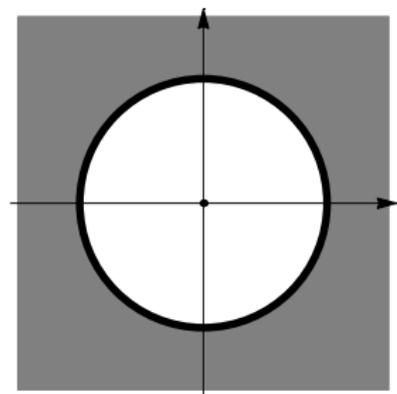
Fold singularities



Fold singularities



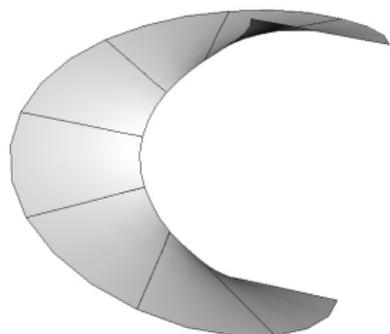
Fold singularities



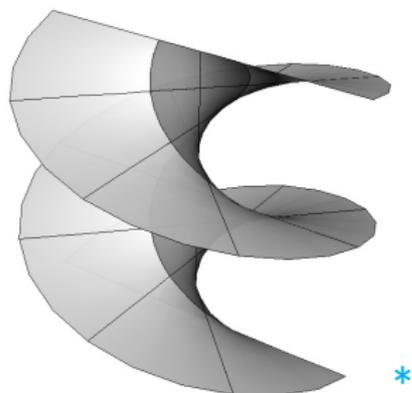
Fact (cf. Fujimori-Kim-Koh-Rossman-Shin-Umehara-Yang-Y, 2015)

The conjugate of a conelike singularity is a fold singularity.

The analytic extension of the helicoid



Lorentz-Helicoid in L^3



The analytic extension *

The dark part is a **timelike minimal surface**.

Remark:

The image of the right-hand figure coincides with the Euclidean helicoid.

Analytic extensions of along fold singularities

Fact

$f: M^2 \rightarrow L^3$: a maxface with fold singularities $\gamma(t)$.

\Rightarrow

- The image $\hat{\gamma}(t) = f \circ \gamma(t)$ is a null curve in L^3 which is **non-degenerate** (i.e. $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are linearly independent for all t).
- The map

$$\tilde{f}(u, v) := \frac{\hat{\gamma}(u + v) + \hat{\gamma}(u - v)}{2}$$

gives a timelike minimal surface.

- The image of \tilde{f} gives the analytic extension of the image of f along $\hat{\gamma}$.
- The union of the images of f and \tilde{f} is an immersed surface near $\hat{\gamma}$.

C. Gu (1985); V. A. Klyachin (2003); Kim-Koh-Shin-Yang (2006).

The analytic extension

- Take $\gamma(t)$: a null curve,



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- which is nondegenerate, i.e. $\ddot{\gamma}(t)$ is not proportional to $\dot{\gamma}(t)$.



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- which is nondegenerate, i.e. $\ddot{\gamma}(t)$ is not proportional to $\dot{\gamma}(t)$.
- Take the midpoint of two points: $f(s, t) := \frac{1}{2}(\gamma(s) + \gamma(t))$



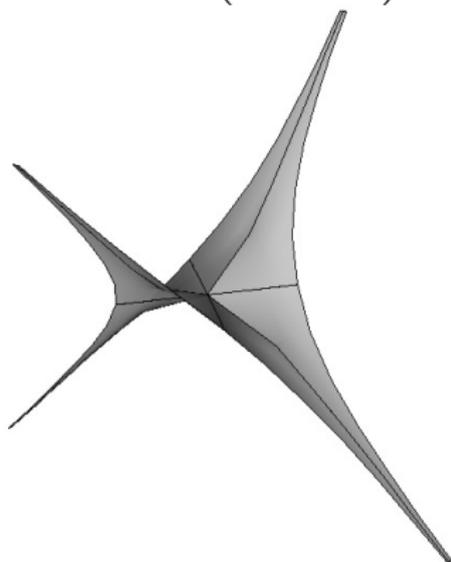
The analytic extension

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- Take the midpoint of two points: $f(s, t) := \frac{1}{2}(\gamma(s) + \gamma(t))$
- f gives a timelike minimal surface.



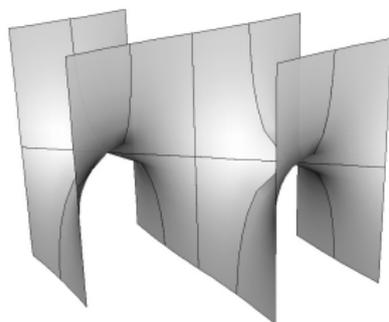
Example (Scherk-type surface)

$$M^2 = \mathbf{C} \setminus \{\pm 1, \pm\sqrt{-1}\}$$
$$(g, \omega) = \left(z, \frac{\sqrt{-1}dz}{1-z^4} \right)$$



Lorentz-Scherk in L^3

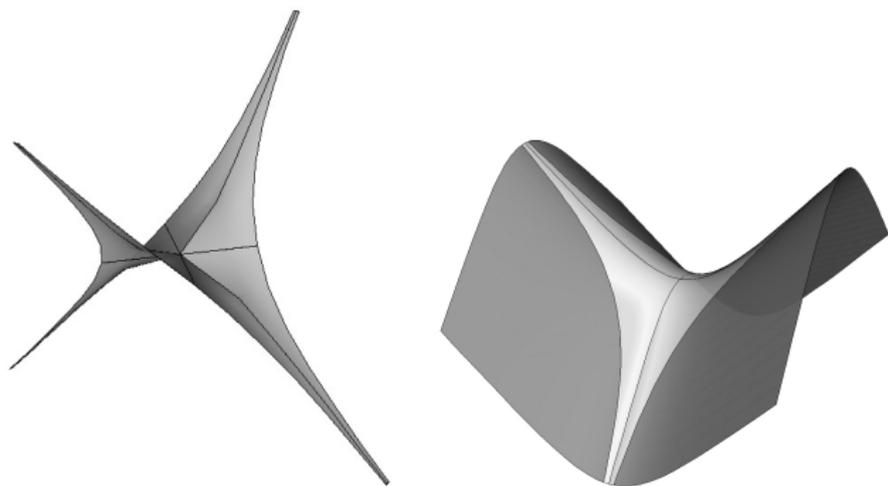
$$M^2 = \mathbf{C} \setminus \{\pm 1, \pm\sqrt{-1}\}$$
$$(g, \omega) = \left(z, \frac{dz}{1-z^4} \right)$$



Scherk \mathbf{R}^3

The graph $x_3 = \log \frac{\cos x_1}{\cos x_2}$.

