Super-edge-magic labelings of trees with large diameter

Akito Oshima*

Let $G$ be a simple undirected graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. A bijection $f$ from $V(G) \cup E(G)$ to \{1, 2, \ldots, |V(G)| + |E(G)|\} is called an edge-magic labeling of $G$ if there exists a constant $C$ such that $f(x) + f(y) + f(xy) = C$ for every edge $xy \in E(G)$. An edge-magic labeling $f$ of $G$ is called a super-edge-magic labeling if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\}$. We say that $G$ is edge-magic (resp. super-edge-magic) if there exists an edge-magic (resp. super-edge-magic) labeling of $G$.

In [1], Kotzig and Rosa made a conjecture that every tree is edge-magic, and proved that caterpillars are edge-magic. Here a caterpillar is a tree with the property that the removal of its pendant vertices leaves a path. Thus by definition, a path is a caterpillar. In [[2]; Conjecture 3.2], Enomoto, Llado, Nakamigawa and Ringel made a stronger conjecture that every tree is super-edge-magic (both conjectures still remain open). The following additional property is introduced in [3]. Let $T$ be a tree. For a vertex $v \in V(T)$, write $A(T, v) = \{w \in V(T) \mid d_T(v, w) \equiv 0 \pmod{2}\}$ and $B(T, v) = \{w \in V(T) \mid d_T(v, w) \equiv 1 \pmod{2}\}$. A super-edge-magic labeling $f$ of $T$ is said to be consecutive with respect to $v$ if $f(A(T, v)) = \{1, \ldots, |A(T, v)|\}$ and $f(B(T, v)) = \{|A(T, v)| + 1, \ldots, |V(T)|\}$. The property that $T$ has a super-edge-magic labeling which is consecutive with respect to a vertex $v \in V(T)$ is independent of the choice of $v$ (see [3]). Thus a super-edge-magic labeling which is consecutive with respect to a certain vertex is often referred to as a consecutively super-edge-magic labeling, and we say that $T$ is consecutively super-edge-magic if $T$ has a consecutively super-edge-magic labeling.

Excellent sources for more information in this area are found in the survey by Gallian [4], and the books by Bača and Miller [5] and Wallis [6].

The labelings of caterpillars constructed in [[1]; Theorem 4] are consecutively super-edge-magic labelings.

Since trees with diameter at most 3 are caterpillars, such trees are consecutively super-edge-magic. Thus it is natural to consider trees with small diameter. In fact, in [7] and [8], edge-magic labelings of trees having diameter 4 are studied, and some partial results are obtained. In [9], we took up another extremal case, i.e., the case where trees have large diameter, and proved the following theorem.

---

*Graph Theory and Applications Research Group, School of Electrical Engineering and Computer Science, Faculty of Engineering and the Built Environment, The University of Newcastle, NSW 2308, AUSTRALIA. E-mail: akitoism@yahoo.co.jp*
**Theorem 1.** Let \( n \geq 2 \) be an integer, and let \( T \) be a tree of order \( n \) such that \( T \) has diameter greater than or equal to \( n - 5 \). Then \( T \) has a super-edge-magic labeling.

Arguments used in the proof of Theorem 1 are, for the most part, unusual ones. That is, we try to avoid giving an explicit labeling of the tree under consideration. Thus the proof of Theorem 1 shows that we can construct a consecutively super-edge-magic labeling of a tree with large order from such labelings of trees with smaller order if certain conditions are satisfied. Moreover, it shows that consecutively super-edge-magic labelings are of great use when we study trees with large order and diameter.

This is the main result of my doctoral thesis.

In this presentation, we take up this memorable method and prove the following theorem.

**Theorem 2.** Let \( n \geq 2 \) be an integer, and let \( T \) be a tree of order \( n \) such that \( T \) has diameter greater than or equal to \( n - 6 \). Then \( T \) has a super-edge-magic labeling.

(This is joint work with Syunsuke Sawaki.)

**References**


