The mean curvature flow for an invariant hypersurface in a Hilbert space with an almost free group action

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Contents

25. The mean curvature flow for a regularizable submanifold

The mean curvature flow for a regularizable submanifold

V : an ∞-dimensional (separable) Hilbert space

$f : M \hookrightarrow V$: a proper Fredholm submanifold

$$\left(\begin{array}{l} \bullet\ \mathrm{codim}\, M < \infty, \\ \bullet\ \exp^\perp|_{B^\perp 1(M)} \ : \text{proper map} \\ \bullet\ \exp^\perp_{*v} \ : \text{Fredholm op.}\ (\forall\, v \in T^\perp M) \end{array}\right)$$

The mean curvature flow for a regularizable submanifold

Furthermore, assume that f is regularizable, that is,

$$\left\{ \begin{array}{l} \forall\, v \in T^{\perp}M, \\ \exists\, \mathrm{Tr}_r A_v \,(<\infty), \quad \exists\, \mathrm{Tr}(A_v^2)\,(<\infty) \\ \left(\begin{array}{l} \mathrm{Tr}_r A_v := \displaystyle\sum_{i=1}^{\infty}(\lambda_i + \mu_i) \\ (\mathrm{Spec}\, A_v = \{\mu_1 \leq \mu_2 \leq \cdot\cdot \leq 0 \leq \cdot\cdot \leq \lambda_2 \leq \lambda_1\}) \\ \mathrm{Tr}(A_v^2) := \displaystyle\sum_{i=1}^{\infty}\nu_i \\ (\mathrm{Spec}\, A_v^2 = \{\nu_1 \geq \nu_2 \geq \cdots > 0\}) \end{array} \right) \end{array} \right. .$$

The mean curvature flow for a regularizable submanifold

Definition(regularized mean curvature vector).

$$H \underset{\text{def}}{\iff} \langle H, v\rangle = \mathrm{Tr}_r A_v \ (\forall\, v \in T^{\perp}M)$$

This normal vector field H is called **the regularized mean curvature** of f.

$$\triangle_r f \underset{\text{def}}{\iff} \langle \triangle_r f, v\rangle = \mathrm{Tr}_r \langle (\nabla df)(\cdot,\cdot), v\rangle^{\sharp} \quad (\forall\, v \in T^{\perp}M)$$

(∇ : the Riemannian connection of the metric of M induced by f)

Then we have $H = \triangle_r f$.

The mean curvature flow for a regularizable submanifold

$f_t : M \hookrightarrow V \ (0 \leq t < T)$: C^∞-family of regularizable submanifolds

$$F : M \times [0, T) \to V$$

$$\underset{\text{def}}{\Longleftrightarrow} F(x, t) := f_t(x) \ ((x, t) \in M \times [0, T))$$

The mean curvature flow for a regularizable submanifold

Definition

$f_t \ (0 \leq t < T)$: **a (regularized) mean curvature flow**

$$\underset{\text{def}}{\Longleftrightarrow} \ \frac{\partial F}{\partial t} = H_t \ (0 \leq t < T)$$

Question.

For any regularizable submanifold f, does the mean curvature flow for f uniquely exist in short time?

The mean curvature flow for a regularizable submanifold

G : a Hilbert Lie group

$G \curvearrowright V$: an almost free isometric action such that the orbits are minimal regularizable submanifolds

V/G : the orbit space of the G-action (which is an orbifold)

$\phi : V \to V/G$: the orbit map of the G-action

The mean curvature flow for a regularizable submanifold

M : a Hilbert manifold

$f : M \hookrightarrow V$: a regularizable submanifold

Fact

If $f(M)$ is G-invariant and if $(\phi \circ f)(M)$ is compact, then the mean curvature flow for f uniquely exists in short time.

The mean curvature flow for a regularizable submanifold

Example

(G, K) : a compact symmetric pair

Γ : a discrete subgroup of G

$$P(G, \Gamma \times K) := \{g \in H^1([0,1], G) \,|\, ((g(0), g(1)) \in \Gamma \times K\}$$

The mean curvature flow for a regularizable submanifold

$P(G, \Gamma \times K)$ **acts on** $H^0([0,1], \mathfrak{g})$ **isometrically and almost freely as the Gauge action (on the space of the connections) and its orbits are minimal regularizable submanifolds.**

Also, we have $H^0([0,1], \mathfrak{g})/P(G, \Gamma \times K) = \Gamma \backslash G/K$.

2. The mean curvature flow for a Riemannian suborbifold

The mean curvature flow for a Riemannian suborbifold

M : a paracompact Hausdorff space

$(U, \phi, \widehat{U}/\Gamma)$: a triple s.t.

$$\begin{cases} \text{(i) } U \text{ is an open set of } M \\ \text{(ii) } \widehat{U} \text{ is an open set of } \mathbb{R}^n \\ \text{(iii) } \Gamma \text{ is a finite subgroup of } \mathrm{Diff}^\infty(\widehat{U}) \\ \text{(iv) } \phi \text{ is a homeomorphism of } U \text{ onto } \widehat{U}/\Gamma \end{cases}$$

Such a triple $(U, \phi, \widehat{U}/\Gamma)$ is called an **n-dimensional orbifold chart**.