The analytic extension of Scherk-type surface



$$t = \log \frac{\cosh x}{\cosh y}$$

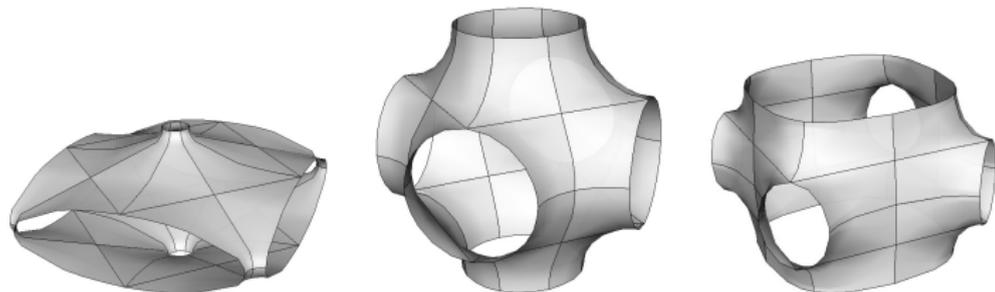
Entire ZMC graph
(O. Kobayashi (1983))

Embedded Examples: Schwarz type (Euclidan case)

$$M_a := \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2; w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\}, (0 < a < 1)$$

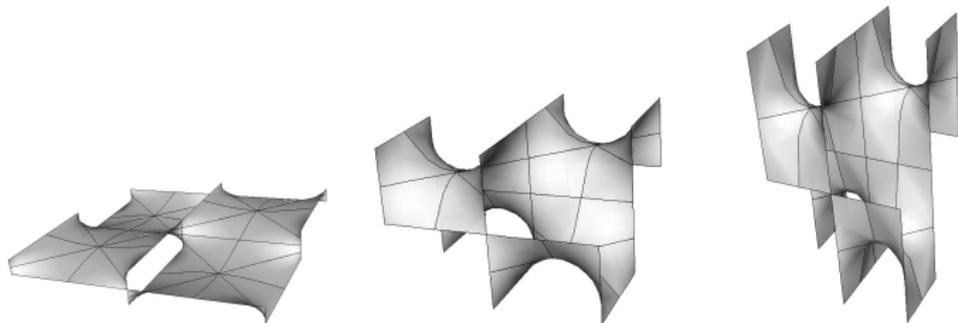
Schwarz P

$$g = z$$
$$\omega = \frac{dz}{w}$$



Schwarz D

$$g = z$$
$$\omega = \frac{dz}{\sqrt{-1}w}$$

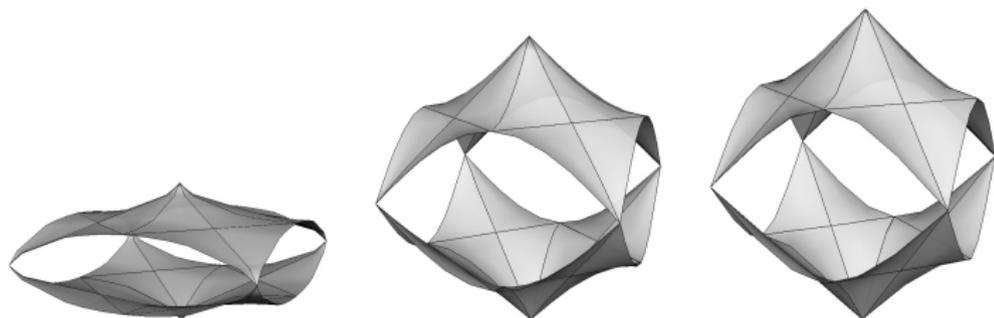


Embedded Examples: Schwarz type (Lorentzian case)

$$M_a := \{(z, w) \in (\mathbf{C} \cup \{\infty\})^2; w^2 = z^8 + (a^4 + a^{-4})z^4 + 1\}, \\ (0 < a < 1)$$

