Certain kind of isoparametric submanifolds in symmetric spaces of non-compact type and Hermann actions

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Content

1. Complex focal radius

Complex focal radius

G/K : a symmetric space of non-compact type

M : a submanifold in G/K

$\psi : T^\perp M \to M$: the normal bundle of M

$\exp^\perp$: the normal exponential map of M

v : a unit normal vec. of M at x

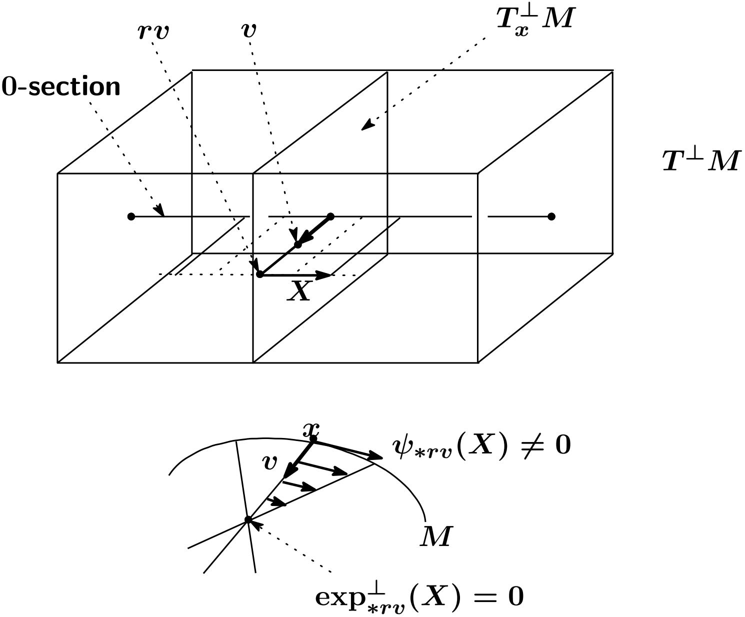
γ_v : the normal geod. s.t. $\gamma_v'(0) = v$
(i.e., $\gamma_v(s) := \exp^\perp(sv)$)

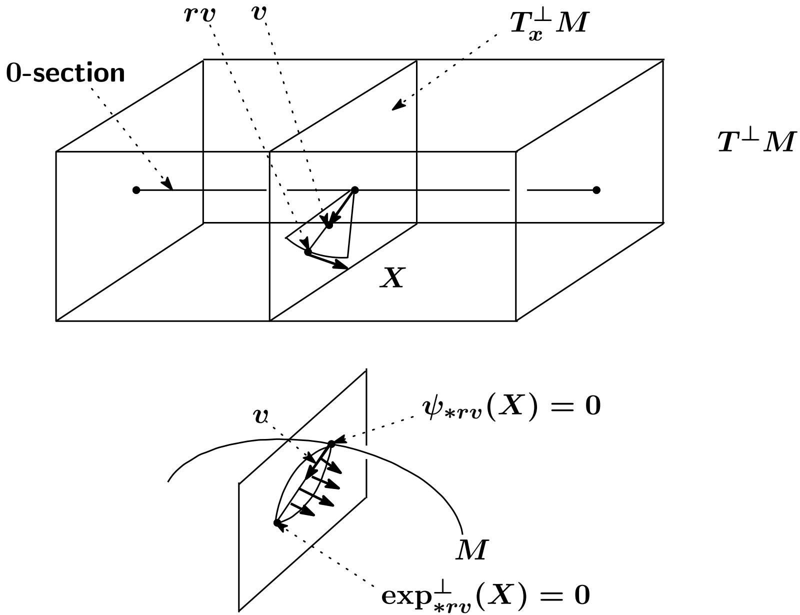
Complex focal radius

Definition

When $\psi_*(\mathrm{Ker}\,\exp^\perp_{*rv}) \neq \{0\}$,
$\exp^\perp(rv)$ is called a **focal point of M along γ_v**,
r is called a **focal radius of M along γ_v**,
$\psi_*(\mathrm{Ker}\,\exp^\perp_{*rv})$ is called **the nullity sp. for r** and its dimension is called **the multiplicity of r**.

Complex focal radius




rv
v
$T_x^\perp M$
0-section
$T^\perp M$
X
v
$\psi_{*rv}(X) = 0$
M
$\exp^\perp_{*rv}(X) = 0$

$\mathcal{FR}^{\mathbf{R}}_{M,v}$: **the set of all focal radii of M along γ_v**

$\mathcal{F}^{\mathbf{R}}_{M,x}$: **the tangential focal set of M at x**

$$\left(\mathcal{F}^{\mathbf{R}}_{M,x} = \bigcup_{v \in T^{\perp}_x M \text{ s.t. } ||v||=1} \{rv \mid r \in \mathcal{FR}^{\mathbf{R}}_{M,v}\}\right) \ (\subset T^{\perp}_x M)$$

Complex focal radius

$$M \subset G/K$$

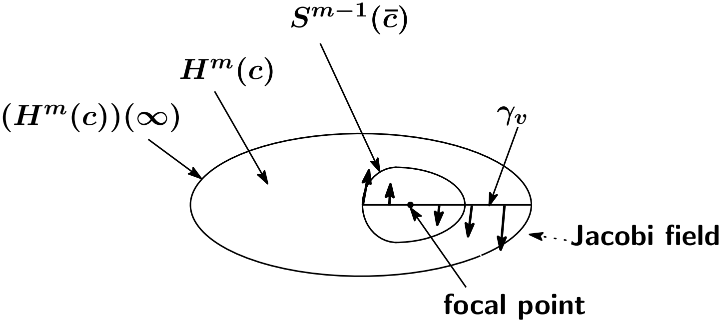
<u>The case of $G/K = H^m(c)$</u>

$$\mathcal{FR}^{\mathrm{R}}_{M,v} = \left\{ \frac{1}{\sqrt{-c}} \operatorname{arctanh} \frac{\sqrt{-c}}{\lambda} \;\middle|\; \lambda \in \operatorname{Spec} A_v \text{ s.t. } |\lambda| > \sqrt{-c} \right\}$$