The mean curvature flow for a Riemannian suborbifold

Let $\mathcal{O} := \{(U_\lambda, \phi_\lambda, \widehat{U}/\Gamma_\lambda) \,|\, \lambda \in \Lambda\}$ be a family of n-dimensional orbifold charts of M s.t.

$$\begin{cases} \text{(O1) } \{U_\lambda \,|\, \lambda \in \Lambda\} \text{ is an open covering of } M \\ \text{(O2) For } \lambda, \mu \in \Lambda \text{ with } U_\lambda \cap U_\mu \neq \emptyset, \\ \quad \text{there exists an } n\text{-dimensional orbifold chart} \\ \quad (W, \psi, \widehat{W}/\Gamma') \text{ s.t. } C^\infty\text{-embeddings } \rho_\lambda : \widehat{W} \hookrightarrow \widehat{U}_\lambda \\ \quad \text{and } \rho_\mu : \widehat{W} \hookrightarrow \widehat{U}_\mu \text{ s.t. } \phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} \circ \rho_\lambda = \psi^{-1} \circ \pi_{\Gamma'} \\ \quad \text{and } \phi_\mu^{-1} \circ \pi_{\Gamma_\mu} \circ \rho_\mu = \psi^{-1} \circ \pi_{\Gamma'}, \end{cases}$$

The mean curvature flow for a Riemannian suborbifold

where π_{Γ_λ}, π_{Γ_μ} and $\pi_{\Gamma'}$ are the orbit maps of Γ_λ, Γ_μ and Γ', respectively.

Such a family $\mathcal{O}$ is called

an n-dimensional (C^∞-)orbifold atlas of M

and the pair $(M, \mathcal{O})$ is called

an n-dimensional (C^∞-)orbifold.

The mean curvature flow for a Riemannian suborbifold

$(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$: an n-dimensional orbifold chart around $x \in M$.

$(\Gamma_\lambda)_{\widehat{x}} := \{b \in \Gamma_\lambda \mid b(\widehat{x}) = \widehat{x}\}$

The conjugate class of $(\Gamma_\lambda)_{\widehat{x}}$ is called the local group at x. If $(\Gamma_\lambda)_{\widehat{x}}$ is not trivial, then x is called a singular point of $(M, \mathcal{O})$.

Denote by $\mathrm{Sing}(M)$ the set of all singular points of $(M, \mathcal{O})$.

The mean curvature flow for a Riemannian suborbifold

$(M, \mathcal{O}_M)$, $(N, \mathcal{O}_N)$: orbifolds

f : a map from M to N

If, for each $x \in M$ and each pair of an orbifold chart $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathcal{O}_M)$ around x and an orbifold chart $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathcal{O}_N)$ around $f(x)$ $(f(U_\lambda) \subset V_\mu)$, there exists a C^k-map $\widehat{f}_{\lambda,\mu} : \widehat{U}_\lambda \to \widehat{V}_\mu$ with $f \circ \phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} = \psi_\mu^{-1} \circ \pi_{\Gamma'_\mu} \circ \widehat{f}_{\lambda,\mu}$, then f is called a C^k-**orbimap**.

The mean curvature flow for a Riemannian suborbifold

Also $\widehat{f}_{\lambda,\mu}$ is called a local lift of f with respect to $(U_\lambda, \phi_\lambda, \widehat{U}_\lambda/\Gamma_\lambda)$ and $(V_\mu, \psi_\mu, \widehat{V}_\mu/\Gamma'_\mu)$.
Furthermore, if each local lift $\widehat{f}_{\lambda,\mu}$ is an immersion, then f is called a C^k-orbiimmersion and $(M, \mathcal{O}_M)$ is called a C^k-(immersed) suborbifold in $(N, \mathcal{O}_N, g)$. Similarly, if each local lift $\widehat{f}_{\lambda,\mu}$ is a submersion, then f is called a C^k-orbisubmersion.

The mean curvature flow for a Riemannian suborbifold

For an orbifold $(M, \mathcal{O})$, the orbitangent bundle $T_{\mathrm{orb}}M$ and the (r, s)-orbitensor bundle $T^{(r,s)}_{\mathrm{orb}}M$ over $(M, \mathcal{O})$ are defined naturally.

pr : the natural proj. of $T_{\mathrm{orb}}M$ (or $T^{(r,s)}_{\mathrm{orb}}M$) onto M

A C^k-orbimap $X : M \to T_{\mathrm{orb}}M$ s.t. $\mathrm{pr} \circ X = \mathrm{id}$ is called a C^k-orbitangent vector field on $(M, \mathcal{O}_M)$ and a C^k-orbimap $S : M \to T^{(r,s)}M$ s.t. $\mathrm{pr} \circ S = \mathrm{id}$ is called a (r, s)-orbitensor field of class C^k on $(M, \mathcal{O}_M)$.

The mean curvature flow for a Riemannian suborbifold

Definition.

If a (r, s)-orbitensor field g of class C^k on $(M, \mathcal{O}_M)$ is positive definite and symmetric, then we call g a C^k-Riemannian orbimetric and $(M, \mathcal{O}_{\mathcal{M}}, g)$ a C^k-Riemannian orbifold.

The mean curvature flow for a Riemannian suborbifold

f : a C^∞-orbiimmersion of C^∞-orbifold $(M, \mathcal{O}_M)$ into C^∞-Riemannian orbifold $(N, \mathcal{O}_N, g)$

Then, the orbinormal bundle $T^\perp_{\mathrm{orb}} M$ of f and the orbitensor bundle $T^{(r,s)}_{\mathrm{orb}} M \otimes T^\perp_{\mathrm{orb}} M$ are defined naturally.