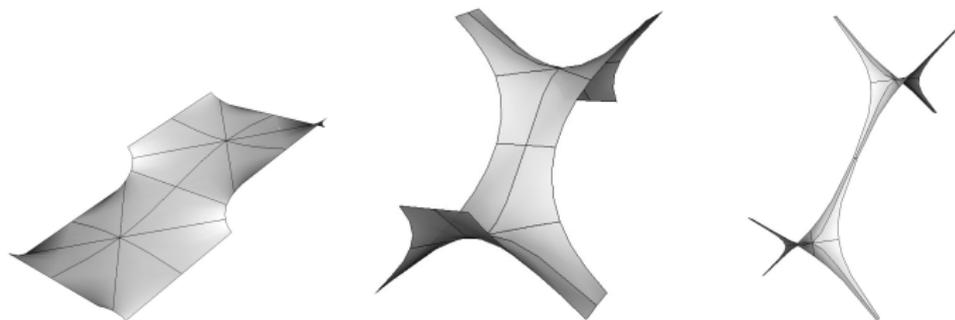
Schwarz P

$$g = z \\ \omega = \frac{dz}{w}$$

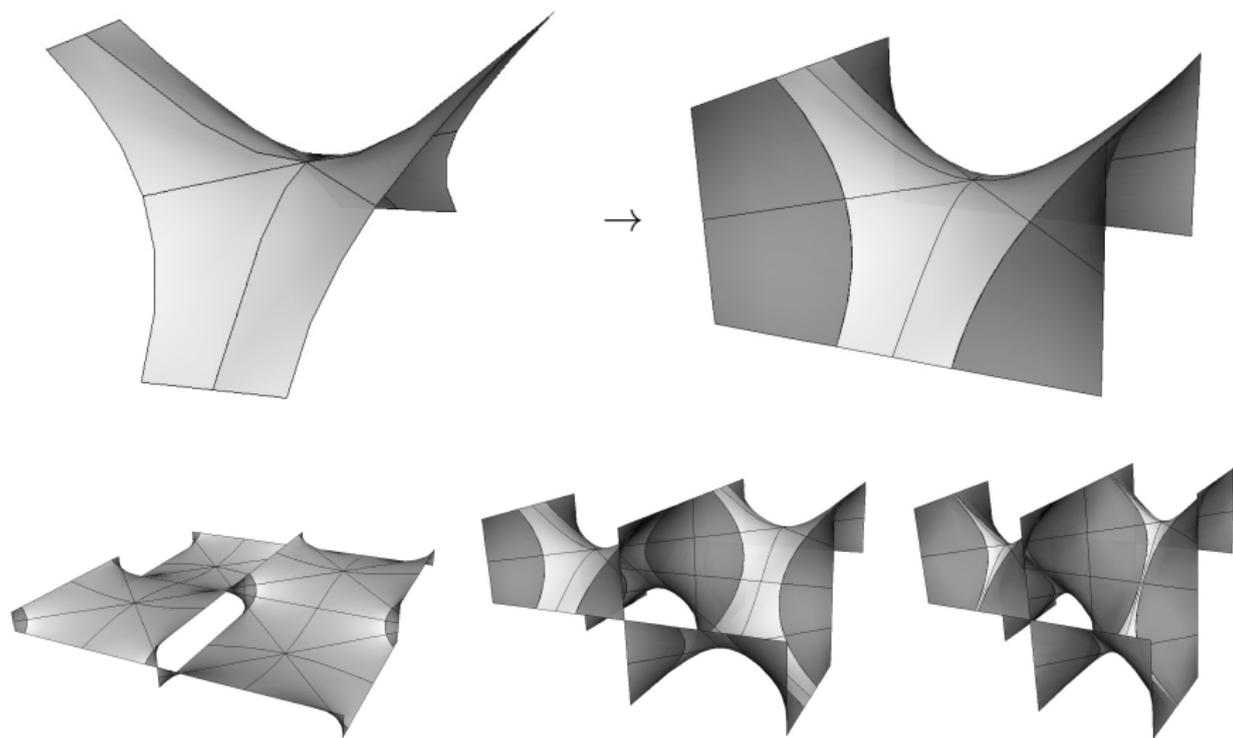


Schwarz D

$$g = z \\ \omega = \frac{dz}{\sqrt{-1}w}$$



The analytic extension of the Schwarz D-type maxface

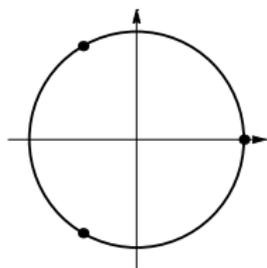


Embedded (Fujimori-Rossmann-Umehara-Yang-Y (2014))

Jorge-Meeks' surfaces in \mathbf{R}^3

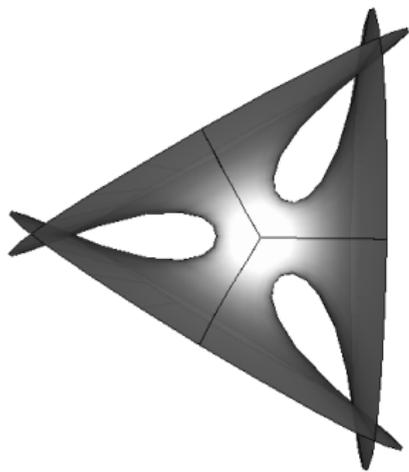
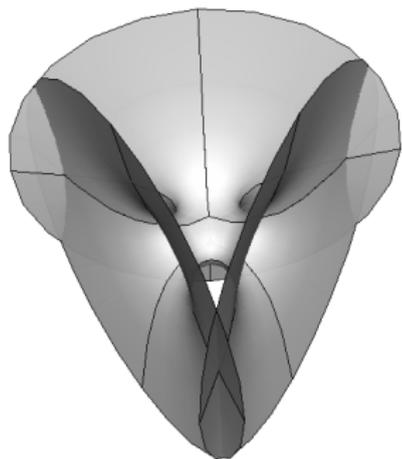
$$M^2 = \mathbf{C} \cup \{\infty\} \setminus \{1, \zeta, \dots, \zeta^{n-1}\}$$

$$(g, \omega) = \left(z^{n-1}, \frac{dz}{(z^n - 1)^2} \right) \quad (\zeta = e^{2\pi\sqrt{-1}/n})$$



$n = 3$ (trinoid)