"The nullity sp. for $\frac{1}{\sqrt{-c}} \operatorname{arctanh} \frac{\sqrt{-c}}{\lambda}$" $= \operatorname{Ker}(A_v - \lambda\, \mathrm{id})$

Complex focal radius

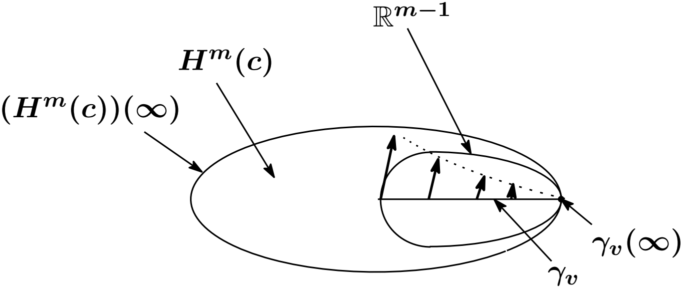
$M = S^{m-1}(\bar{c}) \subset H^m(c)$



$$\lambda = \sqrt{\bar{c} - c} (> \sqrt{-c})$$

Complex focal radius

$\underline{M = \mathbb{R}^{m-1} \subset H^m(c)}$

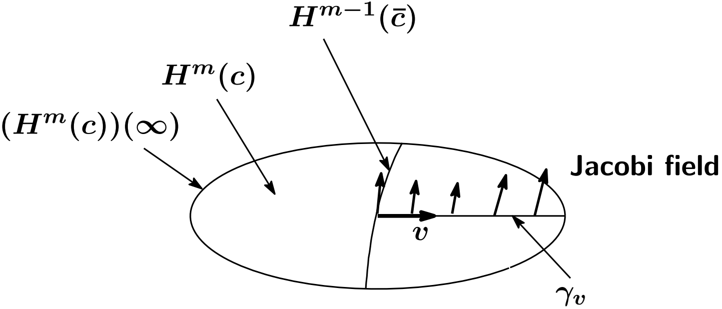


$\lambda = \sqrt{-c}$

No focal radius along γ_v exists.

Complex focal radius

$\underline{M = H^{m-1}(\bar{c}) \subset H^m(c)}$



$$\lambda = \sqrt{\bar{c} - c} \; (< \sqrt{-c})$$

No focal radius along γ_v exists.

Complex focal radius

From these facts, we recognize the following fact:

> When a submfd M in a symmetric sp. G/K of non-compact type deforms as $||A||$ decreases, the focal set of M vanishes beyond the ideal boundary $(G/K)(\infty)$ of G/K.

From this fact, it is considered that the focal radii of M along a normal geodesic should be defined in $\mathbb{C}$.
So the notion of a complex focal radius was introduced.

Complex focal radius

G/K : a symmetric sp. of non-compact type

M : a C^ω-submanifold in G/K

$M^{\mathbf{c}} \subset G^{\mathbf{c}}/K^{\mathbf{c}}$: the complexification of $M(\subset G/K)$

Definition (K, 2005)

$z = s + t\mathrm{i}$: a complex focal radius of M along γ_v

$\underset{\text{def}}{\Longleftrightarrow}$ $\gamma_{sv+tJv}(1)$: a focal point of $M^{\mathbf{c}}$ along γ_{sv+tJv}

(J : the complex structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$)

Complex focal radius

$\mathfrak{g}^{\mathbf{c}} = \mathfrak{k}^{\mathbf{c}} + \mathfrak{p}^{\mathbf{c}}$: the canonical decomp. ass. with $(G^{\mathbf{c}}, K^{\mathbf{c}})$

$B(: \mathfrak{g}^{\mathbf{c}} \times \mathfrak{g}^{\mathbf{c}} \to \mathbb{C})$: the Killing form of $\mathfrak{g}^{\mathbf{c}}$

$B_A := 2\mathrm{Re}B\,(: \mathfrak{g}^{\mathbf{c}} \times \mathfrak{g}^{\mathbf{c}} \to \mathbb{R})$

$\langle\ ,\ \rangle$: the $G^{\mathbf{c}}$-inv. neutral metric of $G^{\mathbf{c}}/K^{\mathbf{c}}$ arising from $B_A|_{\mathfrak{p}^{\mathbf{c}} \times \mathfrak{p}^{\mathbf{c}}}$

J : the $G^{\mathbf{c}}$-inv. complex str. of $G^{\mathbf{c}}/K^{\mathbf{c}}$ arising from the complex str. of $\mathfrak{p}^{\mathbf{c}}$

- $(J, \langle\ ,\ \rangle)$ is an anti-Kaehler structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$ (i.e., $\langle JX, JY\rangle = -\langle X, Y\rangle$ $(\forall\, X, Y)$, $\nabla J = 0$).
- $(G^{\mathbf{c}}/K^{\mathbf{c}}, \langle\ ,\ \rangle)$ is a semi-simple pseudo-Riemannian symmetric space.