The mean curvature flow for a Riemannian suborbifold

g : the induced metric of $f|_{M\setminus\mathrm{Sing}(M)}$

h : the second fundamental form of $f|_{M\setminus\mathrm{Sing}(M)}$

A : the shape operator of $f|_{M\setminus\mathrm{Sing}(M)}$

H : the mean curvature vec. of $f|_{M\setminus\mathrm{Sing}(M)}$

ξ : a unit normal vec. fd. of $f|_{M\setminus\mathrm{Sing}(M)}$

It is easy to show that g, h, A, H extend a $(0,2)$-orbitensor field of class C^∞ on $(M, \mathcal{O}_M)$, a C^k-section of $T^{(0,2)}_{\mathrm{orb}}M \otimes T^\perp_{\mathrm{orb}}M$, a C^k-section of $T^{(1,1)}_{\mathrm{orb}}M \otimes (T^\perp_{\mathrm{orb}}M)^{(0,1)}$ and a C^∞-orbinormal vector field on $(M, \mathcal{O}_M)$.

The mean curvature flow for a Riemannian suborbifold

We denote these extensions by the same symbols. We call these extensions g, h, A and H the induced orbimetric, the second fundamental orbiform, the shape orbitensor and the mean curvature orbivector of f.
Here we note that ξ does not necessarily extend a C^∞-orbinormal vector field on $(M, \mathcal{O})$.

The mean curvature flow for a Riemannian suborbifold

N : an $(n+r)$-dimensional Riemannian orbifold

M : an n-dimensional orbifold

$f : M \hookrightarrow N$: an orbiimmersion

f_t $(0 \leq t < T)$: a C^∞-family of orbiimmersions of M into N s.t. $f_0 = f$

$T^{\perp_t}_{\mathrm{orb}} M$: the normal orbibundle of f_t

$T^{\perp_F}_{\mathrm{orb}} M$: the orbisubbundle of $F^*(T_{\mathrm{orb}} N)$ given by $T^{\perp_t}_{\mathrm{orb}} M$'s

H_t : the mean curvature orbivector of f_t

H : the section of $T^{\perp_F}_{\mathrm{orb}} M$ given by H_t's

The mean curvature flow for a Riemannian suborbifold

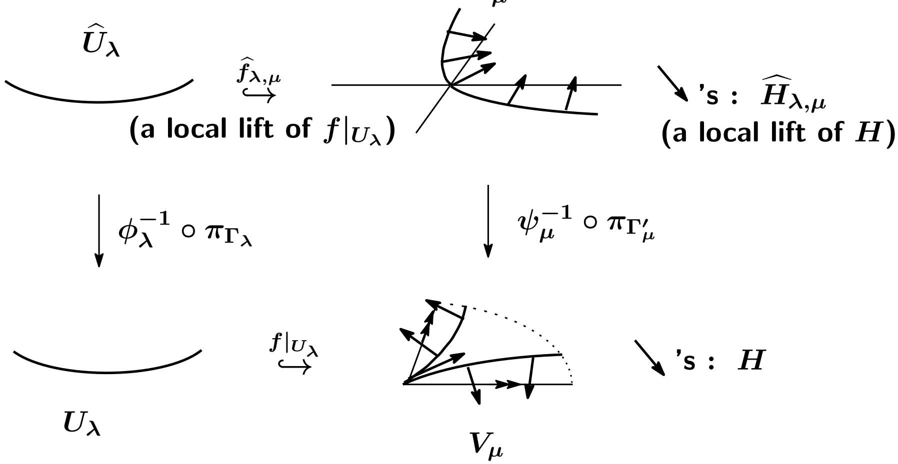
Definition

$f_t\ (0 \leq t < T)$: **a mean curvature flow**

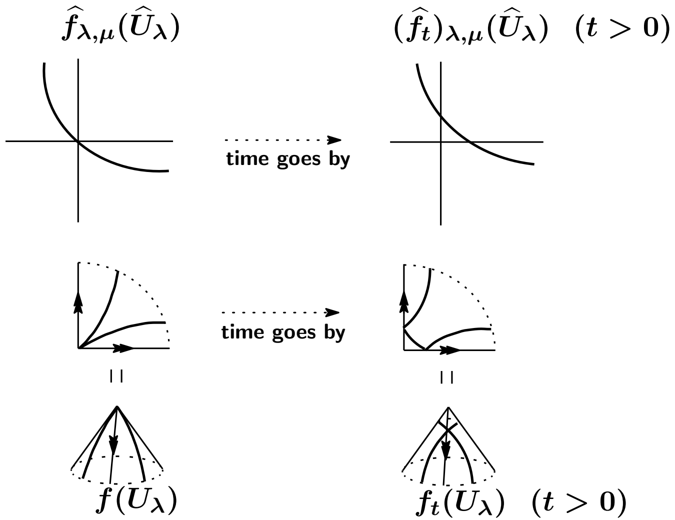
$$\underset{\text{def}}{\Longleftrightarrow} \quad \frac{\partial F}{\partial t} = H \qquad \text{(MCFE)}$$

The mean curvature flow for a Riemannian suborbifold

Example 1.