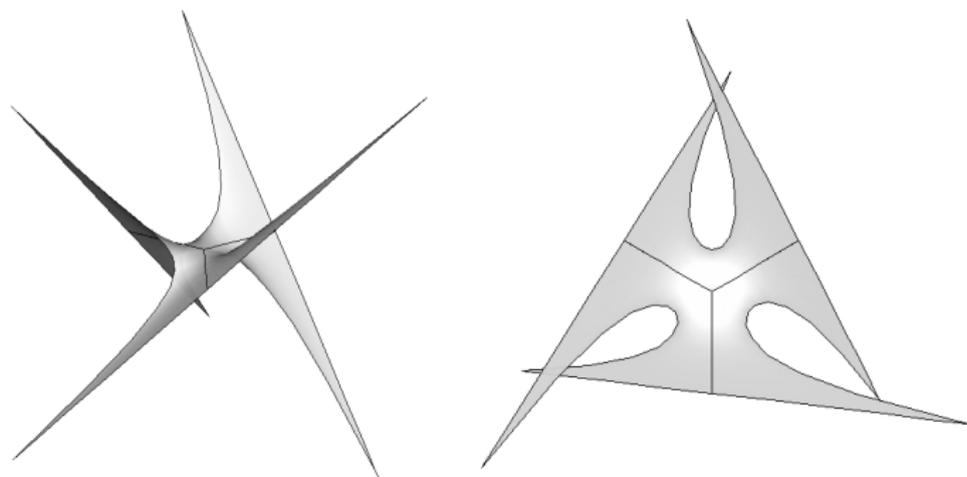
Jorge-Meeks 3-noid in \mathbb{R}^3



Embedded Examples: A Jorge-Meeks' type maxface in L^3

$$M^2 = \mathbf{C} \cup \{\infty\} \setminus \{1, \zeta, \dots, \zeta^{n-1}\}$$

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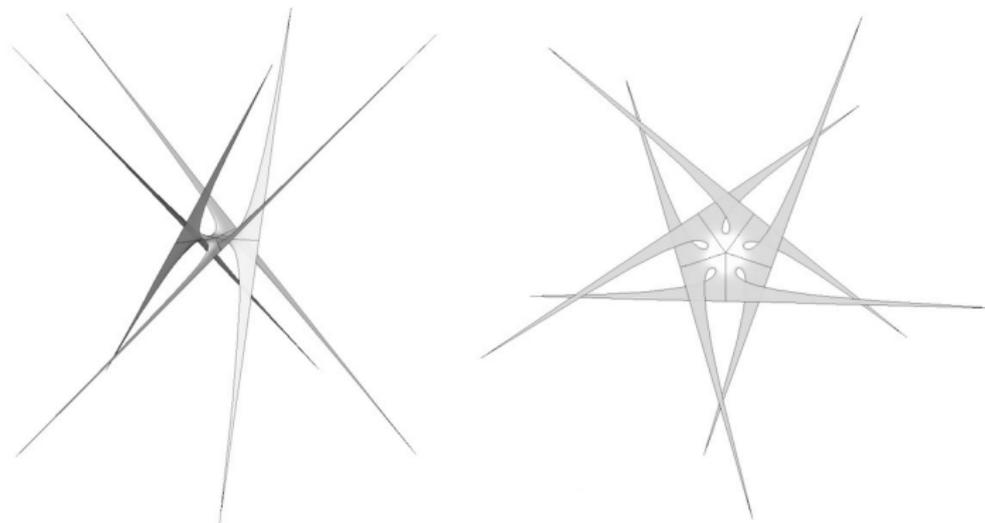


$$n = 3$$

Embedded Examples: A Jorge-Meeks' type maxface in L^3

$$M^2 = \mathbf{C} \cup \{\infty\} \setminus \{1, \zeta, \dots, \zeta^{n-1}\}$$

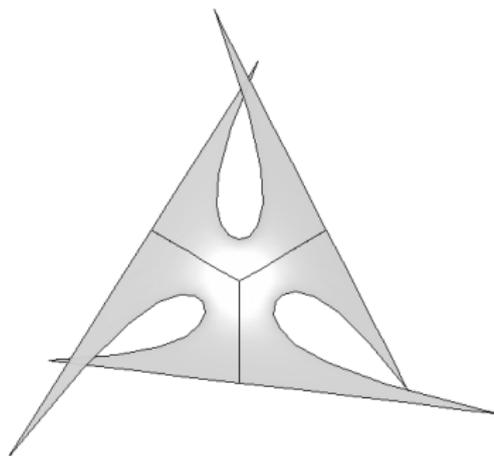
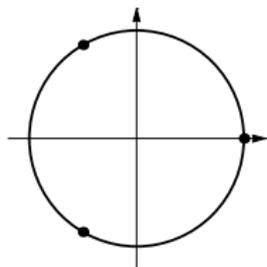
$$(g, \omega) = \left(z^{n-1}, \frac{\sqrt{-1}dz}{(z^n - 1)^2} \right) \quad (\zeta = e^{2\pi\sqrt{-1}/n})$$



$n = 5$

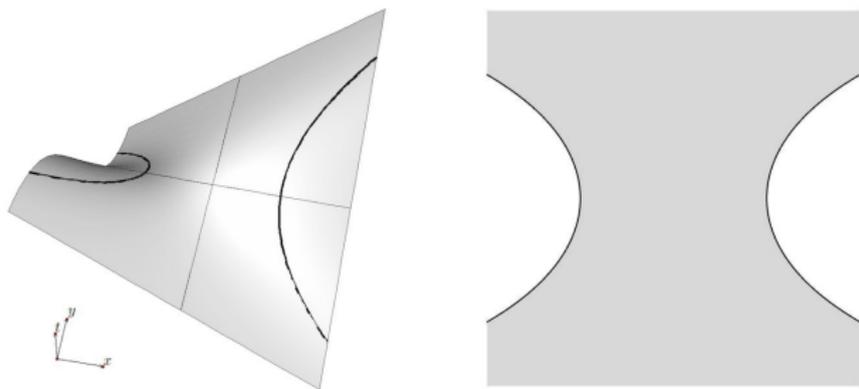
Embedded Examples: A Jorge-Meeks' type maxface in L^3

The singularities are fold singularities:



\Rightarrow analytic extension.

