Maximally ambiguously $k$-colorable graphs ♠

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An anticlique of a graph $G$ is a set of pairwise nonadjacent vertices of $G$, and a $k$-coloring of $G$ is a partition of $V(G)$ into at most $k$ anticliques. Graphs with at least one $k$-coloring are $k$-colorable, and we call those with more than one $k$-coloring ambiguously $k$-colorable. A graph is maximally ambiguously $k$-colorable if it is ambiguously $k$-colorable but adding any edge between distinct nonadjacent vertices produces a graph which is not. We give a full description of the maximally ambiguously $k$-colorable graphs in terms of quadratic matrices.

Let $A$ be a $k \times k$-matrix where all entries are non-negative integers. $A$ is tiny if it is a diagonal matrix with exactly one entry 2, all others at most 1, and at least two diagonal entries 0. $A$ is small if it is a diagonal matrix with at least one entry 2, all others at most 2, and exactly one diagonal entry 0. $A$ is special if all diagonal entries are nonzero, exactly one off-diagonal entry is 1, and all others are 0. $A$ is normal if it is a block diagonal matrix with quadratic blocks $M, D$, where $D$ is a diagonal matrix with nonzero diagonal entries and $M$ has each of the following properties:

(i) All diagonal entries are nonzero,
(ii) $M$ is of size $r \geq 2$ and fully indecomposable, that is it does not admit an $s \times (r-s)$ zero submatrix, where $s \in \{1, \ldots, r-1\}$, and
(iii) whenever $M(i,j) \geq 2$ for $i \neq j$ then there exists a sequence $f_0, \ldots, f_{\ell}$ from $\{1, \ldots, r\}$ with $\ell \geq 3$, $f_{h-1} \neq f_h$ and $M(f_{h-1}, f_h) \geq 1$ for all $h \in \{1, \ldots, \ell\}$, and $(f_0, f_1) = (f_{\ell-1}, f_{\ell}) = (i,j)$.

Finally, $A$ is desirable if it is tiny or small or special or normal.

Given a matrix $A$ with non-negative integer entries, we associate a graph $G(A)$ on $\{(i, j, t) : i, j \in \{1, \ldots, k\}, t \in \{1, \ldots, A(i, j)\}\}$, where $(i, j, t)$ and $(i', j', t')$ are adjacent if and only if $i \neq i'$ and $j \neq j'$. Our main theorem can be formulated as follows.

**Theorem 1.** Given $k \geq 1$, a graph is maximally ambiguously $k$-colorable if and only if it is isomorphic to $G(A)$ for some desirable $k \times k$-matrix $A$.

As a (nontrivial) application of this theorem we get a Turán type result. Given integers $r, n$, the Turán number of $n$ and $K_{r+1}$ is the largest number $\text{ex}(n, K_{r+1})$ of edges a (simple) graph on $n$ vertices without $K_{r+1}$ as a subgraph can have. A desirable $k \times k$-matrix $A$ is called row-balanced if $|\sum_{j=1}^{k} A(i, j) - A(i', j)| \leq 1$ for all $i, i' \in \{1, \ldots, k\}$, that is, the difference of any two row-sums is 0 or ±1. Likewise,
A is \textit{column-balanced} if $|\sum_{i=1}^{k} A(i,j) - A(i,j')| \leq 1$ for all $j, j' \in \{1, \ldots, k\}$. $A$ is \textit{balanced} if it is both row- and column-balanced. Let us call a special matrix $A$ \textit{very special} if it is row-balanced and the sum of the entries in row $j$ is $\lfloor \frac{n}{k} \rfloor$, where $j$ is the index of the unique column with an off-diagonal entry, or, symmetrically, if $A$ is column-balanced and the sum of the entries in column $i$ is $\lfloor \frac{n}{k} \rfloor$, where $i$ is the index of the unique row with an off-diagonal entry. Observe that we can realize a very special matrix for all $n \geq k + 1$. If $k$ divides $n$ or $n+1$ then, up to isomorphism, they all induce one and the same graph, whereas if $k$ does neither divide $n$ nor $n+1$ then, up to isomorphism, they all induce one among two nonisomorphic graphs. Finally, suppose that $A$ is a normal matrix, and let $M, D$ be as in the definition of \textit{normal}; we call $A$ \textit{mininormal}, if $A$ is balanced, $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $2k \leq n < 3k$.

\textbf{Theorem 2.} Let $n, k$ be integers. Then the maximum number of edges an ambiguously $k$-colorable graph on $n$ vertices can have is $ex(n, K_{k+1}) - \max\{1, \lfloor \frac{n}{k} \rfloor\}$. The graphs where equality is attained are of the form $G(A)$, where $A$ is a desirable $k \times k$-matrix such that $A$ is tiny or small or very special or mininormal.