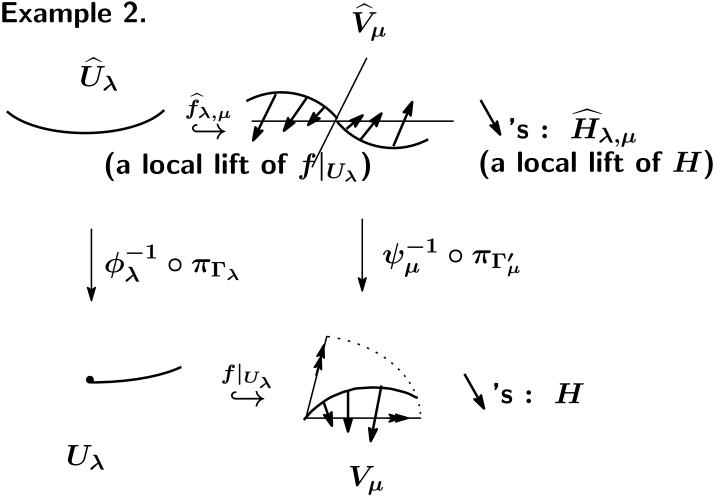


The mean curvature flow for a Riemannian suborbifold

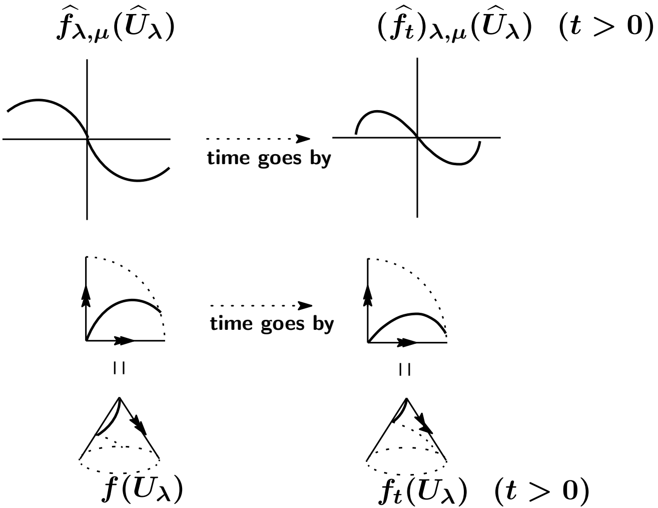


The mean curvature flow for a Riemannian suborbifold

Example 2.



The mean curvature flow for a Riemannian suborbifold



The mean curvature flow for a Riemannian suborbifold

Fact

Assume that M is compact. Then, for any orbiimmersion f of M into N, the mean curvature flow for f uniquely exists in short time.

The mean curvature flow for a Riemannian suborbifold

Proof

Let $f : M \hookrightarrow N$ be an orbiimmersion.

Since M is compact,

there exists a finite open covering $\{U_i \mid i = 1, \cdots, k\}$ of M s.t. each $f|_{U_i}$ admits a local lift $\widetilde{f}_i : \widetilde{U}_i \hookrightarrow \widetilde{V}_i$.

Take the mean curvature flow $(\widetilde{f}_i)_t$ $(0 \leq t < T_i)$ for $\widetilde{f}_i$.

Let $(f_i)_t \, (0 \leq t < T_i)$ be the mean curvature flow for f_i arising from $(\widetilde{f}_i)_t$ $(0 \leq t < T_i)$.

The mean curvature flow for a Riemannian suborbifold

Set $T := \min\{T_i \mid i = 1, \cdots, k\}$.

By patching $(f_i)_t\,(0 \leq t < T)$'s $(i = 1, \cdots, k)$, we obtain the mean curvature flow $f_t\,(0 \leq t < T)$ for f. q.e.d.

27. The evolutions of the geometric quantities

The evolutions of the geometric quantities

V : an ∞-dimensional (separable) Hilbert space

G : a Hilbert Lie group

$G \curvearrowright V$: an almost free isometric action s.t. the orbits are minimal regularizable submanifolds

V/G : the orbit space of the G-action (which is an orbifold)

$\phi : V \to V/G$: the orbit map of the G-action

The evolutions of the geometric quantities

$f : M \hookrightarrow V$: a G-invariant regularizable submanifold such that $(\phi \circ f)(M)$ is compact

$f_t : M \hookrightarrow V \ (0 \leq t < T)$: the mean curvature flow for f

$$F : M \times [0, T) \to V$$

$$\underset{\text{def}}{\Longleftrightarrow} \ F(x,t) := f_t(x) \ \ ((x,t) \in M \times [0,T))$$

The evolutions of the geometric quantities

$\mathcal{H}_t$: **the horizontal distribution of the Riemannian submersion** $\phi \circ f_t$ (: $M \to (\phi \circ f_t)(M)$)

$\mathcal{H}$: **the subbundle of** $\pi_M^*(TM)$ **given by** $\mathcal{H}_t$**'s**

$\mathrm{pr}_{\mathcal{H}}$: **the bundle orthog. proj. of** $\pi_M^*(TM)$ **onto** $\mathcal{H}$

The evolutions of the geometric quantities

g_t : **the metric of M induced by f_t**

h_t : **the second fundamental form of f_t**

A_t : **the shape tensor of f_t**

g : **the section of $\pi_M^*(T^{(0,2)}M)$ given by g_t's**

h : **the section of $\pi_M^*(T^{(0,2)}M)$ given by h_t's**

A : **the section of $\pi_M^*(T^{(1,1)}M)$ given by A_t's**

The evolutions of the geometric quantities

$g_{\mathcal{H}}, h_{\mathcal{H}}, A_{\mathcal{H}}$: **the horizontal components of** g, h, A

$$\left(\begin{array}{c} g_{\mathcal{H}} := g \circ (\mathrm{pr}_{\mathcal{H}} \times \mathrm{pr}_{\mathcal{H}}), \;\; h_{\mathcal{H}} := h \circ (\mathrm{pr}_{\mathcal{H}} \times \mathrm{pr}_{\mathcal{H}}), \\ A_{\mathcal{H}} := \mathrm{pr}_{\mathcal{H}} \circ A \circ \mathrm{pr}_{\mathcal{H}} \end{array} \right)$$