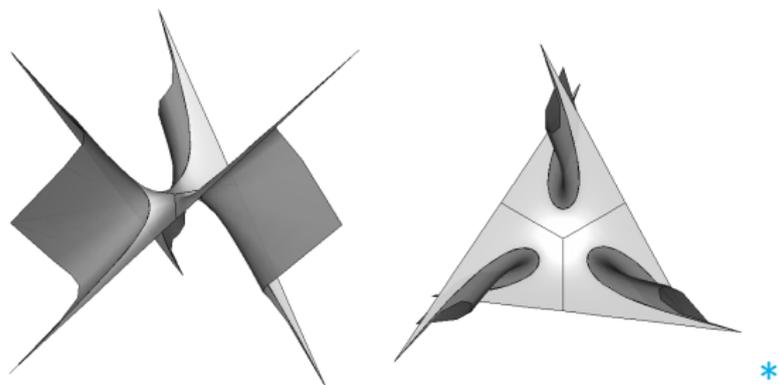
Analytic extension of Jorge-Meeks type surfaces ($n = 2$)



$$t = x \tanh y$$

An entire graph (O. Kobayashi, 1983)

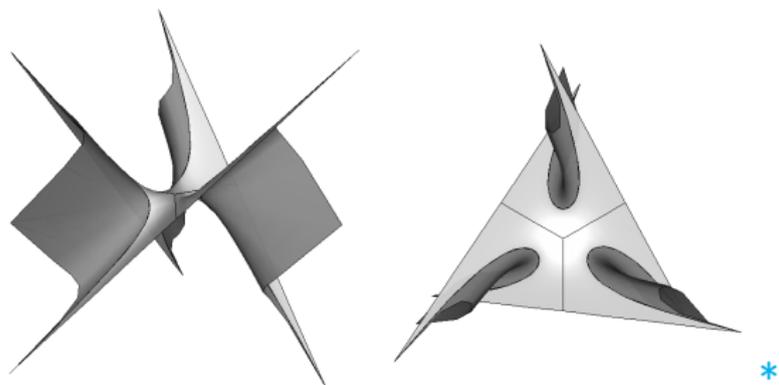
Analytic extension of Jorge-Meeks type surfaces



Theorem (Fujimori-Kawakami-Kokubu-Rossmann-Umehara-Y)

The analytic maximal extension of the Jorge-Meeks type n -noid is

Analytic extension of Jorge-Meeks type surfaces

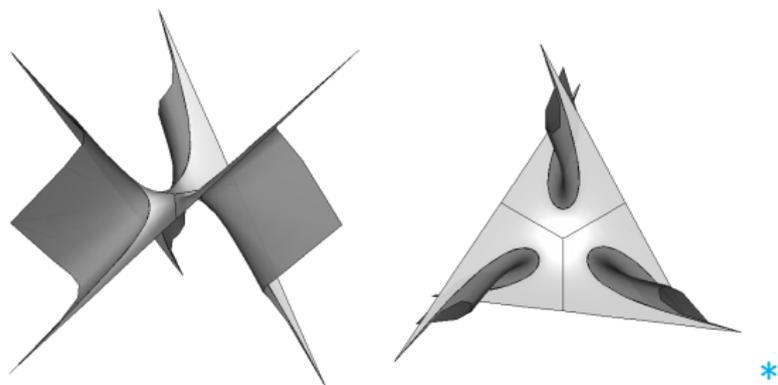


Theorem (Fujimori-Kawakami-Kokubu-Rossmann-Umehara-Y)

The analytic maximal extension of the Jorge-Meeks type n -noid is

- *properly immersed,*

Analytic extension of Jorge-Meeks type surfaces

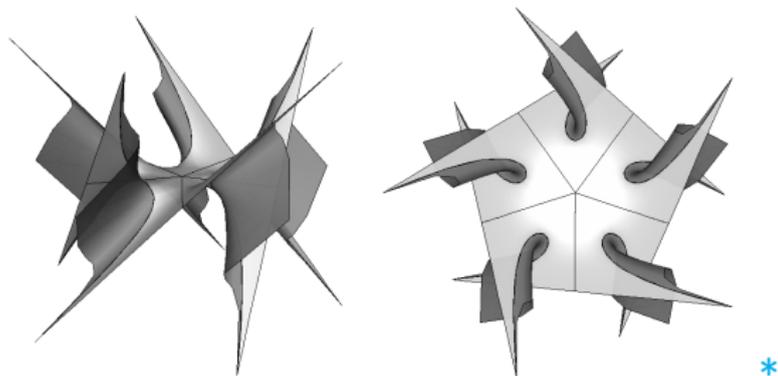


Theorem (Fujimori-Kawakami-Kokubu-Rossmann-Umehara-Y)

The analytic maximal extension of the Jorge-Meeks type n -noid is

- *properly immersed,*
- *and **embedded**.*

Analytic extension of Jorge-Meeks type surfaces



Theorem (Fujimori-Kawakami-Kokubu-Rossmann-Umehara-Y)

The analytic maximal extension of the Jorge-Meeks type n -noid is

- *properly immersed,*
- *and embedded.*

A generalization: Kobayashi surfaces

Definition

A **Kobayashi surface** is a weakly complete maxface

$$f: \mathbf{C} \cup \{\infty\} \setminus \{p_1, \dots, p_N\} \longrightarrow L^3$$

with Weierstrass data (g, ω) such that

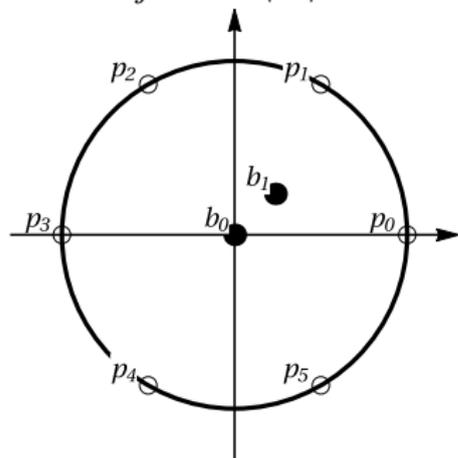
- The Gauss map g is meromorphic on $\mathbf{C} \cup \{\infty\}$,
- $\exists I: \mathbf{C} \cup \{\infty\} \rightarrow \mathbf{C} \cup \{\infty\}$: anti-holo. involution such that $f \circ I = f$,
- $p_1, \dots, p_N \in \Sigma := \{\text{fixed points of } I\}$,
- $\Sigma \setminus \{p_1, \dots, p_N\}$ consists of fold singularities.

