Survey on an equifocal submanifold and a proper complex equifocal submanifold

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Abstract

This is a survey for an equifocal submanifold in a symmetric space of compact type and a proper complex equifocal submanifold in a symmetric space of non-compact type.

Keywords: equifocal submanifold, proper complex equifocal submanifold, isoparametric submanifold, proper complex isoparametric submanifold

1 Equifocal submanifolds and complex equifocal submanifolds

In 1995, C.L. Terng and G. Thorbergsson [TT] introduced the notion of an equifocal submanifold in a (Riemannian) symmetric space. This notion is defined as a compact submanifold with flat section, trivial normal holonomy group and parallel focal structure. Here "with flat section" means that the images of the normal spaces of the submanifold by the normal exponential map are flat totally geodesic submanifolds and the parallelity of the focal structure means that, for any parallel normal vector field v of the submanifold, the focal radii along the normal geodesic γ_{v_x} with $\gamma'_{v_x}(0) = v_x$ are independent of the choice of x (with considering the multiplicities), where $\gamma'_{v_x}(0)$ is the velocity vector of γ_{v_x} at 0. Note that the focal radii of the submanifold along the normal geodesic γ_{v_x} coincide with the zero points of the real valued function

$$F_{v_x}(s) := \det\left(\cos(s\sqrt{R(v_x)}) - \frac{\sin(s\sqrt{R(v_x)})}{\sqrt{R(v_x)}} \circ A_{v_x}\right)$$

over $\mathbb{R}$ defined in terms of the shape operator A_{v_x} and the normal Jacobi operator $R(v_x)(:= R(\cdot, v_x)v_x)$, where R is the curvature tensor of the ambient symmetric space. In particular, in the case where G/K is a Euclidean space, we have $F_{v_x}(s) = \det(\mathrm{id} - sA_{v_x})$ and hence the focal radii along γ_{v_x} coincide with the inverse numbers of the eigenvalues of A_{v_x} (i.e., the principal curvature radii of direction v_x). Compact isoparametric submanifolds in a Euclidean space and compact isoparametric hypersurfaces in a sphere or a hyperbolic space are equifocal. When a non-compact submanifold M in a symmetric space G/K of non-compact type variates as its principal curvatures approach to zero, its focal set vanishes beyond the ideal boundary $(G/K)(\infty)$ of G/K (see Fig. 1).

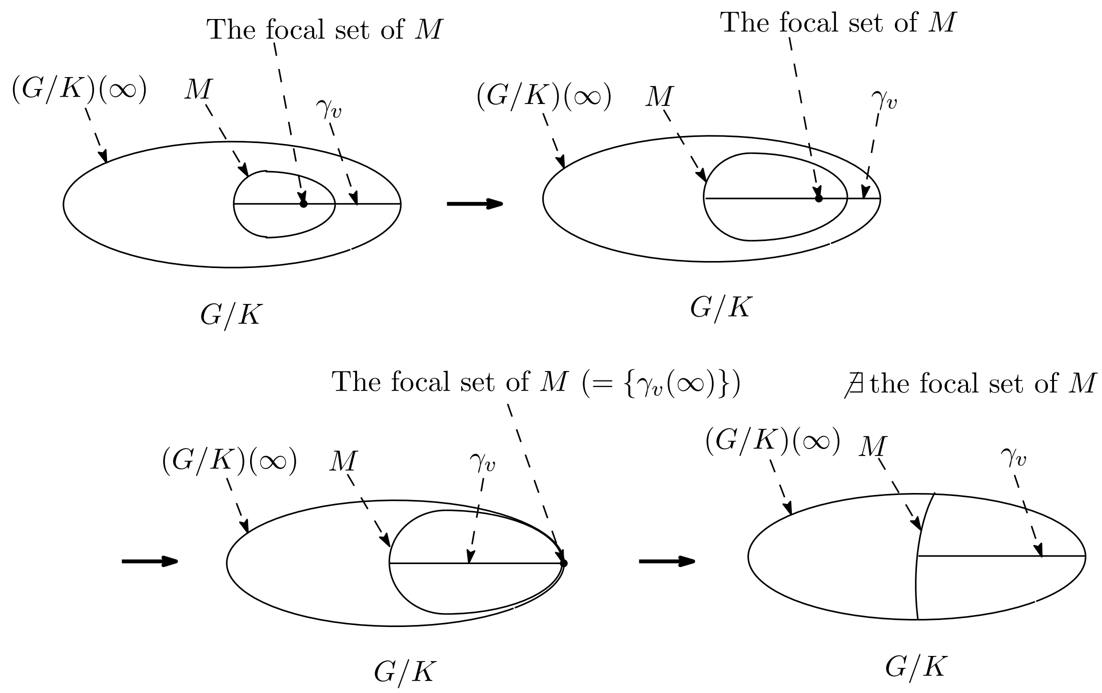


Fig. 1.

From this fact, we recognize that, for a non-compact submanifold in a symmetric space of non-compact type, the parallelity of the focal structure is not an essential condition. So, we ([Koi3]) introduced the notion of a complex focal radius of the submanifold along the normal geodesic γ_{v_x} as the zero points of the complex valued function $F^{\mathbf{c}}_{v_x}$ over $\mathbb{C}$ defined by

$$F^{\mathbf{c}}_{v_x}(z) := \det\left(\cos(z\sqrt{R(v_x)^{\mathbf{c}}}) - \frac{\sin(z\sqrt{R(v_x)^{\mathbf{c}}})}{\sqrt{R(v_x)^{\mathbf{c}}}} \circ A^{\mathbf{c}}_{v_x}\right)$$