The evolutions of the geometric quantities

∇ : the connection of $\pi_M^*(TM)$ given by the Riem. conn. ∇^t's of g_t

$$\left(\begin{array}{r} (\nabla_X Y)_{(u,t)} := (\nabla^t_X Y)_u, \;\; (\nabla_{\frac{\partial}{\partial t}} Y)_{(u,t)} = \dfrac{dY_{(u,\cdot)}}{dt} \\ (X,\, Y \in \Gamma(\pi_M^*(TM))) \end{array} \right)$$

$\nabla^{\mathcal{H}}$: the connection of $\mathcal{H}$ given by ∇^t's

$$\left(\begin{array}{r} (\nabla^{\mathcal{H}}_X Y)_{(u,t)} := \mathrm{pr}_{\mathcal{H}_t}((\nabla^t_X Y)_u), \;\; (\nabla^{\mathcal{H}}_{\frac{\partial}{\partial t}} Y)_{(u,t)} = \dfrac{dY_{(u,\cdot)}}{dt} \\ (X \in \Gamma(\pi_M^*(TM)),\, Y \in \Gamma(\mathcal{H})) \end{array} \right)$$

The evolutions of the geometric quantities

Denote by the same symbol ∇ the connection of $\pi_M^*(T^{(r,s)}M)$ induced from ∇.

Similarly, denote by the same symbol $\nabla^{\mathcal{H}}$ the connection of $\mathcal{H}^{(r,s)}$induced from $\nabla^{\mathcal{H}}$.

$\triangle_{\mathcal{H}}$: the Laplace operator defined in terms of $\nabla^{\mathcal{H}}$

$$\left(\begin{array}{c} (\triangle_{\mathcal{H}} S)_{(u,t)} := \sum_{i=1}^{n} \nabla^{\mathcal{H}}_{e_i} \nabla^{\mathcal{H}}_{e_i} S \ \ (S \in \Gamma(\pi_M^*(T^{(r,s)}M))) \\ ((e_1, \cdots, e_n) \text{ : an orthon. base of } \mathcal{H}_{(u,t)} \text{ w.r.t. } (g_t)_u) \end{array} \right)$$

The evolutions of the geometric quantities

$\mathcal{A}^{\phi}$: the O'Neill' fundamental tensor of the Riemannian orbisubmersion ϕ
($\mathcal{A}^{\phi}$ is the obstruction of the integrability of the horizontal distribution $\mathcal{H}^{\phi}$ of ϕ)

$\mathcal{A}$: the section of $\pi_M^*(T^{(1,2)}M)$ induced from the O'Neill's fundamental tensors $\mathcal{A}_t$'s of the Riemannian orbisubmersions $\phi \circ f_t$'s

The evolutions of the geometric quantities

Theorem 27.1.

- $$\frac{\partial g_{\mathcal{H}}}{\partial t} = -2||H||h_{\mathcal{H}}$$
- $$\frac{\partial ||H||}{\partial t} = \triangle_{\mathcal{H}}||H|| + ||H||\mathrm{Tr}(A_{\mathcal{H}})^2 - 3||H||\mathrm{Tr}((\mathcal{A}_{\xi}^{\phi})^2)_{\mathcal{H}}$$

The evolutions of the geometric quantities

Theorem 27.1(continued)

- $$\begin{aligned}\frac{\partial h_{\mathcal{H}}}{\partial t}(X,Y) = & (\triangle_{\mathcal{H}} h_{\mathcal{H}})(X,Y) - 2\|H\| h_{\mathcal{H}}(A_{\mathcal{H}}X,Y) \\ & + \mathrm{Tr}\left((A_{\mathcal{H}})^2 - (\mathcal{A}^{\phi}_{\xi})^2\right) h(X,Y) \\ & - 3\|H\| g_{\mathcal{H}}((\mathcal{A}^{\phi}_{\xi})^2(X),Y) - 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h(\mathcal{A}_{\bullet}X, \mathcal{A}_{\bullet}Y) \\ & + \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}X),Y) + \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}Y),X) \\ & - \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h((\nabla_{\bullet}\mathcal{A})_{\bullet}X,Y) - \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h((\nabla_{\bullet}\mathcal{A})_{\bullet}Y,X) \\ & \qquad\qquad (X,Y \in \mathcal{H})\end{aligned}$$

Remark $\displaystyle \frac{\partial g_{\mathcal{H}}}{\partial t} := \nabla_{\frac{\partial}{\partial t}} g_{\mathcal{H}}, \quad \frac{\partial h_{\mathcal{H}}}{\partial t} := \nabla_{\frac{\partial}{\partial t}} h_{\mathcal{H}}.$

The evolutions of the geometric quantities

The outline of the proof

ξ : the section of $\Gamma(F^*(TV))(= C^\infty(M \times [0,T), V))$
given by the unit normal vector fields ξ_t's of f_t's

For $X \in \Gamma(TM)$,

$$\overline{X} \in \Gamma(\pi_M^*(TM)) \underset{\text{def}}{\Longleftrightarrow} \overline{X}_{(u,t)} := X_u \ \ ((u,t) \in M \times [0,T))$$

