Counterexamples to the List Square Coloring Conjecture

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A graph $G$ is called chromatic-choosable if $\chi_l(G) = \chi(G)$. It is an interesting problem to find graphs that are chromatic-choosable. There are several famous conjectures that some classes of graphs are chromatic-choosable including the List Coloring Conjecture, which states that any line graph is chromatic-choosable.

The square $G^2$ of a graph $G$ is the graph defined on $V(G)$ such that two vertices $u$ and $v$ are adjacent in $G^2$ if the distance between $u$ and $v$ in $G$ is at most 2. Let $\chi(H)$ and $\chi_l(H)$ be the chromatic number and the list chromatic number of $H$, respectively. Kostochka and Woodall [4] proposed the following conjecture, which is called List Square Coloring Conjecture.

(List Square Coloring Conjecture) For any graph $G$, $\chi_l(G^2) = \chi(G^2)$.

The List Square Coloring Conjecture has attracted a lot of attention and been cited in many papers related with coloring problems so far, and it has been widely accepted to be true. The List Square Coloring Conjecture has been proved for several small classes of graphs.

In this paper, we disprove the List Square Coloring Conjecture by showing that there exists a graph $G$ such that $\chi_l(G^2) \neq \chi(G^2)$. We show that for each prime $n \geq 3$, there exists a graph $G$ such that $G^2$ is the complete multipartite graph $K_{n^2(2n-1)}$, where $K_{n^2(2n-1)}$ denotes the complete multipartite graph with $(2n-1)$ partite sets in which each partite set has size $n$. Note that $\chi_l(K_{n^2(2n-1)}) > \chi(K_{n^2(2n-1)})$ for every integer $n \geq 3$. Thus there exist infinitely many counterexamples to the List Square Coloring Conjecture. Moreover, we show that the gap between $\chi_l(G^2)$ and $\chi(G^2)$ can be arbitrary large, using the property that $\chi_l(K_{n^2(2n-1)}) - \chi(K_{n^2(2n-1)}) \geq n - 1$ for every integer $n \geq 3$.

References


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