over $\mathbb{C}$, where $A_{v_x}^{\mathbf{c}}$ and $R(v_x)^{\mathbf{c}}$ are the complexifications of A_{v_x} and $R(v_x)$, respectively. Here we note that complex focal radii along γ_{v_x} are directly calculated from datas of A_{v_x} and $R(v_x)$ according to this definition. In the case where M is of class C^ω (i.e., real analytic), we can catch the geometrical essence of complex gfocal radii as follows. We ([Koi4]) defined the complexification $M^{\mathbf{c}}$ of M as an anti-Kaehlerian submanifold in the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$, where we note that $G^{\mathbf{c}}/K^{\mathbf{c}}$ is a space including both G/K and its compact dual G_κ/K as submanifolds transversal to each other and that it is interpreted as the complexification of both G/K and G_κ/K, where we note that the induced metric on G/K coincides with the original metric of G/K and that the induced metric on G_κ/K is the (-1)-multiple of the metric of G_κ/K. Also, we note that an anti-Kaehlerian manifold means a manifold M equipped with a pseudo-Riemannian metric g and a complex structure J satisfying $g(JX, JY) = -g(X, Y)$ $(\forall\, X, Y \in TM)$ and $\nabla J = 0$, and an anti-Kaehlerian submanifold in the space means a J-invariant submanifold, where ∇ is the Levi-Civita connection of g. We ([Koi4]) showed that z is a complex focal radius of M along γ_{v_x} if and only if $\gamma_{v_x}^{\mathbf{c}}(z)$ is a focal point of $M^{\mathbf{c}}$ along the complexified geodesic $\gamma_{v_x}^{\mathbf{c}}$ (see Fig. 2,3). Here $\gamma_{v_x}^{\mathbf{c}}$ is defined by $\gamma_{v_x}^{\mathbf{c}}(z) := \gamma_{av_x+bJv_x}(1)$ $(z = a+b\sqrt{-1} \in \mathbb{C})$, where $\gamma_{av_x+bJv_x}$ is the geodesic in $G^{\mathbf{c}}/K^{\mathbf{c}}$ with $\gamma'_{av_x+bJv_x}(0) = av_x + bJv_x$. Thus the complex focal radii of M are the quantities indicating the positions of focal points of $M^{\mathbf{c}}$.

When M variates as above and real analytically, its focal set vanishes beyond $(G/K)(\infty)$ but the focal set of $M^{\mathbf{c}}$ (i.e., the complex focal set of M) does not vanish (see Fig. 4). From this fact, for non-compact submanifolds in a symmetric space of non-compact type, we recognize that the parallelity of the complex focal structure is an essential condition (even if M is not of C^ω). So, we [Koi3] defined the notion of a complex equifocal submanifold as a (properly embedded) complete submanifold with flat section, trivial normal holonomy group and parallel complex focal structure, where we note that this submanifold should be called an equi-complex focal submanifold but that we called it a complex equifocal submanifold for simplicity. Note that equifocal submanifolds in the symmetric space are complex equifocal. In fact, since they are compact, their principal curvatures are not close to zero and hence the parallelity of their focal structure leads to that of their complex focal structure.

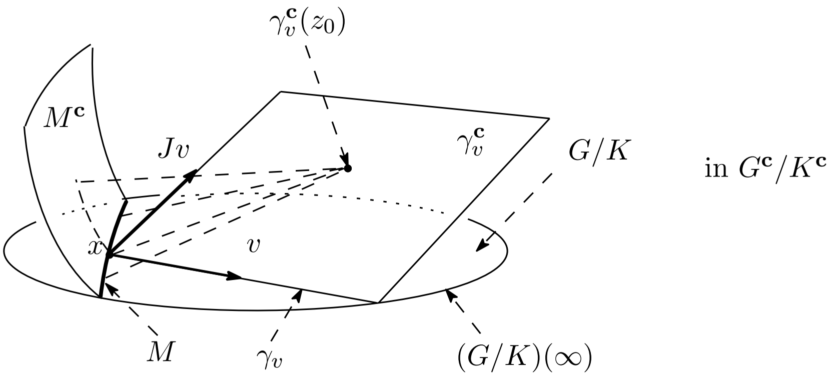
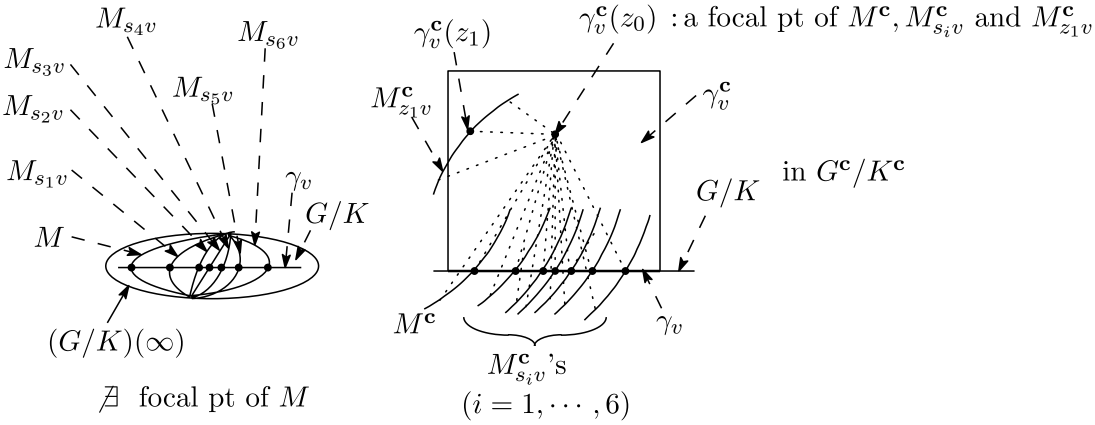


Fig. 2.



$M_{s_i v}$: the parallel submanifold of M through $\gamma_v(s_i)$
$M_{s_i v}^{\mathbf{c}}$: the parallel submanifold of $M^{\mathbf{c}}$ through $\gamma_v^{\mathbf{c}}(s_i)$
$M_{z_1 v}^{\mathbf{c}}$: the parallel submanifold of $M^{\mathbf{c}}$ through $\gamma_v^{\mathbf{c}}(z_1)$

Fig. 3.