Weierstrass data for Kobayashi Surfaces

A Kobayashi surface is expressed by the following Weierstrass data (Fujimori-Kawakami-Kokubu-Rossmann-Umehara-Y):

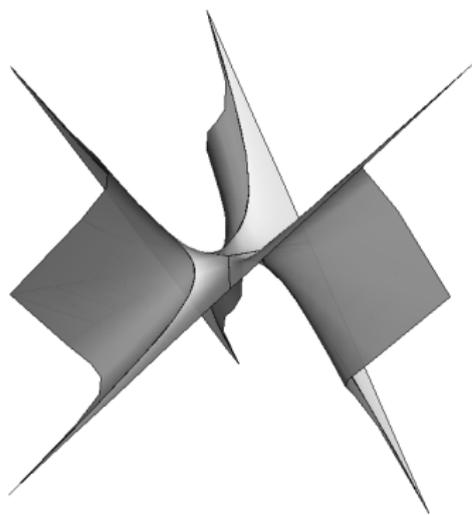
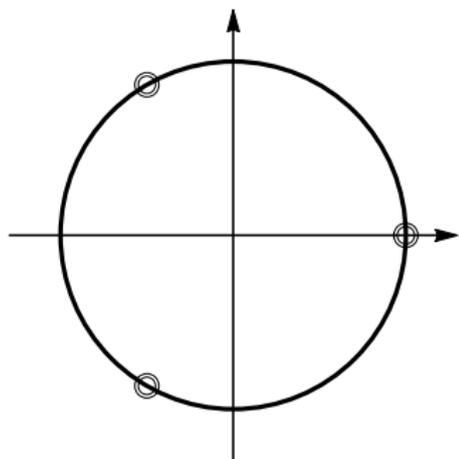
$$g = \prod_{i=1}^{n-1} \frac{z - b_i}{1 - \bar{b}_i z}, \quad \omega = \frac{\sqrt{-1} \prod_{i=1}^{n-1} (1 - \bar{b}_i z)^2}{\prod_{j=0}^{2n-1} (e^{-\sqrt{-1}\alpha_j/2} z - e^{\sqrt{-1}\alpha_j/2})} dz$$

where $\alpha_j \in \mathbf{R}$, $|b_i| < 1$.



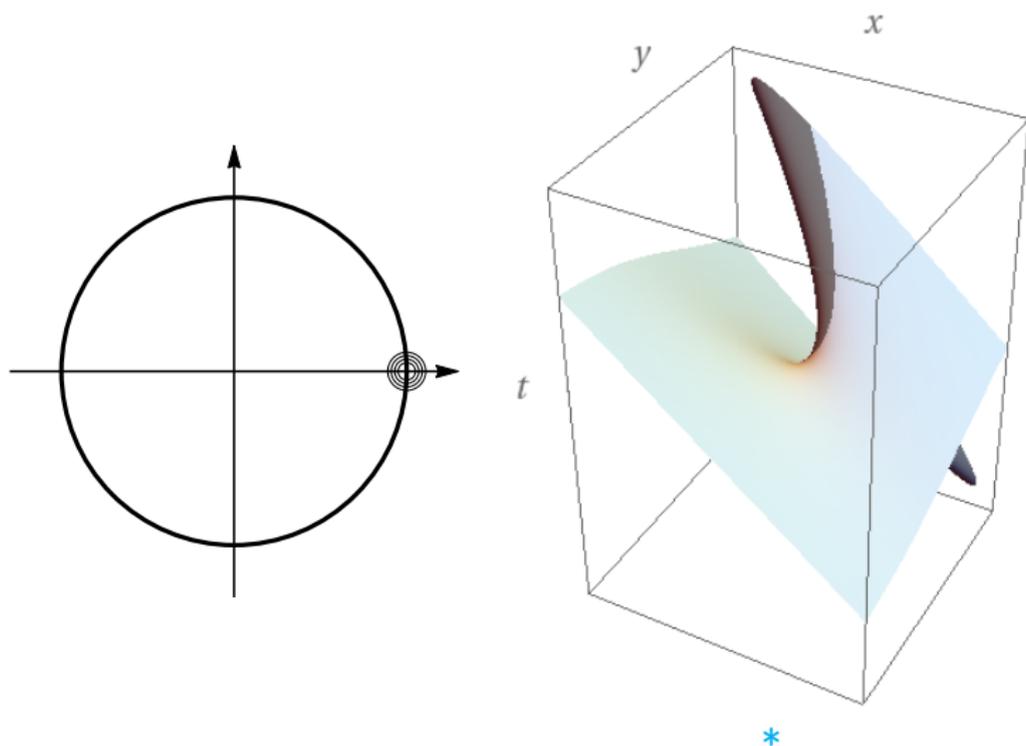
$$p_j = e^{\sqrt{-1}\alpha_j}, \quad I(z) = \frac{1}{\bar{z}}$$

Jorge-Meeks type surface as a Kobayashi surface



*

Example



*
Properly embedded; The Ruled Enneper surface (O. Kobayashi, 1983)

ZMC graph

Fact (ZMC equation)

