The vertex weighted complexity of a graph

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1. Introduction

Graphs and digraphs treated here are finite simple. Let $G$ be a connected (unoriented) graph with vertex set $V(G)$ and edge set $E(G)$, where $E(G)$ is the set of unoriented edges of $G$. Let $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. The complexity $\kappa(G)$ of $G$ is the number of spanning trees in $G$. The complexities for various graphs were given in [1,2].

Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$. The adjacency matrix $A(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and $a_{ij} = 0$ otherwise. Let $D = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_G v_i$, and $Q = D - I$. For a connected graph $G$, let $f_G(u) = \det(I - uA(G) + u^2Q)$.

For a graph $G$, Northshield [5] showed that the complexity of $G$ is given by the derivative of the function above.

**Theorem 1**[Northshield] For a connected graph $G$, $f_G'(1) = 2(l - n)\kappa(G)$, where $n = |V(G)|$ and $l = |E(G)|$.

Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$. Furthermore, let $p : V(G) \rightarrow \mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$ be a function. We say that each vertex $v$ of $G$ has weight $p_v = p(v)$. Then we define the $n \times n$ matrix $L = (L(u, v))_{u,v \in V(G)}$ as follows: $L(u, v) := \sum_{(v,z) \in D(G)} p_z$ if $u = v$, $L(u, v) := -\sqrt{p_u p_v}$ if $(u, v) \in D(G)$, and and $L(u, v) := 0$ otherwise.

Let $T$ denote a spanning tree in $G$. For a vertex $v$ in $T$, we define the rooted directed tree $T_v$ towards the root $v$: $A(T_v) = \{(x, y) \mid xy \in E(T) \text{ and } d_T(v, x) > d_T(v, y)\}$, where $d_T(v, x)$ is the distance between $v$ and $x$ in $T$. Next, for each rooted directed tree $T_v$, the weight $w(T_v)$ is defined as follows: $w(T_v) = \prod_{(x,y) \in A(T_v)} p_y$. Also, we define $\kappa_v(G) = \sum_T w(T_v)$ and $\kappa_p(G) = \sum_{v \in V(G)} \kappa_v(G)$, where $T$ runs over all spanning trees in $G$. Then $\kappa_p(G)$ is called the vertex weighted complexity of $G$. If $p_v = 1$ for all $v \in V(G)$, then the vertex weighted complexity of $G$ is the number of rooted directed spanning trees in $G$.


**Theorem 2**[Chung and Langlands] The cofactor of $L$ obtained by deleting the $u$ th row and the $v$ th column has determinant $(p_u p_v)^{1/2} (\sum z \in V(G) p_z)^{-1} \kappa_p(G)$.

2. Vertex weighted complexities of graphs

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Let $G$ be a connected graph with $n$ vertices $v_1, \ldots, v_n$. Then we consider a $n \times n$ matrix $A_p = A_p(G) = (a_{ij})_{1 \leq i, j \leq n}$ as follows: $a_{ij} := \sqrt{p_{v_i} p_{v_j}}$ if $(v_i, v_j) \in E(G)$, and $a_{ij} := 0$ otherwise. This is called the (vertex) weighted matrix of $G$. Furthermore, let $A_p = (d_{ij})$ be the diagonal matrix with $d_{ii} = \sum_{(v_i, v_j) \in E(G)} p_{v_j}$, and $Q_p = A_p - I$. Note that $\mathcal{L} = D_p - A_p$. Then we introduce the following function: \( f_G(u, p) = \text{det}(I - uA_p(G) + u^2 Q_p). \)

The following theorem is a generalization of Theorem 1.

**Theorem 3** Let $G$ be a connected graph with $n$ vertices, and $p$ a vertex weight of $G$. Then $f_G(1, p) = 2^{|E(G)|} |p(V(G))| \kappa_p(G)$, where $p(E(G)) = \sum_{u \in E(G)} p_u p_v$ and $p(V(G)) = \sum_{v \in V(G)} p_v$.

3. Vertex weighted complexities of regular coverings

**Theorem 4** Let $G$ be a connected graph with $n$ vertices and $l$ unoriented edges, $\Gamma$ a finite group and $\alpha : D(G) \rightarrow \Gamma$ an ordinary voltage assignment. Moreover, let $p$ be a vertex weight of $G$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_l$ be all inequivalent irreducible representations of $\Gamma$, and $f_i$ the degree of $\rho_i$ for each $i$, where $f_1 = 1$. Then $f_G(u, \rho) = f_G(u, p) \cdot \prod_{i=2}^{l} \text{det}(I_{n, f_i} - u \sum_{g \in \Gamma} \rho_i(g) \otimes A_g + u^2 (f_i \circ Q_p))^{f_i}$.

Under the hypothesis of Theorem 4, $f_G(u, p)$ divides $f_{G^\alpha}(u, \rho)$.

We explicitly express the vertex weighted complexity of a connected regular covering of $G$ by using the vertex weighted complexity of $G$.

**Theorem 5** Let $G$ be a connected graph with $n$ vertices, $\Gamma$ be a finite group and $\alpha : D(G) \rightarrow \Gamma$ an ordinary voltage assignment. Moreover, let $p$ be a vertex weight of $G$. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_l$ be the irreducible representations of $\Gamma$, and $f_i$ the degree of $\rho_i$ for each $i$, where $f_1 = 1$. Suppose that the the $\Gamma$-covering $G^\alpha$ of $G$ is connected. Then the vertex weighted complexity of $G^\alpha$ is $\kappa_p(G^\alpha) = \kappa_p(G) \cdot \prod_{i=2}^{l} \text{det}(I_{n, f_i} - u \sum_{g \in \Gamma} \rho_i(g) \otimes A_g + (f_i \circ Q_p))^{f_i}$.

In the case that $p_v = 1$ for all $v \in V(G)$, we have $\kappa_p(G) = |V(G)| \kappa(G)$. From Theorem 5, we obtain a decomposition formula for the complexity of any connected regular covering of a graph $G$ by using that of $G$ (see [4]).

**References**