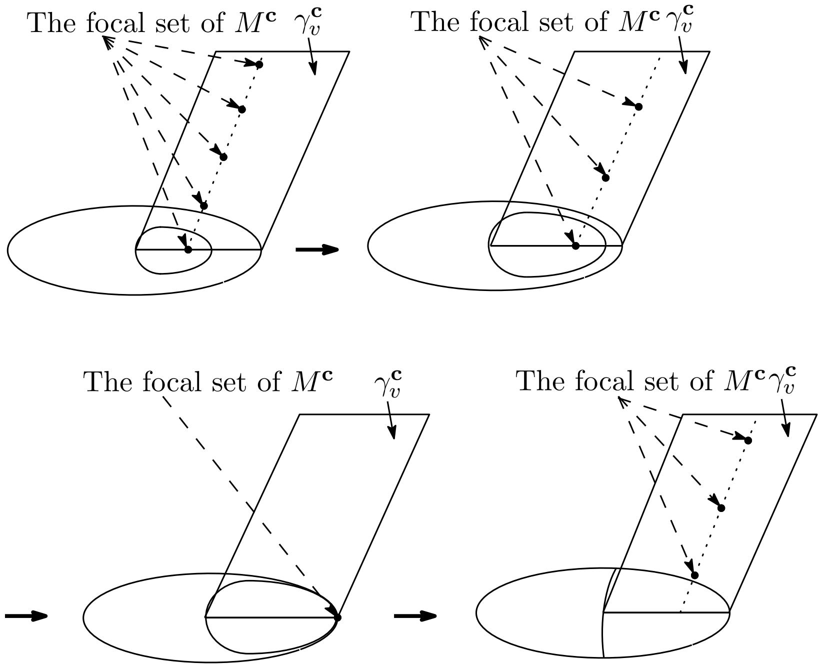


Fig. 4.

2 Isoparametric submanifolds in a Hilbert space, complex isoparametric submanifolds and anti-Kaehlerian isoparametric submanifolds

In 1989, Terng [T2] introduced the notion of an isoparametric submanifold in a (separable) Hilbert space. This notion is defined as a (proper) Fredholm submanifold with trivial normal holonomy group and constant principal curvatures, where a (proper) Fredholm submanifold means a (properly embedded) submanifold of finite codimension such that the normal exponential map $\exp^{\perp}$ of the submanifold is a Fredholm map (i.e., the differential of $\exp^{\perp}$ at each point is a Fredholm operator) and that the restriction of $\exp^{\perp}$ to unit ball normal bundle of M is proper. Note that the shape operators of this submanifold are compact operators (beacuse $\exp^{\perp}$ is a Fredholm map) and that they are simultaneously diagonalizable with respect to an orthonormal base. Also she [T2] introduced the notion of the parallel transport map for a compact semi-simple Lie group G. This map is defined as a Riemannian submersion of a (separable) Hilbert space $H^0([0,1],\mathfrak{g})$ onto G, where $H^0([0,1],\mathfrak{g})$ is the space of all L^2-integrable paths in the Lie algebra $\mathfrak{g}$ of G. Let G/K be a symmetric space of compact type, π the natural projection of G onto G/K and ϕ the parallel transport map for G. Let M be a submanifold in G/K and $\widetilde{M}$ a component of the lifted submanifold $(\pi \circ \phi)^{-1}(M)$. The relation between the focal structures of M and $\widetilde{M}$ is as in Fig. 5.

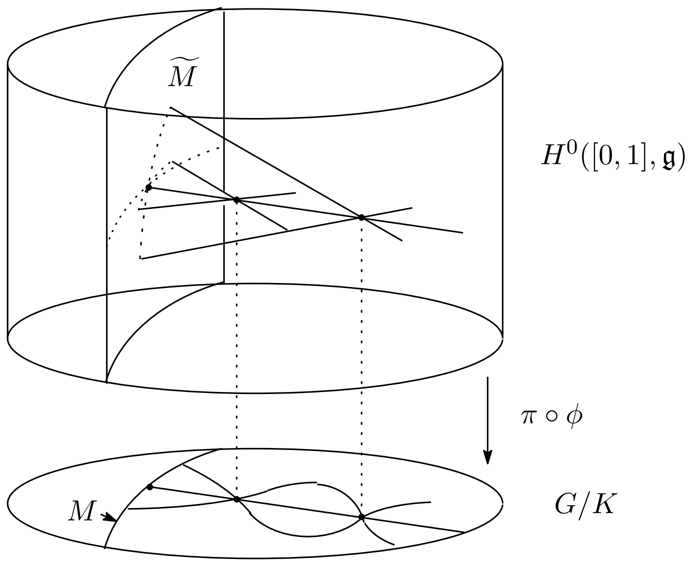


Fig. 5.

In 1995, Terng-Thorbergsson [TT] showed that M is equifocal if and only if $\widetilde{M}$ is isoparametric. Thus the research of an equifocal submanifold in a symmetric space of compact type is reduced to that of an isoparametric submanifold in a (separable) Hilbert space. An advantage of this reducement of the research is as follows. The symmetric space is of non-trivial holonomy group but the Hilbert space is a linear space, that is, it is of trivial holonomy group and is identified with its tangent space at each point. By using this reducement of the research, they proved some facts for an equifocal submanifold in the symmetric space (see [TT]). In [TT], they proposed the following problem:

Problem. *Is there a similar method of research for equifocal submanifolds in symmetric spaces of non-compact type?*

By private discussion with Thorbergsson at Nagoya University in 2002, I knew that this problem is important and began to tackle to this problem. In 2004-2005, we [Koi3,4] constructed a similar method of research for complex equifocal submanifolds in symmetric spaces of non-compact type in more general. We shall explain this method of research. First we shall recall the notions of an isoparametric submanifold, a real isoparametric submanifold, a complex isoparametric submanifold and a proper complex isoparametric submanifold in a (finite dimensional) pseudo-Euclidean space. Let M be a (properly embedded) complete submanifold in a pseudo-Euclidean space. Denote by A the shape tensor of M. Assume that the normal holonomy group of M is trivial. Let v be a parallel normal vector field of M. For each $x \in M$, the shape operator A_{v_x} is expressed as

$$\left(\bigoplus_{i=1}^{n}\bigoplus_{j\in S_i^x}\underbrace{\begin{pmatrix}\lambda_{ij}^x & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_{ij}^x\end{pmatrix}}_{(i,i)-\text{type}}\right)$$