Complex focal radius

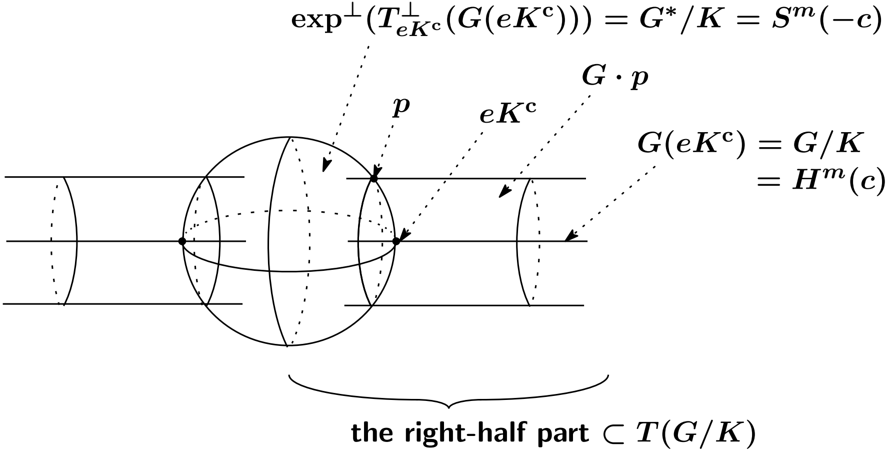
$G \curvearrowright G^{\mathbf{c}}/K^{\mathbf{c}}$ (This is a Hermann action.)

- $G(eK^{\mathbf{c}}) = G/K$
- $\exp^{\perp}(T^{\perp}_{eK^{\mathbf{c}}} G(eK^{\mathbf{c}})) = G^*/K$

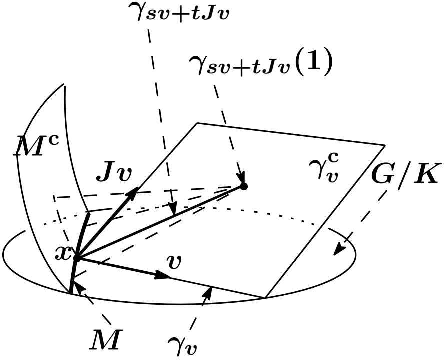
  (the compact dual of G/K)

Complex focal radius

The case of $G/K = H^m(c)$



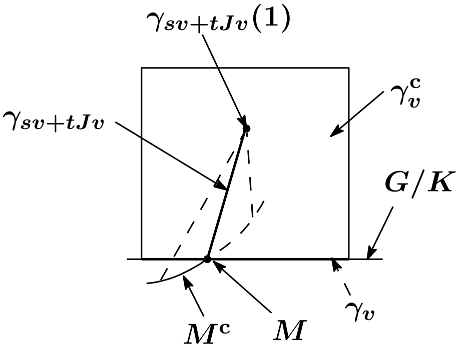
Complex focal radius



$s + t\mathrm{i}$: **a complex focal radius along** γ_v

$\mathrm{Im}\,\gamma_v^{\mathbf{c}} \approx \mathbb{R}^2$ or $S^1 \times \mathbb{R}$

Complex focal radius



Complex focal radius

$\mathcal{FR}^{\mathbf{C}}_{M,v}$: the set of all complex focal radii of M along γ_v

We define $\mathcal{F}^{\mathbf{C}}_{M,x}$ $(\subset T^{\perp}_x(M^{\mathbf{c}}))$ by

$$\mathcal{F}^{\mathbf{C}}_{M,x} := \bigcup_{v \in T^{\perp}_x M \text{ s.t. } \|v\|=1} \{av + bJv \mid a + bi \in \mathcal{FR}^{\mathbf{C}}_{M,v}\}$$

This set is equal to the tangential focal set of $M^{\mathbf{c}}$ at x.
We call this set **tangential complex focal set of M at x**.

Complex focal radius

The case of $G/K = H^m(c)$

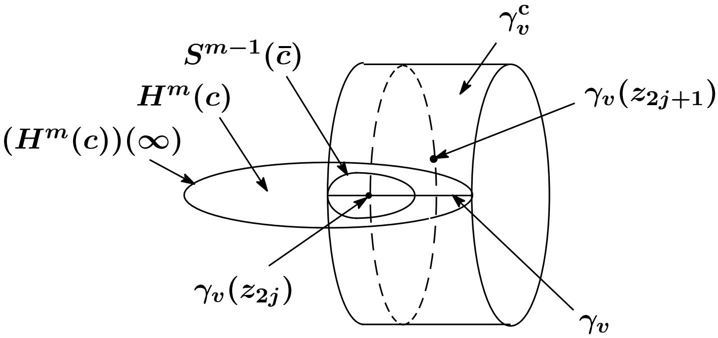
$\mathcal{FR}^{\mathbf{C}}_{M,v}$ **is equal to the following set:**

$$\left\{\frac{1}{\sqrt{-c}}\left(\operatorname{arctanh}\frac{\sqrt{-c}}{\lambda}+j\pi\mathrm{i}\right)\,\middle|\,\lambda\in S_+,\ j\in\mathbb{Z}\right\}\cup$$
$$\left\{\frac{1}{\sqrt{-c}}\left(\operatorname{arctanh}\frac{\lambda}{\sqrt{-c}}+(j+\frac{1}{2})\pi\mathrm{i}\right)\,\middle|\,\lambda\in S_-,\ j\in\mathbb{Z}\right\}$$

$$\left(\begin{array}{l} S_+ := \{\lambda\in\operatorname{Spec}A_v \mid |\lambda|>\sqrt{-c}\} \\ S_- := \{\lambda\in\operatorname{Spec}A_v \mid |\lambda|<\sqrt{-c}\}\end{array}\right)$$

