On a problem for a kinked crack

Hiromichi Itou

Faculty of Engineering, Gunma University
1-5-1, Tenjin-cho, Kiryu, 376-8515
E-mail address: h-itou@math.sci.gunma-u.ac.jp

Propagation of cracks is a phenomenon which leads to the brittle failure of materials. Analysis of the crack growth has been a major subject of fracture mechanics since Griffith's work [2] in 1920. After that the formulation of fracture mechanics began with Irwin [3] and his associates around 1950's. And the continuum field approach to fracture of solids was launched with the introduction of the elastic stress intensity factor K_i as a crack tip field characterizing parameter by Irwin (1957)[3]. In a 2-dimensional elastic body K_i (i=1, 2) is defined by coefficients of the expansion of the stress in the neighborhood of the crack tip, where K_1 and K_2 correspond to normal and tangential deformation along the crack surface respectively. He proposed that a crack begins to grow in a cracked body with limited plastic deformation when the elastic stress intensity factor reaches at a value called the fracture toughness of the material. Irwin also introduced the energy release rate G which means a rate of the energy, per unit length along the crack edge, that is supplied by the elastic energy in the body and by the loading system in creating the new fracture surface. Then, he showed that the energy release rate is described by the elastic stress intensity factor under the state of plane strain.

$$G = \frac{1}{E}(K_1^2 + K_2^2), \tag{1}$$

where E is Young's modulus. However, about a problem of determining the direction of crack propagation in the elastic plate there only exist many criteria from engineering sense and there are hardly any mathematical results. In particular, three famous criteria of them are the following:

1. *Maximum stress criterion* is to find the angle θ_1 such that the stress of the θ_1 direction, as the tip of an initial crack is the origin, attains the maximum value in any directions,

2. *Maximum energy release rate criterion* is to find the angle θ_2 such that

$$G(\theta_2) = \max_{\theta} G(\theta), \tag{2}$$

3. *Local symmetry criterion* is to find the angle θ_3 such that

$$K_2(\theta_3) = 0. \tag{3}$$

Unfortunately, we cannot know which criterion is true because it is very difficult to measure the angle of crack propagation by experiment. Although the difference among three criteria is discussed in [1] and [4], it seems to remain an open problem. In the case of applying 1 we can precisely calculate the angle θ_1 from the initial stress field near the tip of a crack. In the cases of applying 2 and 3 we must calculate the energy release rate G at the tip of a crack. Then, we need to seek the solution in the elastic plate with virtual kinked crack extension (see Figure 1) because G is defined by the released potential energy as the crack increases a unit length. Accordingly, in this talk we consider the formulation of the kinked crack problem in a 2-dimensional elastic body and the procedure for solving it.

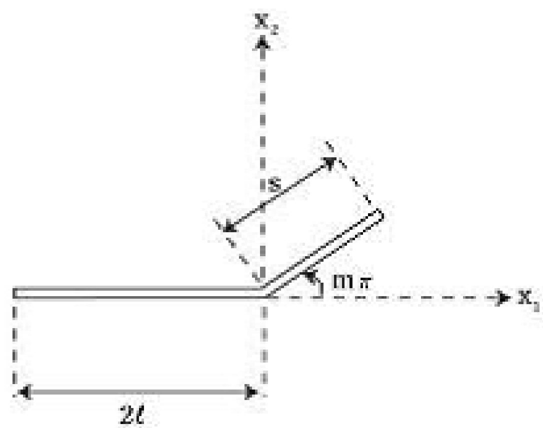


Figure 1: Kinked crack extension

References

[1] M. Amestoy and J. B. Leblond, Crack paths in plane situations – 2. Detailed form of the expansion of the stress intensity factors. *Int. J. Solids. Structures*, **29** (1992), 465 – 501.

[2] A. A. Griffith, The phenomenon of rupture and flow in solid, *Philos. Trans. Roy. Soc. Lond.* , **A221** (1920), 163 – 198.