$$\oplus\left(\bigoplus_{i=1}^{[\frac{n}{2}]}\bigoplus_{j\in {S'}_i^x}\underbrace{\begin{pmatrix}\alpha_{ij}^x & -\beta_{ij}^x & 1 & 0 & & & & \\ \beta_{ij}^x & \alpha_{ij}^x & 0 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & 1 & 0 \\ & & & & \ddots & & 0 & 1 \\ & & & & & & \alpha_{ij}^x & -\beta_{ij}^x \\ & & & & & & \beta_{ij}^x & \alpha_{ij}^x\end{pmatrix}}_{(2i,2i)-\text{type}}\right)$$

with respect to a pseudo-orthonormal base of the tangent space (see [Pe] in detail), where blank components in each matrix mean zero. If $S_i^x = \emptyset$ $(i \geq 2)$ and ${S'}_i^x = \emptyset$ $(i \geq 2)$, that is, the complexification $A_{v_x}^{\mathbf{c}}$ of A_{v_x} is diagonalizable with respect to a pseudo-orthonormal base, then A_{v_x} is called be *proper*. If, for each parallel normal vector field v of M, the set $\{\lambda_{ij}^x \,|\, 1 \leq i \leq n,\ j \in S_i^x\}$ of all real eigenvalues of A_{v_x} is independent of the choice of $x \in M$ (with considering the multiplicities), then M is called a *real isoparametric submanifold*. Also, if, for each parallel normal vector field v of M, the set $\{\lambda_{ij}^x \,|\, 1 \leq i \leq n,\ j \in S_i^x\} \cup \{\alpha_{ij}^x + \sqrt{-1}\beta_{ij}^x \,|\, 1 \leq i \leq [\frac{n}{2}],\ j \in {S'}_i^x\}$ of all complex eigenvalues of A_{v_x} is independent of the choice of $x \in M$ (with considering the multiplicities), then M is called a *complex isoparametric submanifold*. In particular, if M is complex isoparametric and each shape operator of M is proper, then M is called a *proper complex isoparametric submanifold*. Also, if, for any parallel normal vector field v of M, the characterisitic polynomials of A_{v_x} are independent of the choice of $x \in M$, then M is called an *isoparametric submanifold* (see [Ka], [Ha1,2], [M] for example). Clearly we have

$$\begin{aligned}&M\ :\ \text{proper complex isoparametric} \ \Rightarrow\ M\ :\ \text{isoparametric}\\ &\Rightarrow\ M\ :\ \text{complex isoparametric}\ \Rightarrow\ M\ :\ \text{real isoparametric}.\end{aligned}$$

In 2004, we [Koi3] defined the notions of a real isoparametric submanifold, a complex isoparametric submanifold and a proper complex isoparametric submanifold in a pseudo-Hilbert space as Fredholm submanifolds satisfying the similar conditions, where a pseudo-Hilbert space means a topological vector space equipped with a (weak-sense) non-degenerate continuous symmetric bilinear form which is Hilbertable. See [Koi3] about the meaning of the Hilbertability and the definition of a Fredholm submanifold in a pseudo-Hilbert space. Also, we [Koi3] introduced the notion of the parallel transport map for a (not necessarily compact) semi-simple Lie group G. This map is defined as a pseudo-Riemannian submersion of a pseudo-Hilbert space $H^0([0,1],\mathfrak{g})$ onto G, where $H^0([0,1],\mathfrak{g})$ is the space of all paths in the Lie algebra $\mathfrak{g}$ of G which are L^2-integrable with respect to the positive definite inner product associated with the $\mathrm{Ad}(G)$-invariant non-degenerate inner product of $\mathfrak{g}$. Let G/K be a symmetric space of non-compact type, π the natural projection of G onto G/K and ϕ the parallel transport map for G. Also, let M be a (properly embedded) complete submanifold in G/K and $\widetilde{M}$ a component of the lifted submanifold $(\pi\circ\phi)^{-1}(M)$. We [Koi3] showed that M is complex equifocal if and only if $\widetilde{M}$ is complex isoparametric. Thus the research of complex equifocal submanifolds in symmetric spaces of non-compact type is reduced to that of complex isoparametric submanifolds in pseudo-Hilbert spaces. If $\widetilde{M}$ is proper complex isoparametric, then we ([Koi5]) called M a *proper complex equifocal submanifold*. Since the shape operators of a proper complex isoparametric submanifold is simultaneously diagonalizeble with respect to a pseudo-orthonormal base, the complex focal set of the submanifold at any point u consists of infinitely many complex hyperplanes in the complexified normal space at u and the group generated by the complex reflections of order two with respect to the complex hyperplanes is discrete. From this fact, it follows that the same fact holds for the complex focal set of a proper complex equifocal submanifold. In 2005, we [Koi4] introduced the notions of an anti-Kaehlerian isoparametric submanifold and a proper anti-Kaehlerian isoparametric submanifold in an infinite dimensional anti-Kaehlerian space, where an infinite dimensional anti-Kaehlerian space means a topological complex vector space (V,J) equipped with a non-degenerate continuous symmetric bilinear form $\langle\ ,\ \rangle$ such that $\langle JX,JY\rangle=-\langle X,Y\rangle$ for any $X,Y\in V$ and that $(V,\langle\ ,\ \rangle)$ is Hilbertable. See [Koi4] about the definitions of these notions. Let $\pi^{\mathbf{c}}$ the natural projection of $G^{\mathbf{c}}$ onto $G^{\mathbf{c}}/K^{\mathbf{c}}$ and $\phi^{\mathbf{c}}$ the parallel transport map for $G^{\mathbf{c}}$. Let $\widetilde{M^{\mathbf{c}}}$ be a component of the lifted submanifold $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ of the complexification $M^{\mathbf{c}}$ of M. We [Koi4] showed that M is complex equifocal (resp. proper complex equifocal) if and only if $\widetilde{M^{\mathbf{c}}}$ is anti-Kaehlerian isoparametric (resp. proper anti-Kaehlerian isoparametric) in the infinite dimensional anti-Kaehleian space $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$. Thus, in the case where M is of class C^ω, the research of complex equifocal (resp. proper complex equifocal) submanifolds is reduced to that of anti-Kaehlerian isoparametric (resp. proper anti-Kaehlerian isoparametric) submanifolds.

