On the multipartite hypergraphs

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Throughout this abstract, we use the terminology of [1]. A **hypergraph** $\mathcal{H}$ with finite vertex set $V(\mathcal{H})$ is a family of nonempty subsets of $V(\mathcal{H})$ whose union is $V(\mathcal{H})$, called edges. The set of vertices and edges of $\mathcal{H}$ are denoted by $V(\mathcal{H})$ and $E(\mathcal{H})$, respectively. If all the edges of a hypergraph $\mathcal{H}$ have the same cardinality t, then it is said that $\mathcal{H}$ is t-**uniform**. If none of the edges of $\mathcal{H}$ is included in another, then $\mathcal{H}$ is called a **simple** hypergraph. Here, we mean by a hypergraph, a simple one. A **vertex cover** of a hypergraph $\mathcal{H}$ is a subset A of its vertex set, with the property that for every edge e, $e \cap A \neq \emptyset$. A **minimal vertex cover** of $\mathcal{H}$ is a subset A of vertices such that A is a vertex cover and no proper subset of A is a vertex cover of $\mathcal{H}$. The number of vertices in a minimum vertex cover of $\mathcal{H}$ is called the **covering number** of $\mathcal{H}$, and denoted by $\tau(\mathcal{H})$. An **independent** set of $\mathcal{H}$ is a set of vertices which does not contain any edges of $\mathcal{H}$. One can see that a subset of vertices F is maximal independent if and only if $V(\mathcal{H}) \setminus F$ is minimal transversal. The number of vertices in a maximum independent set of $\mathcal{H}$ is called the **independent number** of $\mathcal{H}$, and denoted by $i(\mathcal{H})$. A **simplicial complex** Δ on the vertex set $V(\Delta)$ is a collection of subsets of $V(\Delta)$ such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. An element in Δ is called a **face** of Δ, and $F \in \Delta$ is said to be a **facet** if F is maximal with respect to inclusion. We denote the set of facets of Δ by $\mathcal{F}(\Delta)$. Let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_q\}$. The **independent complex** of a hypergraph $\mathcal{H}$ is defined as

$$\mathrm{Ind}(\mathcal{H}) = \langle F \subseteq V(\mathcal{H}) \ : \ F \text{ is a maximal independent set of } \mathcal{H} \rangle.$$

Let $R = K[x_1, \ldots, x_n]$, where K is a field. The **edge ideal** of a hypergraph $\mathcal{H}$ on n vertices, is the monomial ideal

$$I(\mathcal{H}) = (\mathbf{x}^e \ : \ e \in \mathcal{H}).$$

By $\mathbf{x}^e$, we mean $x_{i_1} \cdots x_{i_s}$, where $e = \{x_{i_1}, \ldots, x_{i_s}\}$. The **Stanley-Reisner ideal** of a simplicial complex Δ is the monomial ideal

$$I_\Delta = (\mathbf{x}^F \ : \ F \notin \Delta).$$

We have $I(\mathcal{H}) = I_{\mathrm{Ind}(\mathcal{H})}$. The **Stanley-Reisner ring** of Δ is defined as R/I_Δ. An s-uniform hypergraph $\mathcal{H}$ is said to be t-**partite**, if its vertex set $V(\mathcal{H})$ can be

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partitioned into sets $V_1, V_2, \ldots, V_t$, called the **sides** of $\mathcal{H}$, such that every edge in the edge set $E(\mathcal{H})$ of $\mathcal{H}$ consists of a choice of precisely one vertex from each side. An s-uniform t-partite hypergraph consisting all possible edges in this way, is called the **complete** s-uniform t-partite hypergraph. A t-partite hypergraph is said to be m-**balanced** if $|V_i| = m$ for every $i = 1, \ldots, t$.

Here, we classify all unmixed, vertex decomposable and shellable multipartite hypergraphs. We also determine all such hypergraphs which are chordal. Moreover, we characterize some of the algebraic properties (like Cohen-Macaulay and (almost) complete intersection) of such hypergraphs with respect to combinatorial terms. Also, we determine when the independent complex of such a hypergraph is a matroid and tight complex. The following are some of our main results:

Theorem 1. *Let $\mathcal{H}$ be a complete s-uniform t-partite hypergraph with $2 \leq s \leq t$. Then $R/I(\mathcal{H})$ is almost complete intersection if and only if one of the following holds:*
(a) *$\mathcal{H}$ is C_3, i.e. the 3-cycle, or*
(b) *$\mathcal{H}$ is the hypergraph over $t+1$ vertices $V = \{v_1, \ldots, v_{t+1}\}$, and with edges $\{v_1, \ldots, v_{t-1}, v_t\}$ and $\{v_1, \ldots, v_{t-1}, v_{t+1}\}$.*

Theorem 2. *Let $\mathcal{H}$ be a complete s-uniform t-partite hypergraph with $2 \leq s \leq t$, and let $r \geq 2$. Then the following conditions are equivalent:*
(a) $\operatorname{Ind}(\mathcal{H})$ *is vertex decomposable.*
(b) $\operatorname{Ind}(\mathcal{H})$ *is shellable.*
(c) *$R/I(\mathcal{H})$ is sequentially Cohen-Macaulay.*
(d) *$R/I(\mathcal{H})$ is sequentially S_r.*
(e) *$\mathcal{H}$ has $(t-1)$ sides consisting of exactly one vertex.*
(f) *$\mathcal{H}$ is a chordal hypergraph.*

References

[1] C. Berge, *Hypergraphs. Combinatorics of finite sets*, North-Holland Mathematical Library 45, North-Holland Publishing Co., Amsterdam, (1989).