[3] G. R. Irwin, Analysis of stresses and strains near the end of a crack traversing a plate, *J. Appl. Mech.* , **24** (1957) 361 – 364.

[4] K. Ohtsuka, Comparison of criteria on the direction of crack extension, *J. Comp. Appl. Math.*, **149** (2002), 335 – 339.

On a problem for a kinked crack

Gunma University
Faculty of Engineering
Hiromichi Itou

Contents

Background

Keywords

- Fracture mechanics
- Crack
- Brittle fracture
- Griffith Theory
  (Energy balance criterion)

(The released potential energy resulted from the extension of the crack)
≧ (The increment of the surface energy)

G: Energy release rate ≧ γ : Fracture toughness

The direction of crack propagation (path) ?

—3 criteria—

1. Maximum stress criterion

σ_θ : The stress of the θ-direction at the crack tip

$$\sigma_{\theta_0} = \max_\theta \sigma_\theta, \quad \sigma_{r\theta_0} = 0$$

2. Maximum energy release rate criterion

$$G(\theta_0) = \max_\theta G(\theta)$$

3. Local symmetry criterion

Expansion of the stress near the crack tip

$$\sigma = K_1 r^{-\frac{1}{2}} h_1(\theta) + K_2 r^{-\frac{1}{2}} h_2(\theta) + O(r)$$

K_i: The stress intensity factors

K_1: The normal deformation along the crack surface (mode 1)

K_2: The tangential deformation along the crack surface (mode 2)

$$K_2(\theta_0) = 0$$

Question

Are 1., 2. and 3. equivalent?

1. ⇒exact

2. 3. ⇒Virtual kinked crack extension

Formulation of the problem

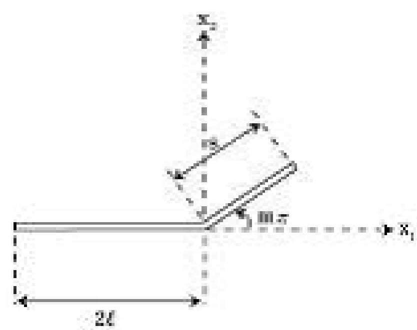


Fig. 1: Kinked crack extension (-1<m<1)

Problem

① The equilibrium equations

$$\frac{\partial}{\partial X_j}\sigma_{ij} = 0 \quad i, j = 1,2$$

② Free traction condition on the crack

$$\sigma_{ij}^{+}\nu_j = \sigma_{ij}^{-}\nu_j = 0$$

ν : The unit outward normal

③ Uniform loads at infinity σ^{∞}

Airy's stress function (U)

$$\sigma_{11} = \frac{\partial^2 U}{\partial X_2^2}, \quad \sigma_{12} = -\frac{\partial^2 U}{\partial X_1 \partial X_2}, \quad \sigma_{22} = \frac{\partial^2 U}{\partial X_1^2}$$

$$① \Rightarrow \Delta^2 U = 0$$

Goursat (Φ, χ), Muskhelishvili (Φ, Ψ)'s stress function

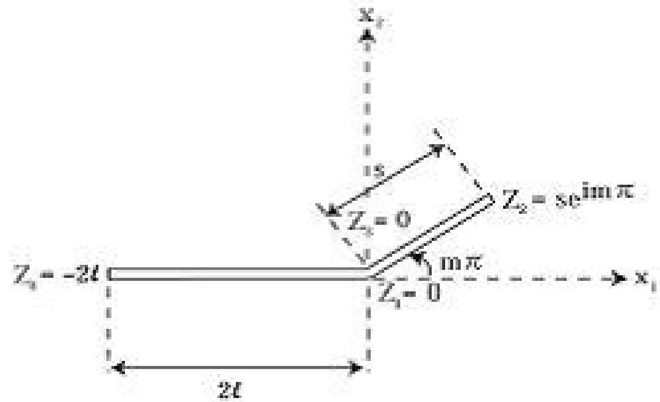
$$U = \mathrm{Re}\{\overline{Z}\Phi(Z) + \chi(Z)\} \quad (Z = X_1 + iX_2)$$