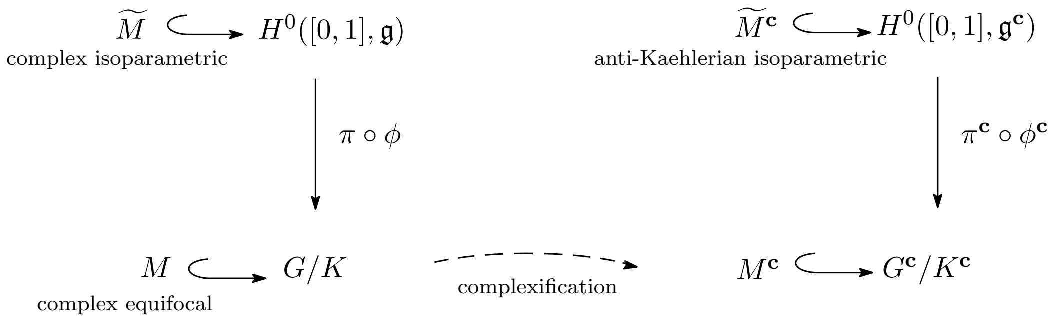


Fig. 6.

3 Hyperpolar actions

Let H be a closed subgroup of G. The H-action on G/K is called a *polar action* if H is compact and if, for each $x \in G/K$, there exists a complete embedded submanifold Σ_x through x meeting all principal H-orbits orthogonally. This submanifold Σ_x is called a *section* of this action through x. Furthermore, if the induced metric on Σ_x is flat, then the H-action is called a *hyperpolar action*. Here we illustrate that the assumption of the compactness of H is indispensable in these definitions. Consider the circle $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ on $\mathbb{R}^2(= \mathbb{C})$ by the multiplication in $\mathbb{C}$. This action $S^1 \curvearrowright \mathbb{R}^2(= \mathbb{C})$ is a compact group action with flat section, that is, a hyperpolar action, and the orbits and the sections of this action give the images of parameter curves of the polar coordinate of $\mathbb{R}^2$ (see Fig. 7). This action has the only fixed point (i.e., pole) $(0,0)$. Define the S^1-action on the unit sphere $S^2 := \{(x,w) \in \mathbb{R} \times \mathbb{C} \mid x^2 + |w|^2 = 1\}$ by $z \cdot (x,w) = (x,\, zw)$ ($(z \in S^1,\ (x,w) \in S^2)$. This action also is hyperpolar and has two fixed points (i.e., poles) $(1,0)$ and $(-1,0)$. On the other hand, the group action $\mathbb{R} \curvearrowright \mathbb{R}^2$ defined by $t \cdot (x,y) := (x+t, y)$ $(t \in \mathbb{R},\ (x,y) \in \mathbb{R}^2)$ is a non-compact action with flat section. However this action has no fixed point (i.e., pole) and the orbits and the sections of this action give the images of the parameter curves of the Euclidean coordinate (i.e., non-polar coordinate) of $\mathbb{R}^2$ (see Fig. 7).

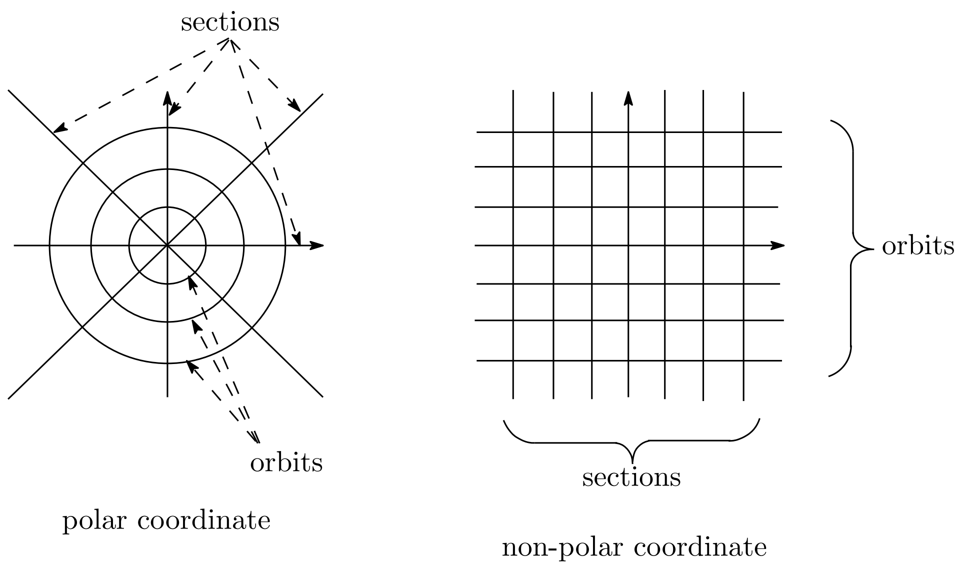


Fig.7.

It is known that principal orbits of a hyperpolar action are equifocal. On the other hand, in 1995, E. Heintze, R.S. Palais, C.L. Terng and G. Thorbergsson ([HPTT]) proved that any homogeneous equifocal submanifold in a simply connected symmetric space of compact type occurs as a principal orbit of a hyperpolar action. If there exists an involution σ of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$, then the H-action on G/K is called a *Hermann action*, where $\mathrm{Fix}\,\sigma$ is the fixed point group of σ and $(\mathrm{Fix}\,\sigma)_0$ is the identity component of $\mathrm{Fix}\,\sigma$. It is easy to show that Hermann actions are hyperpolar. In 2001, A. Kollross ([Kol]) proved that hyperpolar actions of cohomogeneity greater than one on an irreducible simply connected symmetric space of compact type are orbit equivalent to Hermann actions. Also, in 2007, O. Goertsches and G. Thorbergsson ([GT]) proved that any principal orbit M of a Hermann action are curvature-adapted, where the curvature-adaptedness means that, for any unit normal vector v of M (at x), the normal Jacobi operator $R(v)$ preserves the tangent space T_xM and $R(v)|_{T_xM}$ and the shape operator A_v of M commute. Hence we obtain the following fact.