Complex focal radius

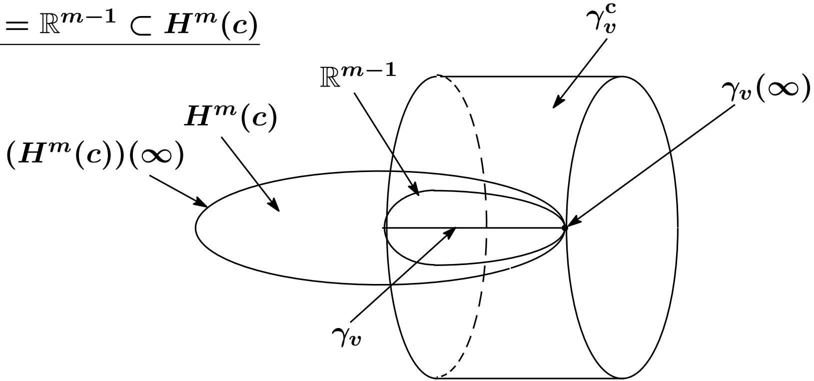
$$M = S^{m-1}(\bar{c}) \subset H^m(c)$$



$$z_j = \frac{1}{\sqrt{-c}}\left(\operatorname{arctanh}\frac{\sqrt{-c}}{\sqrt{\bar{c}-c}} + j\pi\sqrt{-1}\right)$$

Complex focal radius

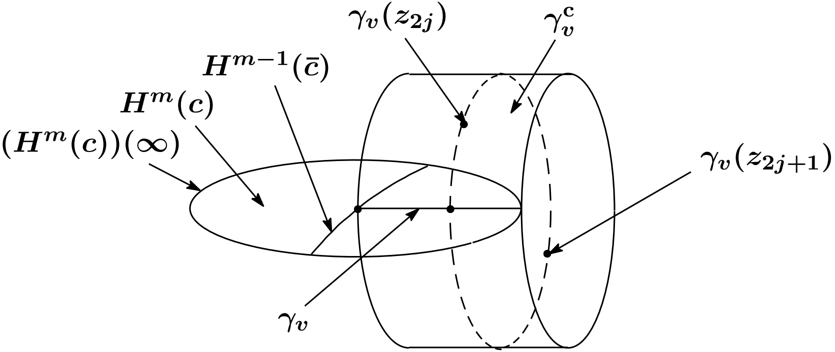
$M = \mathbb{R}^{m-1} \subset H^m(c)$



$\nexists$ a complex focal radius along γ_v

Complex focal radius

$M = H^{m-1}(\bar{c}) \subset H^m(c)$



$$z_j := \frac{1}{\sqrt{-c}}\left(\operatorname{arctanh}\frac{\sqrt{\bar{c}-c}}{\sqrt{-c}} + (j+\frac{1}{2})\pi\sqrt{-1}\right)$$

2. Complex equifocal submanifolds

Complex equifocal submanifolds

G/K : a symmetric space of non-compact type

M : a complete embedded submanifold in G/K

Definition (K, 2004)

M : **complex equifocal submanifold**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{(i)} \quad M \text{ has flat section,} \\ \text{(ii)} \quad \text{the normal holonomy group of } M \text{ is trivial,} \\ \text{(iii)} \quad M \text{ has parallel complex focal structure.} \end{cases}$$

Remark "M has parallel complex focal structure" means

"$\mathcal{F}^{\mathbf{C}}_{M,x}$'s $(x \in M)$ maps to one another by the parallel translations w.r.t. the complexified normal connection."

Complex equifocal submanifolds

Complex equifocal submanifold should be called an equi-complex focal submanifold becasuse it has parallel complex focal structure.

3. Isoparametric submanifolds

Isoparametric submanifold

N : a complete Riemannian manifold

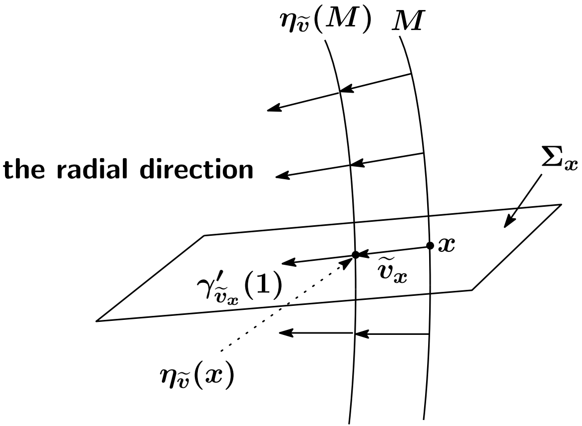
M : a complete embedded submanifold in N

Definition (Heintze-Liu-Olmos, 2006)

M : isoparametric submnaifold

$\overset{\text{def}}{\Longleftrightarrow}$
- (i) M has flat section,
- (ii) the normal holonomy group of M is trivial,
- (iii) sufficiently close parallel submanifolds of M are CMC w.r.t. the radial direction.