A graph $t = f(x, y)$ is a ZMC surface if and only if

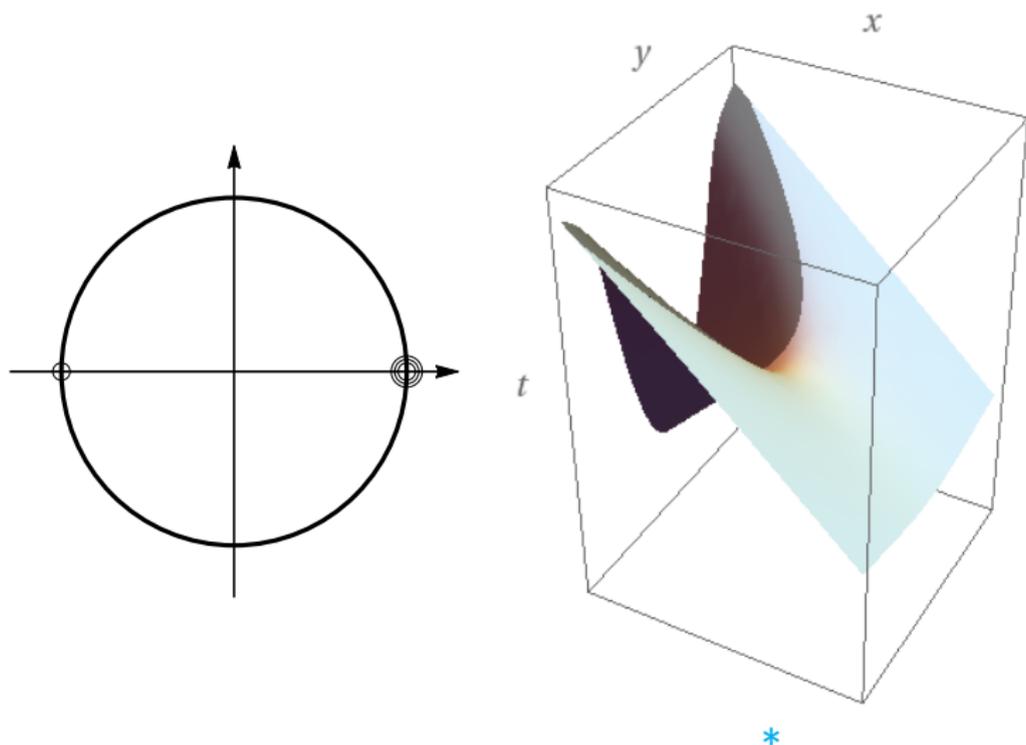
$$(1 - f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1 - f_x^2)f_{yy} = 0. \quad (*)$$

- if $1 - f_x^2 - f_y^2 > 0$: the graph is spacelike maximal surface, and (*) is elliptic.
- if $1 - f_x^2 - f_y^2 < 0$: the graph is timelike minimal surface, and (*) is hyperbolic.

There are entire solutions (O. Kobayashi 1983).

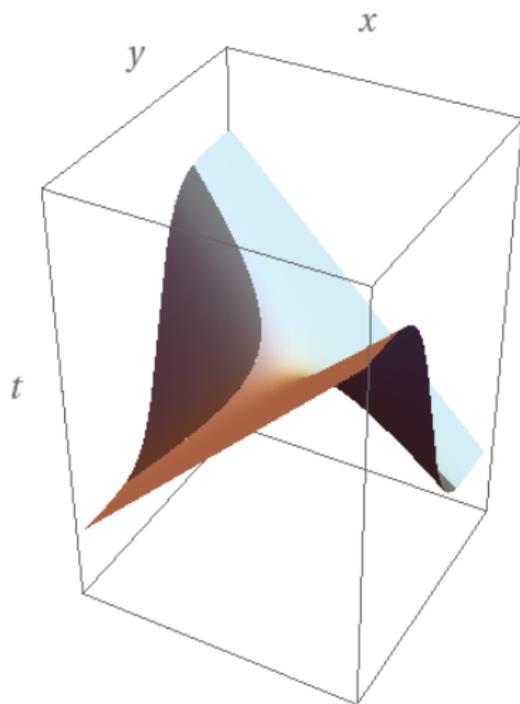
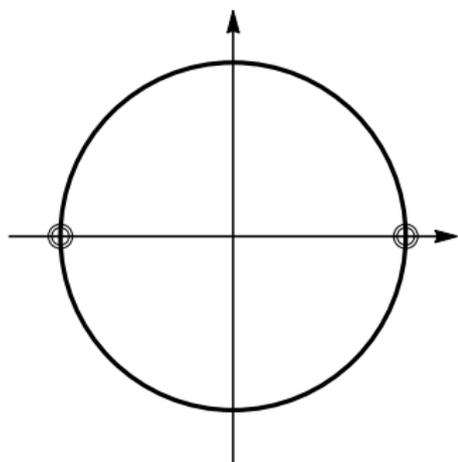
$$f(x, y) = \log \frac{\cosh x}{\cosh y}, \quad f(x, y) = x \tanh y$$

An entire ZMC graph



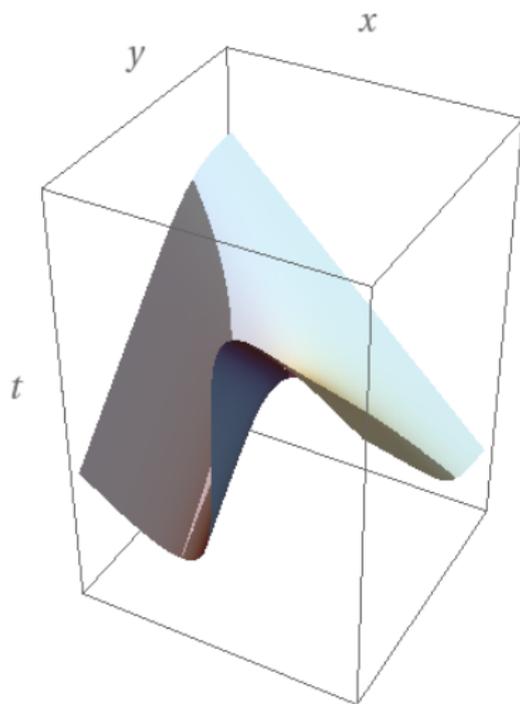
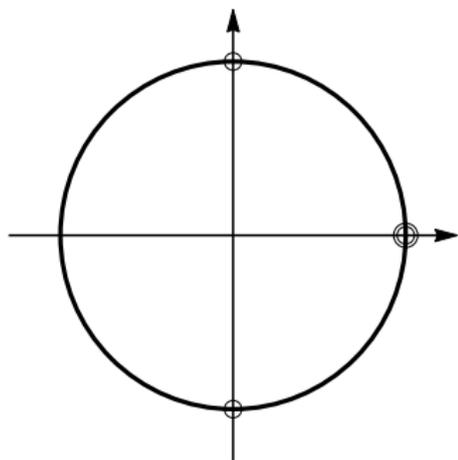
An entire graph, foliated by parabolas
(FKKRUY, see also S. Akamine, arXiv:1510.07451)