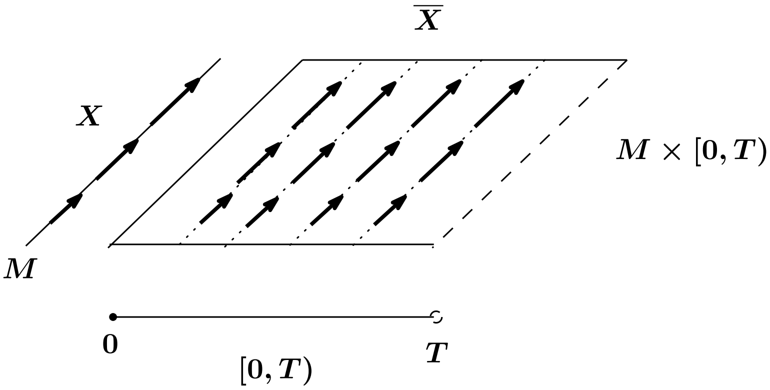
Then we have

$$\left[\frac{\partial}{\partial t}, \overline{X}\right] = 0, \quad \left[\frac{\partial}{\partial t}, \overline{X}_{\mathcal{H}}\right] = 2||H||\mathcal{A}_\xi^\phi \overline{X}_{\mathcal{H}}$$

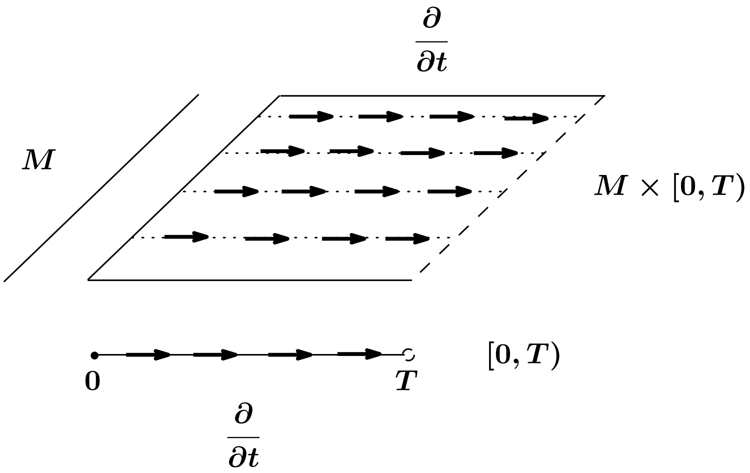
Also we have

$$A\overline{X} = A_{\mathcal{H}}\overline{X} + \mathcal{A}_\xi^\phi \overline{X},$$
$$(A^2)_{\mathcal{H}}\overline{X} = (A_{\mathcal{H}})^2\overline{X} - (\mathcal{A}_\xi^\phi)^2\overline{X}$$

The evolutions of the geometric quantities



The evolutions of the geometric quantities



The evolutions of the geometric quantities

$$\begin{aligned}\frac{\partial h_{\mathcal{H}}}{\partial t}(X,Y) &= \frac{\partial}{\partial t}(h_{\mathcal{H}}(\overline{X},\overline{Y})) - h_{\mathcal{H}}(\left[\frac{\partial}{\partial t},\overline{X}\right],\overline{Y}) \\ &\quad -h_{\mathcal{H}}(\overline{X},\ \left[\frac{\partial}{\partial t},\overline{Y}\right]) \\ &= \frac{\partial}{\partial t}\langle \xi,\ \bar{X}_{\mathcal{H}}(\bar{Y}_{\mathcal{H}}F)\rangle \\ &= \langle \frac{\partial \xi}{\partial t},\ \bar{X}_{\mathcal{H}}(\bar{Y}_{\mathcal{H}}F)\rangle + \langle \xi,\ \bar{X}_{\mathcal{H}}\left(\left[\frac{\partial}{\partial t},\overline{Y}_{\mathcal{H}}\right]F\right)\rangle \\ &\quad +\langle \xi,\ \left[\frac{\partial}{\partial t},\overline{X}_{\mathcal{H}}\right](\bar{Y}_{\mathcal{H}}F)\rangle\end{aligned}$$

(Here we use that V is a linear space.)

Furthermore, by using the previous relations, we obtain

The evolutions of the geometric quantities

$$\frac{\partial h_{\mathcal{H}}}{\partial t}(X,Y) = (\nabla d||H||)(X,Y) - ||H||g((A_{\mathcal{H}})^2X,Y)$$
$$-4||H||g((\mathcal{A}^{\phi}_{\xi})^2X,Y)$$

On the other hand, we have

$$\begin{aligned}
&(\triangle_{\mathcal{H}}h_{\mathcal{H}})(X,Y)\\
=\;&(\nabla d||H||)(X,Y) + ||H||g((A_{\mathcal{H}})^2X,Y) - ||H||g((\mathcal{A}^{\phi}_{\xi})^2X,Y)\\
&-\mathrm{Tr}((A_{\mathcal{H}})^2 - (\mathcal{A}^{\phi}_{\xi})^2)h(X,Y)\\
&+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}((\nabla_{\bullet}h)(\mathcal{A}_{\bullet}X,Y)) + 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}((\nabla_{\bullet}h)(\mathcal{A}_{\bullet}Y,X))\\
&-\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}X),Y) - \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}Y),X)\\
&+\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\nabla_{\bullet}\mathcal{A})_{\bullet}X,Y) + \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\nabla_{\bullet}\mathcal{A})_{\bullet}Y,X)\\
&+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(\mathcal{A}_{\bullet}X,\mathcal{A}_{\bullet}Y)
\end{aligned}$$

From these relations, we obtain the desired evolution eq.

q.e.d.