Isoparametric submanifolds



Isoparametric submanifolds

G/K : a symmetric space of non-compact type

M : a complete submanifold with flat section in G/K

Fact(K, 2005)

M : isoparametric $\Longrightarrow$ M : complex equifocal

Isoparametric submanifolds

Definition (Berndt-Vanhecke, 1992)

M : **curvature-adapted submanifold**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{(i)} \quad R(v)(T_xM) \subset T_xM \quad (\forall\, x \in M,\ \forall\, v \in T_xM) \\ \text{(ii)} \quad [A_v, R(v)] = 0 \quad (\forall\, v \in TM) \end{cases}$$

$\Bigg($ R : the curvature tensor of G/K

$R(v) := R(\cdot, v)v$

A_v : the shape operator of M $\Bigg)$

Fact (K, 2005)

When M is curvature-adapted,

M : isoparametric $\Longleftrightarrow$ M : complex equifocal

4. Focal points on the ideal boundary

Focal points on the ideal boundary

$$v \in T_x^\perp M.$$

Definition (K, 2010)

$\gamma_v(\infty)$: focal point of M on the ideal boundary of G/K along γ_v

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \exists\, Y : M\text{-Jacobi field along } \gamma_v \\ \text{s.t. } \lim_{s\to\infty} \frac{||Y(s)||}{s} = 0 \end{cases}$$

Focal points on the ideal boundary

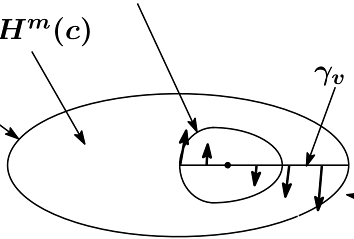
Definition (K, 2010)

$\gamma_v(\infty)$: **non-Euclidean type focal point of M on the ideal boundary of G/K along γ_v**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \exists\, Y \ : \ M\text{-Jacobi field along } \gamma_v \\ \text{s.t.} \quad \lim\limits_{s\to\infty} \dfrac{||Y(s)||}{s} = 0 \ \&\ \mathrm{Sec}(v, Y(0)) \neq 0 \end{cases}$$

Focal points on the ideal boundary

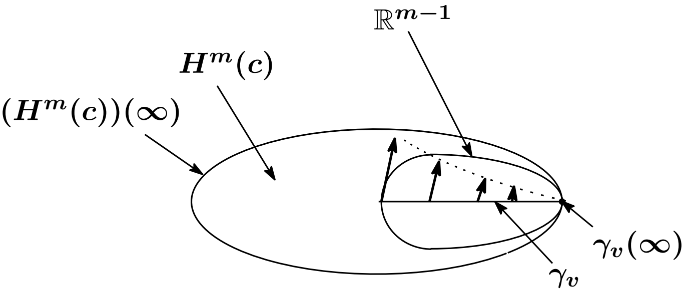
$M = S^{m-1}(\bar{c}) \subset H^m(c)$



$\gamma_v(\infty)$ **is not a focal point of** M **on** $(H^m(c))(\infty)$ **along** γ_v**.**

Focal points on the ideal boundary

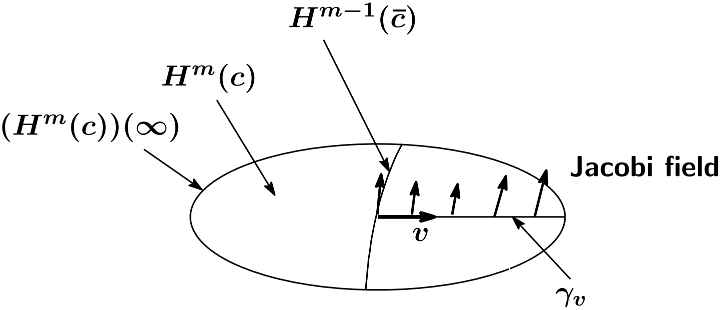
$M = \mathbb{R}^{m-1} \subset H^m(c)$



$\gamma_v(\infty)$ **is a non-Euclidean type focal point of** M **on** $(H^m(c))(\infty)$ **along** γ_v.

Focal points on the ideal boundary

$\underline{M = H^{m-1}(\bar{c}) \subset H^m(c)}$



$\gamma_v(\infty)$ is not a focal point of M on $(H^m(c))(\infty)$ along γ_v.

5. Principal orbits of Hermann actions

Principal orbits of Hermann actions

G/K : a symmetric space of non-compact type

Definition

$H \curvearrowright G/K$: **Hermann action**

$\underset{\text{def}}{\Longleftrightarrow}$ (G, H) : a symmetric pair

Principal orbits of Hermann actions

Fact

Principal orbits of Hermann actions are curvature-adapted isoparametric submanifolds and they have no non-Euclidean type focal point on the ideal boundary of G/K.

6. Results

Results

G/K : a symmetric space of non-compact type

Theorem 6.1.(K, 2012)

Let M be an irreducible curvature-adpated isoparametric C^ω-submanifold of codimension greater than one in G/K. If M has no non-Euclidean type focal point on the ideal boundary of G/K, then it is a principal orbit of a Hermann action on G/K.

Results

Remark In this theorem, are indispensable both the conditions of curvature-adaptedness and the non-existe-nceness of a non-Euclidean type focal point on the ideal boundary of G/K. In fact, we can give examples showing that the conditions are indispensable as principal orbits of subgroup actions of the solvable part AN in the Iwasawa's decomposition $G = KAN$ of G.

Results

Theorem 6.2.(K, 2012)

Let M be as in Theorem 6.1. Furthermore, if M is compact, then it is a principal orbit of the isotropy action on G/K.