$$\frac{\partial U}{\partial X_1} + i\frac{\partial U}{\partial X_2} = \Phi(Z) + Z\overline{\Phi'(Z)} + \overline{\Psi(Z)}, \quad \Psi(Z) = \frac{d\chi}{dZ}$$

$$\sigma_{11} + \sigma_{22} = \Delta U = 2\left[\Phi'(Z) + \overline{\Phi'(Z)}\right] = 4\,\mathrm{Re}\{\Phi'(Z)\}$$

$$② \Rightarrow ④ \quad \Phi(Z) + Z\overline{\Phi'(Z)} + \overline{\Psi(Z)} = Const.$$

Domain



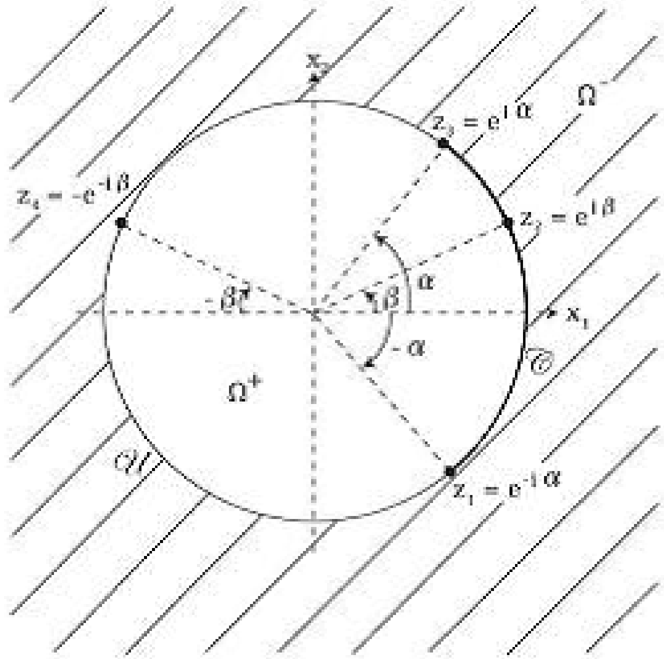
Conformal mapping $Z = w(z)$

$$Z = w(z) = \mathrm{R}e^{im\alpha}\frac{(z-e^{i\alpha})(z-e^{-i\alpha})}{z}\left(\frac{z-e^{-i\alpha}}{z-e^{i\alpha}}\right)^{m}$$

$$\ell = 2R\left(\cos\frac{\alpha+\beta}{2}\right)^{1-m}\left(\cos\frac{\alpha-\beta}{2}\right)^{1+m},$$

$$s = 4R\left(\sin\frac{\alpha+\beta}{2}\right)^{1+m}\left(\sin\frac{\alpha-\beta}{2}\right)^{1-m},$$

$$\sin\beta \equiv m\sin\alpha.$$



$\Phi(Z) = \phi(z)$、$\Psi(Z) = \psi(z)$

⑤ $\varphi(z) + \dfrac{w(z)}{\overline{w'(z)}}\overline{\varphi'(z)} + \overline{\psi(z)} = Const.$ for $z \in U$

$$\frac{w(z)}{\overline{w'(z)}} = \frac{w(z)}{\overline{w(z)}}\overline{\left\{\frac{w(z)}{w'(z)}\right\}}$$

$$\overline{\left\{\frac{w(z)}{w'(z)}\right\}} = -\frac{(z-e^{i\alpha})(z-e^{-i\alpha})}{z(z-e^{i\beta})(z+e^{-i\beta})} \equiv -Q(z)$$

$$\frac{w(z)}{\overline{w(z)}} = \begin{cases} 1 & z \in U \setminus C \\ e^{2\pi i\alpha} & z \in C \end{cases}$$

