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Quantization (matrix regularization) of Lie-Poisson algebra and IKKT matrix model

Abstract: A quantization of Lie-Poisson algebras is studied. Classical solutions of the mass-deformed IKKT matrix model can be constructed from semisimple Lie algebras whose dimension matches the number of matrices in the model. We consider the geometry described by the classical solutions of the Lie algebras in the limit where the mass vanishes and the matrix size tends to infinity. Lie-Poisson varieties are regarded as such geometric objects. We provide a quantization called "weak matrix regularization" of Lie-Poisson algebras (linear Poisson algebras) on the algebraic varieties defined by their Casimir polynomials. In order to define the weak matrix regularization of the quotient space by the ideal generated by the Casimir polynomials, we take a fixed reduced Gröbner basis of the ideal. The Gröbner basis determines remainders of polynomials. The operation of replacing this remainders with representation matrices of a Lie algebra roughly corresponds to a weak matrix regularization.

To describe this process more concretely, we can summarize it as follows. We consider a Lie algebra $\mathfrak{g}$ satisfying certain conditions, which includes all semisimple Lie algebras. Among corresponding Lie–Poisson algebras, we denote by $A_{\mathfrak{g}}$ the one obtained by simply endowing the coordinate polynomial ring of Euclidean space with a Poisson structure. Casimir polynomials correspond with Casimir operators of the Lie algebra by the quantization. Let $I(C)$ denote the ideal generated by a certain Casimir polynomial. Then $A_{\mathfrak{g}}/I(C)$ is also obtained as a Lie–Poisson algebra. Let r_f denote the remainder obtained by dividing a certain $f \in A_{\mathfrak{g}}$ by a Gröbner basis of $I(C)$. Let V^μ be the representation space of an irreducible representation of the Lie algebra $\mathfrak{g}$. Then the matrix regularization $A_{\mathfrak{g}}/I(C) \to gl(V^\mu)$ is constructed by $q_{A/I,\mu} := R_\mu \circ \rho_{U/I,\mu} \circ q_{U/I}$.

$$
\begin{tikzcd}
A_{\mathfrak{g}}/I(C) \arrow[r, "q_{U/I}\in Q"] \arrow[rr, bend left, "q_\mu^{pre}\in Q"] \arrow[rrr, bend left, "q_{A/I\mu}\in Q"] & \mathcal{U}_{\mathfrak{g}}[\hbar]/I(C(X)) \arrow[r, "\rho_{U/I,\mu}"] & gl(V^\mu) \arrow[r, "R_\mu"] & gl(V^\mu) \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\big[f(x)\big] = \big[r_f + h_f\big] \arrow[r, mapsto] & {[q_U(r_f)] = \big[\sum_I a_I X_{(i_1,\cdots,i_m)}\big]} \arrow[r, mapsto] & \sum_I a_I e^{(\mu)}_{(i_1,\cdots,i_m)} \arrow[r, mapsto] & \sum_{|I|=m<n_\mu} a_I e^{(\mu)}_{(i_1,\cdots,i_m)}
\end{tikzcd}
$$

Here $q_{U/I} : A_{\mathfrak{g}}/I(C) \to \mathcal{U}_{\mathfrak{g}}[\hbar]/I(C(X))$ is a quantization from $A_{\mathfrak{g}}/I(C)$ to enveloping algebra devided by the ideal made from $I(C)$. $\rho_{U/I,\mu}$ is an expression of $\mathcal{U}_{\mathfrak{g}}[\hbar]/I(C(X))$ to $gl(V^\mu)$, and R_μ is a projection operator that restricts the degree of the polynomial ring.

As concrete examples, we construct weak matrix regularization for $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$. In the case of $\mathfrak{su}(3)$, we not only construct weak matrix regularization for the quadratic Casimir polynomial, but also construct weak matrix regularization for the cubic Casimir polynomial.

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