Results

Strategy of proof of Theorem 6.1.

$$\widetilde{M}^{\mathbf{c}} := (\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}}) \hookrightarrow H^0([0,1], \mathfrak{g}^{\mathbf{c}})$$

proper anti – Kaeh. isoparametric

$$\downarrow \phi^{\mathbf{c}}$$

$$G^{\mathbf{c}}$$

$$\downarrow \pi^{\mathbf{c}}$$

$$M \hookrightarrow G/K \quad \cdots\cdots\cdots\cdots \to \quad M^{\mathbf{c}} \hookrightarrow G^{\mathbf{c}}/K^{\mathbf{c}}$$

as in Theorem 6.1

Results

$$\widetilde{M}^{\mathbf{c}} \ : \ \textbf{homogeneous}$$
$$\downarrow$$
$$M \ : \ \textbf{homogeneous}$$
$$\downarrow$$
$$M \ : \ \textbf{a principal orbit of a Hermann action}$$

7. Infinite dimensional proper anti-Kaehler isoparametric submanifolds

Proper anti-Kaehler isoparametric submanifolds

V : an ∞-dimensional topological vector space

$\langle\ ,\ \rangle$: a continuous non-degenerate bilinear form of V

J : a continuous linear operator of V s.t.

$J^2 = -\mathrm{id},\ \langle JX, JY\rangle = -\langle X, Y\rangle\ (\forall X, Y \in V)$

Definition

$(V, \langle\ ,\ \rangle, J)$: ∞-dim. anti-Kaehler space

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists V = V_- \oplus V_+$ (orthog. time-space decomp.)

s.t. $(V, \langle\ ,\ \rangle_{V_\pm})$: Hilbert space & $JV_- = V_+$

$$\left(\begin{array}{l} \langle\ ,\ \rangle_{V_\pm} := -\pi_{V_-}^*\langle\ ,\ \rangle + \pi_{V_+}^*\langle\ ,\ \rangle \\ (\pi_{V_\pm} : \text{the orthog. proj. onto } V_\pm) \end{array} \right)$$

Proper anti-Kaehler isoparametric submanifolds

$V = V_- \oplus V_+$: **an orthogonal time-space decomposition**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \langle\ ,\ \rangle|_{V_-\times V_+} = 0 \\ \langle\ ,\ \rangle|_{V_-\times V_-} \ : \ \text{negative definite} \\ \langle\ ,\ \rangle|_{V_+\times V_+} \ : \ \text{positive definite} \end{cases}$$

Proper anti-Kaehler isoparametric submanifolds

$$M \hookrightarrow (V, \langle\ ,\ \rangle, J)$$ (∞-dim. anti-Kaehler sp.)

Definition

M : **Fredholm anti-Kaehler submanifold**

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \bullet\ J(T.M) = T.M \quad \& \quad \operatorname{codim} M < \infty \\ \bullet\ \forall\, v \in T^{\perp}M,\ A_v : \text{cpt op. w.r.t. } \langle\ ,\ \rangle_{V_{\pm}} \end{cases}$$

Definition

$a + bi (\in \mathbb{C})$: **J-eigenvalue of A_v**

$\underset{\text{def}}{\Longleftrightarrow} \exists\, X (\neq 0) \in T_x M$ s.t. $A_v X = aX + bJX$

Also, X is called a **J-eigenvector of A_v for $a + bi$**.

Proper anti-Kaehler isoparametric submanifolds

Definition

M : **anti-Kaehler isoparametric submanifold**

$\overset{\Longleftrightarrow}{\text{def}}$
- M : Fredholm anti-Kaehler
- the normal holonomy gr. of M : trivial
- $\forall\, v \in \Gamma(T^{\perp}M)$ s.t. $\nabla^{\perp} v = 0$, $\mathrm{Spec}_J\, A_{v_x}$: indep. of $x \in M$

($\mathrm{Spec}_J\, A_{v_x}$: the set of all J-eigenvalues of A_{v_x})

Proper anti-Kaehler isoparametric submanifolds

Definition

A_v : **diagonalizable w.r.t. a J-orthonormal base** (or **J-diagonalizable**)

$$\underset{\text{def}}{\Longleftrightarrow} \left\{ \begin{array}{l} \exists \, \{e_i \,|\, i \in I\} \\ \text{s.t.} \left\{ \begin{array}{l} \bullet \;\; e_i\text{'s}: \; J\text{-eigenvectors of } A_v \\ \bullet \;\; \{e_i \,|\, i \in I\} \cup \{Je_i \,|\, i \in I\} \\ \qquad : \text{an orthon. base of } T_xM \end{array} \right. \end{array} \right.$$

Proper anti-Kaehler isoparametric submanifolds

Definition

M : **proper anti-Kaehler isoparametric submanifold**

$\underset{\text{def}}{\Longleftrightarrow}$

- M : anti-Kaehler isoparametric
- $\forall\, v \in T^{\perp}M, \;\; A_v$: diagonalizable w.r.t. a J-orthon. base

Proper anti-Kaehler isoparametric submanifolds

Remark

Geatti-Gorodski defined the notion of an isoparametric submanifold with diagonalizable Weingarten maps in a pseudo-Euclidean space.
[Geatti-Gorodski, J. Algebra (2008)].
Proper anti-Kaehler isoparametric submanifolds belong to the class of infinite dimensional version of this notion.