All homogeneous equifocal submanifolds of codimension greater than one in an irreducible simply connected symmetric space of compact type are curvature-adapted.

4 Homogeneity of equifocal submanifolds

In this section, we shall state a homogeneity theorem for an equifocal submanifold in a symmetric space of compact type. In 1999, E. Heintze and X. Liu proved the following homogeneity theorem for an isoparametric submanifold in a Hilbert space.

Theorem 1([HL2]). *All irreducible isoparametric submanifolds of codimension greater than one in a Hilbert space are homogeneous.*

This result is the infinite dimensional version of the homogeneity theorem for isoparametric submanifolds in a (finite dimensional) Euclidean space by G. Thorbergsson ([Th]), which states that all irreducible isoparametric submanifolds of codimension greater than two in a Euclidean space are homogeneous. In 2002, by using the result of Heintze-Liu, U. Christ [Ch] proved the following homogeneity theorem for an equifocal submanifold in a simply connected symmmetric space of compact type.

Theorem 2([Ch]). *All irreducible equifocal submanifolds of codimension greater than one in a simply connected symmetric space of compact type are homogeneous.*

From this homogeneity theorem and the facts stated in the previous section, we have the following fact.

Theorem 3. *All equifocal submanifolds of codimension greater than one in an irreducible simply connected symmetric space of compact type occur as principal orbits of Hermann actions.*

5 Complex hyperpolar actions

Let G/K be a symmetric space of non-compact type and H be a closed subgroup of G. We ([Koi4]) called the H-action on G/K a *complex polar action* if, for each $x \in G/K$, there exists a complete embedded submanifold Σ_x through x meeting all principal H-orbits orthogonally. Furthermore, if the induced metric on Σ_x is flat, then we ([Koi4]) called H-action a *complex hyperpolar action*, where we note that this action should be called a hyper-complex polar action but that we called it a complex hyperpolar action for simplicity. We illustrate why we named this action thus. Define the $\mathbb{R}$-action on the hyperbolic space $H^2(= SO(1,2)/SO(2) = \{(x_1, x_2, x_3) \,|\, -x_1^2 + x_2^2 + x_3^2 = -1\}(\subset \mathbb{R}^3_1))$ by $\theta \cdot (x_1, x_2, x_3) = (x_1 \cosh\theta + x_2 \sinh\theta,\, x_1 \sinh\theta + x_2 \cosh\theta, x_3)$ $((\theta \in \mathbb{R},\, (x_1, x_2, x_3) \in H^2)$, where $\mathbb{R}^3_1$ is the Lorentzian space equipped with the Lorentzian inner product $-dx_1^2 + dx_2^2 + dx_3^2$. This action is a complex hyperpolar action. By the way, this action has no fixed point (i.e., pole) but the complexified action $\mathbb{C}$ on the anti-Kaehlerian symmetric space $SO(3,\mathbb{C})/SO(2,\mathbb{C})$ (which is the complexification of H^2) has fixed points (i.e. poles). These fixed points should be called complex poles of the original action. In this sense, we named the above action a complex (hyper)polar action. See also Fig. 8.

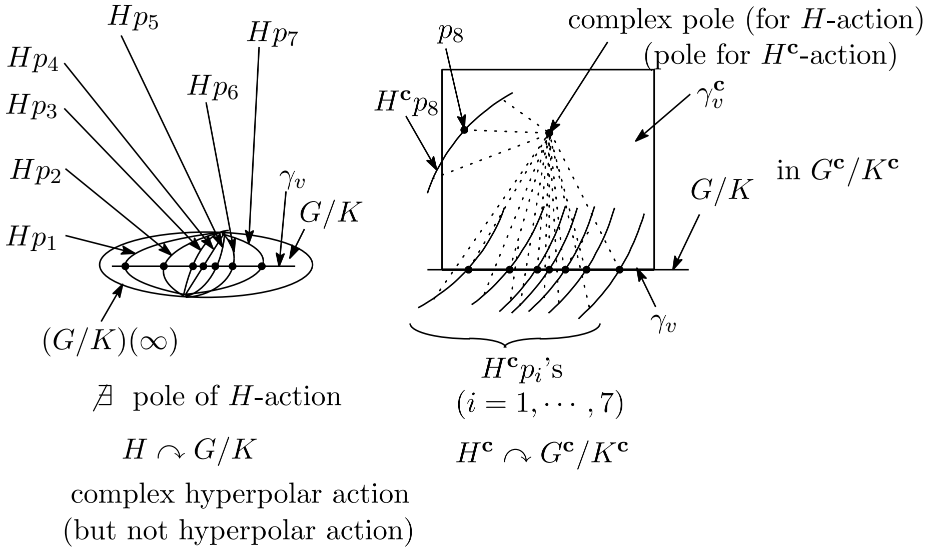


Fig. 8.

We ([Koi4]) showed that principal orbits of a complex hyperpolar action are complex equifocal. Conversely we ([Koi4]) showed that homogeneous complex equifocal submanifolds occur as principal orbits of complex hyperpolar actions. If there exists an involution σ of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$, then we called the H-action on G/K a *Hermann type action*. We ([Koi5]) showed that principal orbits of a Hermann type action are proper complex equifocal and curvature-adapted. Also, we ([Koi7]) showed that all complex hyperpolar actions with totally geodesic orbit are orbit equivalent to Hermann type actions.

6 Homogeneity of proper complex equifocal submanifolds

In this section, we shall state a homogeneity theorem for proper complex equifocal submanifolds. In [Koi13], we first proved the following homogeneity theorem for a proper anti-Kaehlerian isoparametric submanifold in an infinite dimensional anti-Kaehlerian space.

