On Cameron–Erdős problem of Sidon sets

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A set $A$ of positive integers is called a Sidon set if all the sums $a_1 + a_2$, with $a_1 \leq a_2$ and $a_1, a_2 \in A$, are distinct. In this talk we consider Cameron–Erdős problem [1] which was suggested in 1990. The problem is to estimate the number of Sidon sets contained in $[n] := \{1, 2, \ldots, n\}$.

Let $Z_n$ be the family of Sidon sets contained in $[n]$. Results of Chowla, Erdős, Singer, and Turán from the 1940s imply that the maximum size of Sidon sets in $[n]$ is $\sqrt{n}(1 + o(1))$. From this result, one trivially has

$$2^{\sqrt{n}(1+o(1))} \leq |Z_n| \leq \sum_{1 \leq i \leq F(n)} \binom{n}{i} = 2^{(1/2+o(1))\sqrt{n}\log n}. \quad (1)$$

Cameron and Erdős [1] improved the lower bound in (1) by showing that $\limsup_n |Z_n|^{2^{-F(n)}} = \infty$ and asked whether the upper bound could also be strengthened. However, these bounds have not been notably improved for about 20 years. Our result is as follows.

**Theorem 1.** (Kohayakawa, Lee, Rödl, Samotij [2]) There is a constant $c$ for which $|Z_n| \leq 2^{c\sqrt{n}}$ for all large enough $n$.

Our proof method gives that the constant $c$ in Theorem 1 may be taken to be arbitrarily close to $\log_2(32e) = 6.442 \cdots$. We do not make any attempts to optimize this constant as it seems that our approach cannot yield a sharp estimate for $\log_2|Z_n|$ (in particular, we give the proof for constants arbitrarily close to $\log_2(33e) = 6.487 \cdots$).

Very recently, Saxton and Thomason [3] derived Theorem 1 (for $c$ arbitrarily close to 55) from a more general theorem bounding the number of independent sets in certain hypergraphs. They also proved that $|Z_n| \geq 2^{(1.16+o(1))\sqrt{n}}$.

For the proof of Theorem 1, we define a graph from our setting such that, roughly speaking, a Sidon set in $[n]$ corresponds to an independent set of the graph. In addition, the graph satisfies some dense condition. We show that in a graph satisfying the dense condition, the number of independent sets of given size $t$ is much smaller than the trivial bound $\binom{n}{t}$. By applying it repeatedly, we have the

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new upper bound on the number of Sidon sets of a given size contained in \([n]\) as follows.

**Theorem 2.** (Kohayakawa, Lee, Rödl, Samotij [2]) Let \(0 < \sigma < 1\) be a real number. For any large enough \(n\) and \(t \geq 2s_0\), where \(s_0 = \left( \frac{2(1 - \sigma)^{-1}n \log n}{32en} \right)^{1/3}\), we have

\[
|Z_n(t)| \leq n^{3s_0} \left( \frac{32en}{\sigma t^2} \right)^t.
\]

(2)

Theorem 1 follows from Theorem 2 by summing over all \(t\).

References