Proper anti-Kaehler isoparametric submanifolds

$M \hookrightarrow V$: proper anti-Kaehler isoparametric

$x \in M$

Fact

A_v's ($v \in T_x^\perp M$) are simultaneously diagonalizable with respect to a J-orthonormal base.

$$T_x M = \overline{E_0^x \oplus \left(\bigoplus_{i \in I_x} E_i^x \right)} \quad \left(E_0^x := \bigcap_{v \in T_x^\perp M} \mathrm{Ker}\, A_v \right)$$

(common J-eigenspace decomposition of A_v's)

$$\exists 1\ \lambda_i^x \in (T_x^\perp M)^{*c} \text{ s.t. } A_v|_{E_i^x} = \mathrm{Re}(\lambda_i^x(v))\mathrm{id} + \mathrm{Im}(\lambda_i^x(v))J$$
$$(\forall\, v \in T_x^\perp M)$$

By reordering E_i^x's ($x \in, M \ i \in I$) if necessary, we may assume that

$$x \mapsto \lambda_i^x, \ x \mapsto n_i^x \text{ and } x \mapsto E_i^x \text{ are smooth.}$$

Definition

$$\lambda_i \in \Gamma((T^\perp M)^{*c}) \underset{\text{def}}{\Longleftrightarrow} (\lambda_i)_x := \lambda_i^x \ (x \in M)$$

***J*-principal curvature**

$$\mathrm{n}_i \in \Gamma(T^\perp M) \underset{\text{def}}{\Longleftrightarrow} \lambda_i(\cdot) = \langle \mathrm{n}_i, \cdot \rangle - \sqrt{-1}\langle J\mathrm{n}_i, \cdot \rangle$$

***J*-curvature normal**

$$E_i \ : \text{a distribution on } M \underset{\text{def}}{\Longleftrightarrow} (E_i)_x := E_i^x \ (x \in M)$$

***J*-curvature distribution**

$$E_0 \ : \text{a distribution on } M \underset{\text{def}}{\Longleftrightarrow} (E_0)_x := E_0^x \ (x \in M)$$

***J*-curvature distribution**

Proper anti-Kaehler isoparametric submanifolds

Fact

- **The focal set of M at x is equal to $\bigcup_{i\in I}(\lambda_i)_x^{-1}(1)$.**
- **E_i is tot. geod. and each leaf of E_i is a complex sphere.**
- **E_0 is tot. geod. and each leaf of E_0 is an anti-Kaeh. sp.**

8. Outline of proof of the main result

Outline of proof of the main result

Outline of the proof of Theorem 6.1.

$M \subset G/K$: as in Theorem 6.1

$\phi : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$: the parallel transport map for $G^{\mathbf{c}}$

$\pi : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$: the natural projection

$\widetilde{M}^{\mathbf{c}} := (\pi \circ \phi)^{-1}(M^{\mathbf{c}})$

Outline of proof of the main result

(Step I) We show that $\widetilde{M}^{\mathbf{c}}$ is a proper anti-Kaehler isoparametric submanifold (i.e., an anti-Kaehler isoparametric submanifold with J-diagonalizable shape operators).

(Step II) By using the fact in (Step I) and using the fact that $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ is a linear space, we show that $\widetilde{M}^{\mathbf{c}}$ is homogeneous.

(We refer the proof of the homogeneity of inifinite dimensional isoaparametric submanifolds in the Hilbert space [Heintze-Liu, Ann. of Math. (1999)].)

Outline of proof of the main result

(Step III) By using the homogeneity of $\widetilde{M}^{\mathrm{c}}$ and refering the discussions in [Christ, J.D.G. (2002)] and [Gorodski-Heintze, J. Fixed Point Theory Appl. (2012)], we show that M is homogeneous.

(By using the homogeneity theorem by Heintze-Liu, Christ proved the homogeneity of irreducible equifocal submanifolds of codeminesion greater than one in a symmetric space of compact type. Recently Gorodski-Heintze has filled a gap in the proof by Christ.)

Outline of proof of the main result

(Step IV) From the homogeneity of M, it is shown that M is a principal orbit of a (complex) hyperpolar action. Furthermore, we show that the action admits a totally geodesic orbit. As its result, it is shown that the action is orbit equivalent to a Hermann action.

Outline of proof of the main result

complex polar action $=$ **polar action with complex poles**
$\neq$ **the complexifocation of polar action**

complex hyperpolar action $=$ **special complex polar action**

9. On (Step II) in proof of the main theorem

On (Step II) in proof of the main theorem

$\lambda_i \, (i \in I)$: J-principal curvatures of $\widetilde{M^c}$

$\mathbf{n}_i \, (i \in I)$: J-curvature normals of $\widetilde{M^c}$

$E_i \, (i \in I \cup \{0\})$: J-curvature distributions of $\widetilde{M^c}$

On (Step II) in proof of the main theorem

$L_u^{E_i}$: the leaf of E_i through u (focal leaf)

$$v_0 \in l_i \setminus \left(\bigcup_{j\in I\setminus\{i\}} l_j \right) \quad (l_j := ((\lambda_i)_{u_0})^{-1}(1))$$

v : the parallel normal vector field s.t. $v_{u_0} = v_0$

f_v : the focal map for v

$F_v := f_v(\widetilde{M^c})$ (focal submanifold)

Fact

$$L_u^{E_i} = f_v^{-1}(f_v(u))$$

On (Step II) in proof of the main theorem

Set $V := H^0([0,1], \mathfrak{g}^{\mathbf{c}})$.

Take $u_1 \in L_{u_0}^{E_{i_0}}$.