Theorem 4([Koi13]). *All irreducible proper anti-Kaehlerian isoparametric submanifolds of codimension greater than one in the infinite dimensional anti-Kaehlerian space are homogeneous.*

By using this homogeneity theorem, we proved the following homogeneity theorem for a proper complex equifocal C^ω-submanifold in a symmetric space of non-compact type.

Theorem 5([Koi13]). *All irreducible proper complex equifocal C^ω-submanifolds of codimension greater than one in a symmetric space of non-compact type are homogeneous.*

By using this homogeneity theorem and the facts stated in the previous section, we have recently proved the following fact.

Theorem 6([Koi14]). *All irreducible curvature-adapted proper complex equifocal C^ω-submanifolds of codimension greater than one in a symmetric space of non-compact type occur as principal orbits of Hermann type actions.*

Remark 1. In this theorem, we cannot replace "proper complex equifocal" to "complex equifocal". In fact, principal orbits of the N-action on an irreducible symmetric space G/K of non-compact type and rank greater than one are irreducible curvature-adapted complex equifocal submanifolds of codimension greater than one but they do not occur as principal orbits of a Hermann type action, where N is the nilpotent part in the Iwasawa's decomposition $G = KAN$ of G.

Also, we have recently proved the following fact for the curvature-adaptedness of a proper complex equifocal submanifold.

Theorem 7([Koi14]). *Let G/K be a symmetric space of non-compact type and rank r whose root system is reduced. Then all proper complex equifocal submanifolds of codimension r in G/K are curvature-adapted.*

Remark 2. By imitating the discussion in the proof of Theorem 7, we can show the following fact:

Let G/K be a symmetric space of compact type and rank r whose root system is reduced. Then all equifocal submanifolds of codimension r in G/K are curvature-adapted.

From Theorems 6 and 7, the following fact follows directly.

Theorem 8. *Let G/K be a symmetric space of non-compact type and rank $r(\geq 2)$ whose root system is reduced. Then all irreducible proper complex equifocal submanifolds of codimension r in G/K occur as principal orbits of Hermann type actions on G/K.*

7 Isoparametric submanifolds with flat section in the sense of Heintze-Liu-Olmos

In 2006, Heintze-Liu-Olmos [HLO] defined the notion of isoparametric submanifold with flat section in a general Riemannian manifold as a (properly embedded) complete submanifold with flat section and trivial normal holonomy group whose sufficiently close parallel submanifolds have constant mean curvature with respect to the radial direction. For a compact submanifold with trivial holonomy group and flat section in a symmetric space of compact type, they [HLO] showed that it is equifocal if and only if, for each parallel normal vector field v, F_{v_x} is independent of the choice of a point x of the submanifold, where F_{v_x} is the function defined in Page 1. Thus, if it is an isoparametric submanifold with flat section, then it is equifocal. Furthermore, for a compact submanifold in a symmetric space of compact type, they proved te following fact.

Theorem 9([HLO]). *Let M be a compact submanifold in a symmetric space of compact type. Then M is equifocal if and only if it is an isoparametric submanifold with flat section.*

The proof of this fact is performed by investigating the lift $(\pi \circ \phi)^{-1}(M)$ of M to the Hilbert space $H^0([0,1],\mathfrak{g})$.

On the other hand, we [Koi4] showed that, for a (properly embedded) complete submanifold with trivial normal holonomy group and flat section in a symmetric space of non-compact type, it is an isoparametric submanifold with flat section if and only if, for each parallel normal vector field v, $F^{\mathbf{c}}_{v_x}$ is independent of the choice of a point x of the submanifold, where $F^{\mathbf{c}}_{v_x}$ is the function defined in Page 2. Thus if it is an isoparametric submanifold with flat section, then it is complex equifocal. Conversely, we proved the following fact.

Theorem 10([Koi4]). *All curvature-adapted complex equifocal submanifolds in a symmetric space of non-compact type are isoparametric submanifolds with flat section.*

For a submanifold M in a Hadamard manifold N, we ([Koi11]) defined the notion of a focal point of non-Euclidean type on the ideal boundary $N(\infty)$ as follows. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of N and A the shape tensor of M. Let $\gamma_v : [0,\infty) \to N$ be the normal geodesic of M of direction $v \in T^{\perp}_x M$. If there exists a M-Jacobi field (resp. strongly M-Jacobi field) Y along γ_v satisfying $\lim_{t\to\infty} \frac{||Y_t||}{t} = 0$, then we call $\gamma_v(\infty)$ $(\in N(\infty))$ a *focal point* (resp. *strongly focal point*) *on the ideal boundary* $N(\infty)$ *of* M *along* γ_v (see Fig. 9), where $\gamma_v(\infty)$ is the asymptotic class of γ_v. Also, if there exists a M-Jacobi field Y

along γ_v satisfying $\lim_{t\to\infty} \frac{||Y_t||}{t} = 0$ and $\mathrm{Sec}(v, Y(0)) < 0$, then we call $\gamma_v(\infty)$ a *focal point of non-Euclidean type on* $N(\infty)$ *of* M *along* γ_v, where $\mathrm{Sec}(v, Y(0))$ is the sectional curvature for the 2-plane spanned by v and $Y(0)$.

We proved the following fact.

Theorem 11([Koi11]). *Let* M *be a curvature-adapted submanifold in a symmetric space* $N := G/K$ *of non-compact type. Then* M *is proper complex equifocal if and only if it is an isoparametric submanifold with flat section which admits no focal point of non-Euclidean type on the ideal boundary* $N(\infty)$ *of* N.

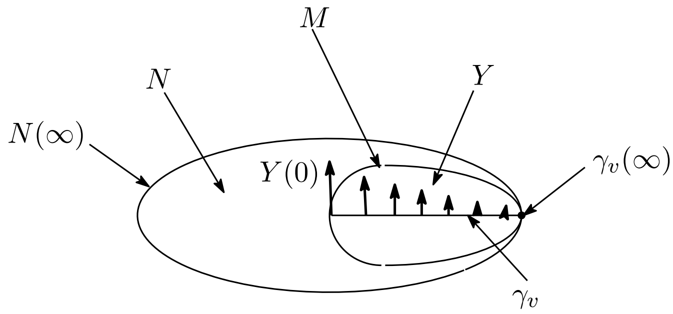


Fig. 9.