The evolutions of the geometric quantities

$\rho_t : G \times M \to M$: the action on M induced from the action $G \curvearrowright V$ by f_t

Set
$$G_t := \rho_t(G).$$

Then we have

Fact

g_t and h_t are C^∞-families of G_t-invariant symmetric $(0,2)$-tensor fields on M.

28. The maximum principle

The maximum principle

G : a Hilbert Lie group

M : a Hilbert manifold

$g_t\ (t \in [0, T))$: a C^∞-family of G-invariant Riemannian metrics of M

g : the section of $\pi_M^*(T^{(0,2)}M)$ given by g_t's

The maximum principle

For $B \in \Gamma(\pi_M^*(T^{(r_0,s_0)}M))$, define $\psi_{B\otimes} : \Gamma(\pi_M^*(T^{(r,s)}M)) \to \Gamma(\pi_M^*(T^{(r+r_0,s+s_0)}M))$ by

$$\psi_{B\otimes}(S) := B \otimes S \quad (S \in \Gamma(\pi_M^*(T^{(r,s)}M))).$$

Define $\psi_{\otimes^k} : \Gamma(\pi_M^*(T^{(r,s)}M)) \to \Gamma(\pi_M^*(T^{(kr,ks)}M))$ by

$$\psi_{\otimes^k}(S) := S \otimes \cdots \otimes S \ (k\text{−times}) \quad (S \in \Gamma(\pi_M^*(T^{(r,s)}M))).$$

The maximum principle

Also, define
$\psi_{g_{\mathcal{H}},ij}:\Gamma(\pi_M^*(T^{(r,s)}M))\to\Gamma(\pi_M^*(T^{(r,s-2)}M))$ by

$$\begin{aligned}&(\psi_{g_{\mathcal{H}},ij}(S))_{(x,t)}(X_1,\cdots,X_{s-2})\\ :=&\sum_{k=1}^{n}S_{(x,t)}(X_1,\cdots,\underset{i}{e_k},\cdots,\underset{j}{e_k},\cdots,X_{s-2})\\ &\qquad(S\in\Gamma(\pi_M^*(T^{(r,s)}M)),\ \ X_1,\cdots,X_{s-2}\in T_xM),\end{aligned}$$

where $\{e_1,\cdots,e_n\}$ is an orthon. base of $\mathcal{H}_{(x,t)}$ w.r.t. $(g_{\mathcal{H}})_t$.

The maximum principle

Also, define
$\psi_{\mathcal{H},i}:\Gamma(\pi_M^*(T^{(r,s)}M))\to\Gamma(\pi_M^*(T^{(r-1,s-1)}M))$ **by**

$$\begin{aligned}&(\psi_{\mathcal{H},i}(S))_{(x,t)}(X_1,\cdots,X_{s-1})\\ &:=\mathrm{Tr}(\mathrm{pr}_{\mathcal{H}_{(x,t)}}\circ S_{(x,t)}(X_1,\cdots,X_{i-1},\bullet,X_i,\cdots,X_{s-1})|_{\mathcal{H}_{(x,t)}})\\ &\qquad(S\in\Gamma(\pi_M^*(T^{(r,s)}M)),\ \ X_1,\cdots,X_{s-1}\in T_xM).\end{aligned}$$

The maximum principle

P : a map from $\Gamma(\pi_M^*(T^{(r,s)}M))$ to $\Gamma(\pi_M^*(\bigoplus_{r',s'=0}^{\infty} T^{(r',s')}M))$

Definition(a map of polynomial type).

If P is given as the sum of the compositions of the above five types of maps $\psi_{B\otimes}$, $\psi_{\otimes B}$, $\psi_{\otimes^k}$, $\psi_{g_{\mathcal{H}},ij}$, $\psi_{\mathcal{H},i}$, then we say that P is **of polynomial type**.

The maximum principle

P : a map of polynomial type from $\Gamma(\pi_M^*(T^{(0,2)}M))$ to oneself

Definition(horizontally null vector condition).

Assume that, for any $S \in \Gamma(\pi_M^*(T^{(0,2)}M))$ and any $(x,t) \in M \times [0,T)$,

$$X \in \mathrm{Ker}\,(S_{\mathcal{H}})_{(x,t)} \implies P(S)_{(x,t)}(X,X) \geq 0.$$

Then we say that

P satisfies the horizontally null vector condition.

The maximum principle

P : a map of polynomial type from $\Gamma(\pi_M^*(M \times \mathbb{R}))$ to oneself

Definition(null vector condition (function version)).

Assume that, for any $\rho \in \Gamma(\pi_M^*(M \times \mathbb{R}))$ and any $(x, t) \in M \times [0, T)$,

$$\rho(x, t) = 0 \implies P(\rho)(x, t) \geq 0.$$

Then we say that P satisfies the null vector condition.

The maximum principle

$\nabla^t\,(0 \le t < T)$: **the Riemannian connection of** g_t

∇ : **the connection of** $\pi_M^*(TM)$ **defined by** ∇^t**'s**

The maximum principle

S : an element of $\Gamma(\pi_M^*(T^{(0,2)}M))$ s.t.
S_t $(0 \leq t < T)$ are G-invariant and symmetric

Theorem 28.1(Maximum principle).

