The metric dimension of comb product graphs

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Throughout this paper, all graphs G are finite, connected, and simple. We denote by V the vertex set of G and by E the edge set of G. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is the length of a shortest path from u to v in G. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. The *representation* of a vertex v of G with respect to W is defined as the k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is called a *resolving set* of G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x|W) \neq r(y|W)$. A *basis* of G is a resolving set of G with the minimum cardinality, and the *metric dimension* of G refers to its cardinality and is denoted by $\beta(G)$.

The metric dimension problems were first studied by Harary and Melter [4]. Khuller *et al.* [5] studied the metric dimension motivated by the robot navigation in a graph space. A resolving set for a graph corresponds to the presence of distinctively labelled "landmark" nodes in the graph. It is assumed that a robot can detect the distance to each node of the landmarks, and hence uniquely determine its location in the graph.

Garey and Johnson [3] showed that determining the metric dimension of an arbitrary graph is an NP-complete problem. However, Chartrand *et al.* [2] have obtained some results as follows.

Theorem 1. (Chartrand *et al.* [2]) *Let G be a connected graph of order $n \geq 2$. Then*

1. $\beta(G) = 1$ *if and only if* $G = P_n$.
2. $\beta(G) = n - 1$ *if and only if* $G = K_n$.
3. *For* $n \geq 3$, $\beta(C_n) = 2$.
4. $\beta(G) = n - 2$ *if and only if G is either* $K_{r,s}$ *for* $r, s \geq 1$, *or* $K_r + \overline{K_s}$ *for* $r \geq 1, s \geq 2$, *or* $K_r + (K_1 \cup K_s)$ *for* $r, s \geq 1$.

There are also some results of the metric dimension problem for graphs resulting from operations on graphs. Some results on certain joint product graphs have been proved in [1]. Caceres *et al.* [1] have determined the metric dimension of graphs which are obtained by Cartesian product of two or more graphs. Some graphs which are constructed by corona product of two graphs have been studied in [7].

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Saputro *et al.* [6] have been showed the metric dimension of lexicographic product of connected graph G and an arbitrary graph H. In this paper, we study the metric dimension of *comb product* of connected graphs G and H.

Let G and H be two connected graphs. Let o be a vertex of H. The *comb product* between G and H, denoted by $G \rhd H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of G. By the definition of comb product, we can say that $V(G \rhd H) = \{(a, v)|a \in V(G), v \in V(H)\}$ and $(a, v)(b, w) \in E(G \rhd H)$ whenever $a = b$ and $vw \in E(H)$, or $ab \in E(G)$ and $v = w = o$.

By considering the structure of the comb product graph, we show the metric dimension of $G \rhd H$ into two theorems. In Theorem 2, we show the metric dimension of $G \rhd H$ when H is not a path, while in Theorem 3, we prove $\beta(G \rhd P_n)$ for $n \geq 2$. We recall the corona product between two graphs G and H, denoted by $G \odot H$, which is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and connecting all vertices of the i-th copy of H to the i-th vertex of G.

Theorem 2. *Let G and H be connected graphs with order at least 2. If $|V(G)| = m$ and H is not a path, then*

$$\beta(G \rhd H) = \begin{cases} m(\beta(H) - 1), & \text{if there exist a basis of } H \text{ containing } o, \\ m\beta(H), & \text{otherwise.} \end{cases}$$

Theorem 3. *Let G be a connected graph of order $m \geq 2$. For $n \geq 2$, if o is a leaf of a path P_n, then $\beta(G) \leq \beta(G \rhd P_n) \leq \beta(G \odot K_1)$. Otherwise $\beta(G \rhd P_n) = m$.*

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