Stack-queue mixed layouts of graph subdivisions

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A vertex ordering of a graph $G$ is a total order of the vertex set $V(G)$. In a vertex ordering $<$ of a graph $G$, let $L(e)$ and $R(e)$ denote the endpoints of each edge $e \in E(G)$ such that $L(e) < R(e)$. Consider two edges $e, f \in E(G)$. If $L(e) < L(f) < R(e) < R(f)$ then $e$ and $f$ cross, and if $L(e) < L(f) < R(f) < R(e)$ then $e$ and $f$ nest. A stack (resp. queue) is a set of edges $E \subset E(G)$ such that no two edges in $E$ cross (nest). Observe that when traversing the vertex ordering, edges in a stack (queue) appear in LIFO (FIFO) order - hence the names.

For an integer $k > 0$, a $k$-stack (queue) layout of $G$ consists of a vertex ordering of $G$ and a partition $\{E_i \mid 1 \leq i \leq k\}$ of $E(G)$, such that each $E_i$ (which is called a page) is a stack (queue). A graph admitting a $k$-stack (queue) layout is called a $k$-stack (queue) graph. The stack number $sn(G)$ of a graph $G$ is the minimum $k$ such that there is a $k$-stack layout of $G$. The queue number $qn(G)$ of a graph $G$ is the minimum $k$ such that there is a $k$-queue layout of $G$. Stack layouts of graphs can be regarded as book embeddings of graphs. Thus, results for book embedding problems also hold true for stack layout problems.

Stack and queue layouts are generalized through the notion of a mixed layout. Here each edge of a graph is assigned to a stack or to a queue that is defined with respect to a common vertex ordering. Such a layout is called an $s$-stack $q$-queue layout, if there are $s$ stacks and $q$ queues, and a graph is called an $s$-stack $q$-queue graph if it has an $s$-stack $q$-queue layout. Part of the motivation for studying stack-queue mixed layouts is that they model the double-ended queue (dequeue) data structure, since a dequeue may be simulated by two stacks and one queue.

This paper studies stack-queue mixed layouts of graph subdivisions. It is interesting to determine the minimum number of division vertices in a stack-queue mixed subdivision of a given graph. Dujmović and Wood [1] showed the following proposition:

**Proposition 1.** (Dujmović and Wood [1])

1. For all integers $s \geq 1$ and $q \geq 1$, every graph $G$ has an $s$-stack $q$-queue mixed subdivision with at most $4\lceil \log_{(s+q)} q sn(G) \rceil$ division vertices per edge.
2. For all integers $s \geq 1$ and $q \geq 1$, every graph $G$ has an $s$-stack $q$-queue mixed subdivision with at most $2 + 4\lceil \log_{(s+q)} qn(G) \rceil$ division vertices per edge.

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Let \( \alpha = \frac{s+q-1+\sqrt{(s+q-1)^2+4q}}{2} \) be the positive root of \( X^2 - (s + q - 1)X - q = 0 \), and \( \beta \) be the negative root. Define

\[
h(n) = \min\{k \mid \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} \geq n\}.
\]

The following theorem improves Proposition 1, because \( \alpha > \sqrt{(s + q)q} \) and \( h(n) \leq \lceil \log_\alpha n \rceil \).

**Theorem 2.** For all integers \( s \geq 1 \) and \( q \geq 1 \), every graph \( G \) has \( s \)-stack \( q \)-queue mixed subdivisions with at most \( 2h(sn(G)) + 2 \) and \( 2h(qn(G)) + 4 \) division vertices per edge, respectively.

We prove Theorem 2 by constructing a 1-stack 1-queue mixed layout of a newly defined \( (s, q) \)-ary tree. This construction method is different from the one given in Dujmović and Wood [1], because not all leaves are laid out at the right end in our layout.

**References**