Assume that S satisfies

$$\frac{\partial S_{\mathcal{H}}}{\partial t} = \triangle_{\mathcal{H}} S_{\mathcal{H}} + \nabla^{\mathcal{H}}_{\bar{X}_0} S_{\mathcal{H}} + P(S)_{\mathcal{H}}$$

(
- X_0 : an element of $\Gamma(TM)$
- P : a map of polynomial type of $\Gamma(\pi_M^*(T^{(0,2)}M))$ to oneself satisfying the horizontally null vector condition

)

If $(S_{\mathcal{H}})_{(\cdot,0)} \geq 0$ (resp. $(S_{\mathcal{H}})_{(\cdot,0)} > 0$), then $(S_{\mathcal{H}})_{(\cdot,t)} \geq 0$ (resp. $(S_{\mathcal{H}})_{(\cdot,t)} > 0$) holds for all $t \in [0, T)$.

The maximum principle

ρ : an element of $\Gamma(\pi_M^*(M\times\mathbb{R}))$ s.t.
$\rho_t\ (0\le t<T)$ are G-invariant

Theorem 28.2(Maximum principle).

Assume that ρ satisfies $\dfrac{\partial\rho}{\partial t}=\triangle_{\mathcal{H}}\rho+d\rho(\bar{X}_0)+P(\rho)$

$$\left(\begin{array}{l}\bullet\ X_0\ :\ \text{an element of }\Gamma(TM)\\ \bullet\ P\ :\ \text{a map of polynomial type of }\Gamma(\pi_M^*(M\times\mathbb{R}))\\ \qquad\qquad \text{to oneself satisfying the null vector condition}\end{array}\right).$$

If $\rho_0\ge 0$ (resp. $\rho_0>0$), then $\rho_t\ge 0$ (resp. $\rho_t>0$) holds for all $t\in[0,T)$.

29. The horizontally strictly convexity-preservability theorem

The horizontally strictly convexity-preservability theorem

G : a Hilbert Lie group

V : a Hilbert space

$G \curvearrowright V$: an almost free isometric action s.t. the orbits are minimal regularizable submanifolds

$\phi : V \to V/G$: the orbit map

The horizontally strictly convexity-preservability theorem

$f : M \hookrightarrow V$: **a regularizable hypersurface such that $f(M)$ is G-invariant and $(\phi \circ f)(M)$ is compact**

$f_t \ (0 \le t < T)$: **the mean curvature flow for f**

The horizontally strictly convexity-preservability theorem

Define $K \in \Gamma((\mathcal{H}^\phi)^{(0,4)})$ by

$$\begin{aligned}
&K(X,Y,Z,W)\\
&:= (\tilde{\nabla}_X\mathcal{A}^\phi)_Y(\mathcal{A}^\phi_Z W) + \mathcal{A}^\phi_Y((\tilde{\nabla}_X\mathcal{A}^\phi)_Z W)\\
&\quad +(\tilde{\nabla}_X\mathcal{A}^\phi)_Z(\mathcal{A}^\phi_W Y) + \mathcal{A}^\phi_Z((\tilde{\nabla}_X\mathcal{A}^\phi)_W Y)\\
&\quad -2(\tilde{\nabla}_X\mathcal{A}^\phi)_W(\mathcal{A}^\phi_Y Z)\\
&\quad -2\mathcal{A}^\phi_W((\tilde{\nabla}_X\mathcal{A}^\phi)_Y Z)\\
&\qquad\qquad\qquad (X,Y,Z,W \in \mathcal{H}^\phi)
\end{aligned}$$

Set $L := \sup_{u\in V} ||K_u||$. Assume that $L < \infty$.

The horizontally strictly convexity-preservability theorem

Theorem 29.1(Horizontally strictly convexity-preservability th.).

Assume that

$$||H_0||^2(h_{\mathcal{H}})_{(\cdot,0)} > 2n^2 L(g_{\mathcal{H}})_{(\cdot,0)}.$$

Then $T < \infty$ **and**

$$||H_t||^2(h_{\mathcal{H}})_{(\cdot,t)} > 2n^2 L(g_{\mathcal{H}})_{(\cdot,t)}$$

holds for any $t \in [0, T)$.

This statement is proved in terms of the evolution equations in Theorem 27.1 and the maximum principle (Theorems 28.1 and 28.2).

The horizontally strictly convexity-preservability theorem

Set $N := V/G$ and $n := \dim N - 1$.

$\overline{M}$ **: a n-dimensional compact manifold**

$\overline{f} : \overline{M} \hookrightarrow N$ **: an orbiimmersion**

$\overline{f}_t \ (0 \leq t < T)$ **: the mean curvature flow for $\overline{f}$**

$\overline{g}_t$, $\overline{h}_t$, $\overline{H}_t$ **: the quantities for $\overline{f}_t$**

The horizontally strictly convexity-preservability theorem

$\overline{R}$: the curvature orbitensor of N

$\overline{\nabla}$: the Riemannian orbiconnection of the orbimetric of N

Set $\overline{L} := \sup_{x \in N} ||\overline{\nabla R}||$. Assume that $\overline{L} < \infty$.

The horizontally strictly convexity-preservability theorem

Theorem 29.2(Strictly convexity-preservability theorem).

Assume that

$$||\overline{H}_0||^2\overline{h}_0 > n^2\overline{L}\overline{g}_0.$$

Then $T < \infty$ **and**

$$||\overline{H}_t||^2\overline{h}_t > n^2\overline{L}\overline{g}_t$$

holds for any $t \in [0, T)$.

This statement is proved by applying Theorem 29.1 to the lift of $\overline{f}_t$ **by** ϕ.

The horizontally strictly convexity-preservability theorem