γ **: a geodesic in** $L_{u_0}^{E_{i_0}}$ **with** $\gamma(0) = u_0$ **and** $\gamma(1) = u_1$

Definition.

$$B_\gamma : T_{u_0}V \to T_{u_1}V$$

$$\underset{\text{def}}{\Longleftrightarrow} \begin{cases} \tau_\gamma & (\text{on } (E_{i_0})_{u_0}) \\ (\tau^\perp_{f_i\circ\gamma})_{*u_0}|_{(E_i)_{u_0}} & (\text{on } (E_i)_{u_0} \ (i \neq i_0)) \\ (h_\gamma)_{*u_0} & (\text{on } (E_0)_{u_0}) \\ \tau^\perp_\gamma & (\text{on } T^\perp_{u_0}\widetilde{M^c}) \end{cases}$$

τ_γ : the parallel translation along γ w.r.t. the conn. of $L^{E_{i_0}}_{u_0}$

$\tau^\perp_{f_i\circ\gamma}$: the parallel translation along $f_i\circ\gamma$ w.r.t. the normal conn. of F_i

h_γ : the isometry from $L^{E_0}_{u_0}$ to $L^{E_0}_{u_1}$ determined by γ

$\tau^\perp_\gamma$: the parallel translation along γ w.r.t. the normal conn. of $\widetilde{M^c}$

On (Step II) in proof of the main theorem

Definition.

$$\psi_\gamma \in I(V) \underset{\mathrm{def}}{\Longleftrightarrow} (\psi_\gamma)_{*u_0} = B_\gamma$$

Remark.

Since V is an anti-Kaehler space (hence a linear space), there uniquely exists the above (holom.) isometry ψ_γ.

Lemma 9.1.

$$\psi_\gamma(\widetilde{M^c}) = \widetilde{M^c}$$

On (Step II) in proof of the main theorem

$Q(u_0)$: the set of all points of $\widetilde{M^c}$ connected with u_0 by a piecewise smooth curve each of whose segment lies on a complex curvature sphere

From Lemma 9.1, we have

Lemma 9.2.

$\forall\, u \in Q(u_0)$,

$\exists\, \psi \in I(V)$ s.t. "$\psi(u_0) = u,\ \psi(\widetilde{M^c}) = \widetilde{M^c}$"

On (Step II) in proof of the main theorem

Proof of Lemma 9.2.

For any $u \in Q(u_0)$,

$\exists\, \alpha := \gamma_1 \cdot \cdots \cdot \gamma_k$: piecewise smooth curve s.t.

- $\alpha(0) = u_0, \ \alpha(1) = u$
- γ_j is a geodesic in a complex curvature sphere $(1 \leq j \leq k)$

Then $\psi_\alpha := \psi_{\gamma_k} \circ \cdots \circ \psi_{\gamma_1}$ is an isometry of V with $\psi_\alpha(u_0) = u$.

q.e.d.

On (Step II) in proof of the main theorem

On the other hand, we have

Lemma 9.3.

$$\overline{Q(u_0)} = \widetilde{M}^{\mathbf{c}}$$

From Lemmas 9.1, 9.2 and 9.3, we have

Proposition 9.4.

$\forall\, u \in \widetilde{M}^{\mathbf{c}}$,
$\exists\, \psi \in I(V)$ **s.t.** "$\psi(u_0) = u,\ \psi(\widetilde{M}^{\mathbf{c}}) = \widetilde{M}^{\mathbf{c}}$"
(i.e., $\widetilde{M}^{\mathbf{c}}$ **is homogeneous.)**

On (Step II) in proof of the main theorem

$$\widetilde{H} := \{\psi \in I(V) \mid \psi(\widetilde{M^c}) = \widetilde{M^c}\}$$

According to Proposition 9.4, we obtain

Fact.

$$\widetilde{H} \cdot u_0 = \widetilde{M^c}$$

10. On (Step III) in proof of the main theorem

On (Step III) in proof of the main theorem

$\mathfrak{o}_{AK}(V)$: **the Banach Lie algebla of all continuous (i.e., bounded) skew symmetric $\mathbb{C}$-linear maps**

$O_{AK}(V) := \exp(\mathfrak{o}_{AK}(V))\,(\subset I(V))$

$I_b(V) := O_{AK}(V) \ltimes V\ (\subset I(V))$

$H^1([0,1], G^{\mathbf{c}}) \curvearrowright V$ **(the gauge action)**
(Under this action, $H^1([0,1], G^{\mathbf{c}}) \subset I(V)$)

It is shown that $H^1([0,1], G^{\mathbf{c}}) \subset I_b(V)$ holds.

$\widetilde{H}_b := \widetilde{H} \cap I_h^b(V)$

Outline of proof of homogeneity

(Step III · 1) **We show that the holomorphic Killing field ass. with one-parameter transformation gr. $\{\psi_{\gamma|_{[0,t]}}\}_{t\in\mathbb{R}}$ of holomorphic isometries constructed in (Step II) is defined on the whole of V (i.e., $\psi_{\gamma|_{[0,t]}} \in I_b(V)$). From this fact, we can show $\widetilde{H_b} \cdot u_0 = \widetilde{M^{\mathbf{c}}}$.**
(We refered work of Gorodski-Heintze)

Outline of proof of homogeneity

(Step III · 2) By using $\widetilde{H}_b \subset I_b(V)$, we show $\widetilde{H}_b \subset H^1([0,1], G^{\mathbf{c}})$.
(We refered work of Christ)

(Step III · 3) By using $\widetilde{H}_b \subset H^1([0,1], G^{\mathbf{c}})$, we find a subgroup H of G s.t. $H \cdot x_0 = M$.
(We refered work of Christ)

11. Classifications

Classifications

G/K : irreducible symmetric space of non-compact type

M : isoparametric submanifold in G/K as in Theorem 6.1

Theorem 11.1.

M occurs as a principal orbit of one of Hermann actions $H \curvearrowright G/K$ as in Table 1∼4.

Thanks!