An entire ZMC graph



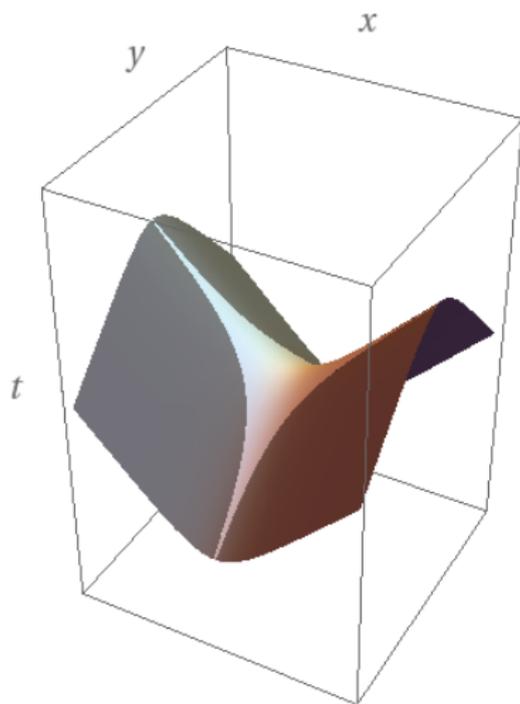
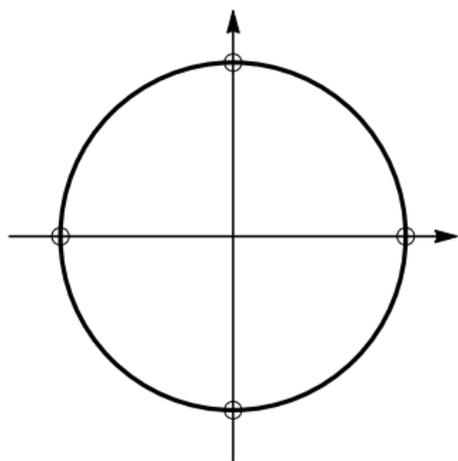
An entire graph $t = x \tanh y$ (O. Kobayashi, 1983)

An entire ZMC graph



An entire graph (FKKRUY)

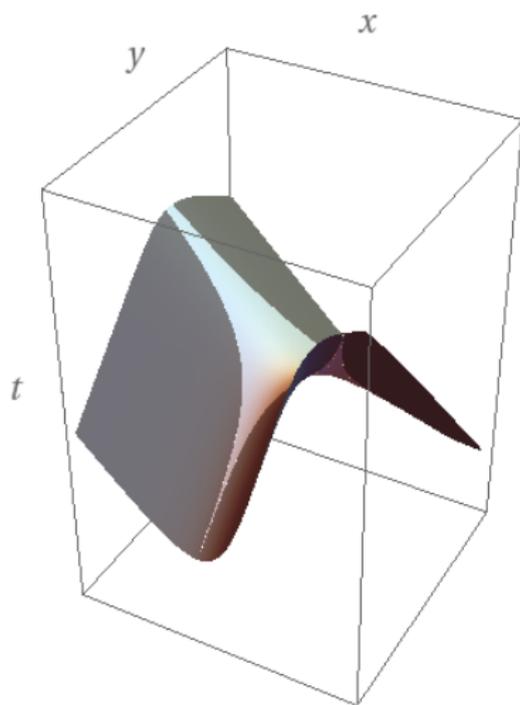
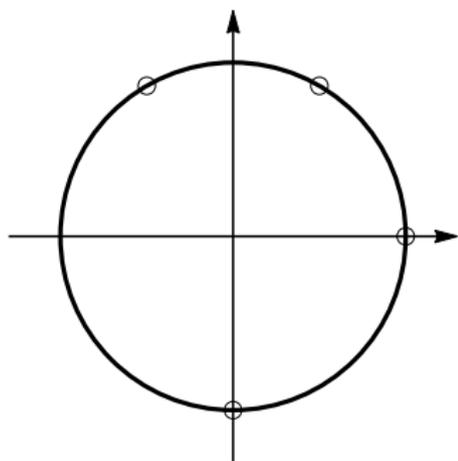
An entire ZMC graph



*

An entire graph $t = \log \frac{\cosh x}{\cosh y}$ (O. Kobayashi, 1983)

An entire ZMC graph



An entire graph (FKKRUY)

Type changes for ZMC graph

Fact (Klyachin (2003))

A type change of ZMC graph occurs along either

- *a non-degenerate null curve*
- *or a null line.*

There are many examples of the first case. The second case?

Theorem (Fujimori-Kim-Koh-Rossmann-Umehara-Yang-Y, 2014)

There exists a ZMC graph which changes type along a null line.



$$f(x, y) = y + \sum_{k=3}^{\infty} \frac{b_k(y)}{k} x^k$$

- Zero mean curvature surfaces in the Lorentz Minkowski space.
 - ▶ Spacelike maximal surfaces: Weierstrass representation
 - ▶ Timelike minimal surfaces.
- ZMC surfaces with type changes:
 - ▶ Analytic extension of fold singularities of maxfaces.
 - ▶ ZMC graph with type changes along a null (lightlike) line
- Many examples of embedded ZMC surfaces:
 - ▶ The analytic extension of Jorge-Meeks type maxface
 - ▶ Kobayashi surfaces
 - ★ entire ZMC graphs
 - ★ Jorge-Meeks type surfaces
- There are many entire solutions of ZMC equation

$$(1 - f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1 - f_x^2)f_{yy} = 0.$$

cf. The minimal surface equation in \mathbf{R}^3 (Bernstein's theorem):

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0.$$

References again



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