Furthermore, we have recently proved the following fact.

Theorem 12([Koi14]). *Let* G/K *be a symmetric space of non-compact type and rank* r. *Then all proper complex equifocal submanifolds of codimension* r *in* G/K *are isoparametric submanifolds with flat section.*

At the end of this section, we propose the following question.

Question. *Let* M *be a (properly embedded) complete submanifold in a symmetric space* $N = G/K$ *of non-compact type. Is* M *a proper complex equifocal submanifold if and only if it is an isoparametric submanifold with flat section which admits no focal point of non-Euclidean type on* $N(\infty)$*?*

8 Duality

In this section, we explain the duality of Hermann actions on symmetric spaces of compact type and Hermann type actions on symmetric spaces of non-compact type. Let G/K be a symmetric space of non-compact type and G_κ/K the compact dual of G/K. Also, let θ be the Cartan involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$, where $\mathrm{Fix}\,\theta$ is the fixed point group of θ and $(\mathrm{Fix}\,\theta)_0$ is the identity component of $\mathrm{Fix}\,\theta$. If H is a symmetric subgroup of G (i.e., $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$ for some involution σ of G), then the H-action on G/K is called a *Hermann type action*. Here we explain the duality between Hermann actions on G_κ/K and Hermann type actions on G/K. We may assume that $\theta \circ \sigma = \sigma \circ \theta$ by replacing H to a its suitable conjugate group if necessary. Then we obtain the involution $\hat{\sigma}$ of G_κ with $\theta \circ \hat{\sigma} = \hat{\sigma} \circ \theta$ from σ. Set $\hat{H} := (\mathrm{Fix}\,\hat{\sigma})_0$. Thus we obtain a Hermann action $\hat{H} \curvearrowright G_\kappa/K$. Conversely, we may assume that $\theta \circ \tau = \tau \circ \theta$ by replacing H' to a its suitable conjugate group if necessary except for three exceptional ones. Then we obtain the involution $\hat{\tau}$ of G with $\theta \circ \hat{\tau} = \hat{\tau} \circ \theta$ from τ. Set $\hat{H}' := (\mathrm{Fix}\,\hat{\tau})_0$. Thus we obtain a Hermann type action $\hat{H}' \curvearrowright G_\kappa/K$. Thus Hermann type actions on G/K correspond almost one-to-one to Hermann actions on G_κ/K.

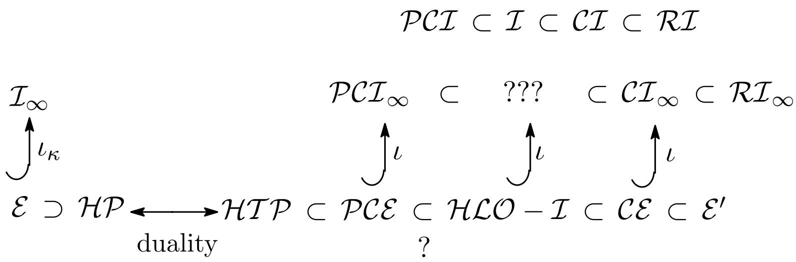


Fig. 10.

Notations in Fig. 10 are as in the next page. The congruence classes of the orbits of the action of the nilpotent group N on G/K belong to $(\mathcal{HLO} - \mathcal{I}) \setminus \mathcal{PCE}$, where N is the nilpotent part in the Iwasawa's decomposition $G = KAN$ of G. See [Koi9] about examples other than these classes belonging to $(\mathcal{HLO} - \mathcal{I}) \setminus \mathcal{PCE}$. Also, for almost all complete submanifolds all of whose principal curvatures are sufficiently close to zero in G/K, the ε-tubes over them belong to $\mathcal{E}' \setminus \mathcal{CE}$, where ε is any positive constant. Thus $\mathcal{E}'$ is a very big class.

$\mathcal{E}$: the set of all congruence classes of equifocal submanifolds in G_κ/K

$\mathcal{HP}$: the set of all congruence classes of principal orbits of a Hermann actions on G_κ/K

$\mathcal{I}_\infty$: the set of all congruence classes of isoparametric submanifolds in $H^0([0,1],\mathfrak{g}_\kappa)$

$\mathcal{E}'$: the set of all congruence classes of equifocal submanifolds in G/K, where they may not be compact

$\mathcal{CE}$: the set of all congruence classes of complex equifocal submanifolds in G/K

$\mathcal{HLO}$-$\mathcal{I}$: the set of all congruence classes of isoparametric submanifolds with flat section in G/K

$\mathcal{PCE}$: the set of all congruence classes of proper complex equifocal submanifolds in G/K

$\mathcal{HTP}$: the set of all congruence classes of principal orbits of Hermann type actions on G/K

$\mathcal{RI}_\infty$: the set of all congruence classes of real isoparametric submanifolds in $H^0([0,1],\mathfrak{g})$

$\mathcal{CI}_\infty$: the set of all congruence classes of complex isoparametric submanifolds in $H^0([0,1],\mathfrak{g})$

$\mathcal{PCI}_\infty$: the set of all congruence classes of proper complex isoparametric submanifolds in $H^0([0,1],\mathfrak{g})$

$\mathcal{RI}$: the set of all congruence classes of real isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\mathcal{CI}$: the set of all congruence classes of complex isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\mathcal{I}$: the set of all congruence classes of isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\mathcal{PCI}$: the set of all congruence classes of proper complex isoparametric submanifolds in $\mathbb{R}^m_\nu$

$\iota\,:\,\mathcal{CE}\to\mathcal{CI}_\infty \underset{\text{def}}{\Longleftrightarrow} \iota([M]) := [(\pi\circ\phi)^{-1}(M)]$

$\iota_\kappa\,:\,\mathcal{E}\to\mathcal{I}_\infty \underset{\text{def}}{\Longleftrightarrow} \iota_\kappa([M]) := [(\pi\circ\phi)^{-1}(M)]$

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