Behavior of ϕ and ψ on U

i) $e^{i\beta}, -e^{-i\beta}$

ϕ : Infinitely differentiable
ψ : Simple pole

ii) $e^{\pm i\alpha}$

ϕ'、ψ' : Weakly singular

$$\lim_{z \to e^{\pm i\alpha}} \left(z - e^{\pm i\alpha}\right)\varphi'(z) = 0$$

Reduction to an integral equation

③Uniform loads at infinity

$$\Phi(Z) = \Gamma Z + O(1), \quad \Psi(Z) = \Gamma' Z + O(1)$$

↓ Z=w(z)

real const: $\Gamma = \frac{1}{4}\left(\sigma_{11}^{\infty} + \sigma_{22}^{\infty}\right)$

complex const: $\Gamma' = \frac{1}{2}\left(\sigma_{22}^{\infty} - \sigma_{11}^{\infty}\right) + i\sigma_{12}^{\infty}$

⑥ $\varphi(z) = \Gamma R e^{im\alpha} z + O(1), \quad \psi(z) = \Gamma' R e^{im\alpha} z + O(1)$

⑤ $\varphi(z) + \dfrac{w(z)}{\overline{w'(z)}}\overline{\varphi'(z)} + \overline{\psi(z)} = Const.$ for $z \in U$

Applying $(z-e^{i\beta})(z+e^{-i\beta})$ to the both sides of ⑤,

$$(z-e^{i\beta})(z+e^{-i\beta})\varphi(z)+(z-e^{i\beta})(z+e^{-i\beta})\frac{w(z)}{\overline{w'(z)}}\overline{\varphi'(z)}$$

⑦

$$+(z-e^{i\beta})(z+e^{-i\beta})\overline{\psi(z)}=C(z-e^{i\beta})(z+e^{-i\beta}).$$

Adding $f_1(z)\equiv-(z-e^{i\beta})(z+e^{-i\beta})(\Gamma \mathrm{Re}^{im\pi}z+O(1))$
$-((\Gamma+\overline{\Gamma'})\mathrm{Re}^{-im\pi}z+O(1)),$

Applying **Cauchy**'s integral theorem to ⑦, we get

⑧

$$(z-e^{i\beta})(z+e^{-i\beta})\varphi(z)-\frac{1-e^{2im\pi}}{2\pi i}\int_C\frac{(t-e^{i\alpha})(t-e^{-i\alpha})\overline{\varphi'(t)}}{t(t-z)}dt$$

$$=f_2(z)\quad for\quad z\in\Omega^-,$$

where $f_2(z)=\int_C\frac{f_1(t)}{t-z}dt+(z-e^{i\beta})(z+e^{-i\beta})(\Gamma \mathrm{Re}^{im\pi}z+O(1))$

By virtue of **Plemelj's** Formula, $z\in\Omega^-\to z\in C$,

⑨

$$(z-e^{i\beta})(z+e^{-i\beta})\varphi(z)+\frac{1-e^{2im\pi}}{2}(z-e^{i\alpha})(z-e^{-i\alpha})\overline{\varphi'(z)}$$

$$-\frac{1-e^{2im\pi}}{2\pi i}v.p.\int_C\frac{(t-e^{i\alpha})(t-e^{-i\alpha})\overline{\varphi'(t)}}{t(t-z)}dt=f(z)\quad for\quad z\in C.$$

Expansion in powers of the crack extension

Noting that $R=\frac{\ell}{2}+O(s),$

$$\alpha=\sqrt{\frac{2}{(1-m^2)\ell}}\left(\frac{1-m}{1+m}\right)^{\frac{m}{2}}\sqrt{s}+O\left(s^{\frac{3}{2}}\right),$$

$$\beta=m\alpha+O\left(s^{\frac{3}{2}}\right).$$

Changing of variable $z = e^{i\alpha\varsigma}$
which maps C to [-1, 1], we can get

⑩
$$(\varsigma - m)\varphi_0(\varsigma) + \frac{1-e^{2im\pi}}{4}(1-\varsigma^2)\varphi_0'(\varsigma) - \frac{1-e^{2im\pi}}{4\pi i} v.p.\int_{-1}^{1} \frac{(1-\lambda^2)\varphi_0'(\lambda)}{(\lambda-\varsigma)} = f_0(\varsigma) \quad for\ \varsigma \in [-1, 1]$$

Andersson's (1969) formula:

$$K_1(s) - iK_2(s) = 2\sqrt{\pi}\varphi'(e^{i\beta})e^{-\frac{im\pi}{2}}\left[w''(e^{i\beta})\right]^{-\frac{1}{2}}$$

Singular integro-differential equation

Example : Aircraft wings of finite span

$$\frac{\Gamma(x)}{B(x)} - \frac{1}{\pi} v.p.\int_{-a}^{a} \frac{\Gamma'(t)}{t-x} dt = f(x),$$

Where $2a$: the span of the wing

$$B(x) = \frac{mb(x)}{8}, \quad f(x) = 4V\alpha(x),$$

$b(x)$: the chord of the profile
$\Gamma(x)$: the circulation of the airflow around this profile
$\alpha(x)$: the geometrical angle of incidence
V : the velocity of the airflow at infinity

Singular integral equation

$$K\gamma \equiv C(x)\gamma(x) + \frac{D(x)}{\pi i} v.p. \int_{-a}^{a} \frac{\gamma(t)}{t-x} dt + \frac{1}{\pi i} \int_{-a}^{a} k(x,t)\gamma(t)\, dt = f(x)$$

The dominant part of the operator K

$$K_0\gamma \equiv C(x)\gamma(x) + \frac{D(x)}{\pi i} v.p. \int_{-a}^{a} \frac{\gamma(t)}{t-x} dt$$

Assume : C(x) ± D(x) do not vanish anywhere on $[-a, a]$

Regular Fredholm equation

$$\gamma(x) + \int_{-a}^{a} k_1(x,t)\gamma(t)\, dt = \tilde{f}(x)$$

Comparison of 3 criteria

In J. B. Leblond (1989) the expansion of the stress intensity factors $K_p(s)$ (p=1, 2) at the extended crack tip in powers of ***s*** is of the general form

$$K_p(s) = K_p^* + K_p^{\frac{1}{2}}\sqrt{s} + K_p^1 s + O\left(s^{\frac{3}{2}}\right),$$

$$K_p^* = F_{pq}(m) K_q, \quad K_p^{\frac{1}{2}} = G_p(m) T.$$

K_p: the stress intensity factors at the original crack tip 0

T : non-singular stress at the original crack tip 0

・2． Maximum energy release rate criterion

In the state of the plane stress, (E: Young's modulus)
$$G = \frac{1}{E}\left(K_1^2(s) + K_2^2(s)\right)$$

$$K_1^* \frac{\partial K_1^*}{\partial m} + K_2^* \frac{\partial K_2^*}{\partial m} = 0.$$

$$F_{11}[m] = 1 - \frac{3\pi^2}{8} m^2 + \left(\pi^2 - \frac{5\pi^4}{128}\right) m^4 + \left(\frac{\pi^2}{9} - \frac{11\pi^4}{72} + \frac{119\pi^6}{15360}\right) m^6 + O(m^8)$$

$$F_{12}[m] = -\frac{3\pi}{2} m + \left(\frac{10\pi}{3} + \frac{\pi^3}{16}\right) m^3 + \left(-2\pi - \frac{133\pi^3}{180} + \frac{59\pi^5}{1280}\right) m^5 + O(m^7)$$

$$F_{21}[m] = \frac{\pi}{2} m - \left(\frac{4\pi}{3} + \frac{\pi^3}{48}\right) m^3 + \left(-\frac{2\pi}{3} + \frac{13\pi^3}{30} - \frac{59\pi^5}{3840}\right) m^5 + O(m^7)$$

$$F_{22}[m] = 1 - \left(4 + \frac{3\pi^2}{8}\right) m^2 + \left(\frac{8}{3} + \frac{29\pi^2}{18} - \frac{5\pi^4}{128}\right) m^4 + \left(-\frac{32}{15} - \frac{4\pi^2}{9} - \frac{1159\pi^4}{7200} + \frac{119\pi^6}{15360}\right) m^6 + O(m^8)$$

・Comparison of 2． and 3． Local symmetry criterion

$$K_2^* = 0 \overset{?}{\Rightarrow} K_1^* \frac{\partial K_1^*}{\partial m} + K_2^* \frac{\partial K_2^*}{\partial m} = 0.$$

$$K_2^* = 0 \overset{?}{\Rightarrow} \frac{\partial K_1^*}{\partial m} = 0.$$

$$F_{21}(m)K_1 + F_{22}(m)K_2 = 0 \overset{?}{\Rightarrow} F'_{11}(m)K_1 + F'_{12}(m)K_2 = 0.$$

$$\frac{F'_{11}(m)}{F_{21}(m)} \overset{?}{=} \frac{F'_{12}(m)}{F_{22}(m)}$$

$$\frac{F'_{11}(m)}{F_{21}(m)} = -\frac{3\pi}{2} m + \left(4\pi - \frac{3\pi^3}{8}\right) m^2 + \left(10\pi - \frac{41\pi^3}{30} + \frac{\pi^5}{32}\right) m^4 + O(m^6)$$

$$\frac{F'_{12}(m)}{F_{22}(m)} = -\frac{3\pi}{2} m + \left(4\pi - \frac{3\pi^3}{8}\right) m^2 + \left(10\pi - \frac{23\pi^3}{18} + \frac{\pi^5}{32}\right) m^4 + O(m^6)$$

• 1. Maximum stress criterion

$$\widetilde{K}_1 = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_\theta, \qquad \widetilde{K}_2 = \lim_{r\to 0}\sqrt{2\pi r}\,\sigma_{r\theta},$$

$$\widetilde{K}_p = \widetilde{F}_{pq}(m)K_q,$$

$$\widetilde{F}_{11}(m)=\frac{3}{4}\cos\left(\frac{m\pi}{2}\right)+\frac{1}{4}\cos\left(\frac{3m\pi}{2}\right)=1-\frac{3\pi^2}{8}m^2+\frac{7\pi^4}{128}m^4-\frac{61\pi^6}{15360}m^6+O(m^7),$$

$$\widetilde{F}_{12}(m)=-\frac{3}{4}\sin\left(\frac{m\pi}{2}\right)-\frac{3}{4}\sin\left(\frac{3m\pi}{2}\right)=-\frac{3\pi}{2}m+\frac{7\pi^3}{16}m^3-\frac{61\pi^5}{1280}m^5+O(m^7),$$

$$\widetilde{F}_{21}(m)=\frac{1}{4}\sin\left(\frac{m\pi}{2}\right)+\frac{1}{4}\sin\left(\frac{3m\pi}{2}\right)=\frac{\pi}{2}m-\frac{7\pi^3}{48}m^3+\frac{61\pi^5}{3840}m^5+O(m^7),$$

$$\widetilde{F}_{22}(m)=\frac{1}{4}\cos\left(\frac{m\pi}{2}\right)+\frac{3}{4}\cos\left(\frac{3m\pi}{2}\right)=1-\frac{7\pi^2}{8}m^2+\frac{61\pi^4}{384}m^4-\frac{547\pi^6}{46080}m^6+O(m^7).$$

1. ⇔

Finding the angle which is determined by conditions

$$\max_\theta \widetilde{K}_1, \quad \widetilde{K}_2 = 0$$

If $K_p^* = \widetilde{K}_p$, the 3 criteria are equivalent !!

$$K_p^* - \widetilde{K}_p \quad ?$$

Future work

- Mathematical (Theoretical) evidence
- What happens if two different types of singularity join ?
- Expansion to the case of general loads

- Nonlinear problem
  (Plasticity, Large deformation, …)
- Expansion to the case of 3-D domain