Topological Superconductivity in Quasicrystals

Masahiro Hori
University of Saskachewan, Canada
Tokyo University of Science

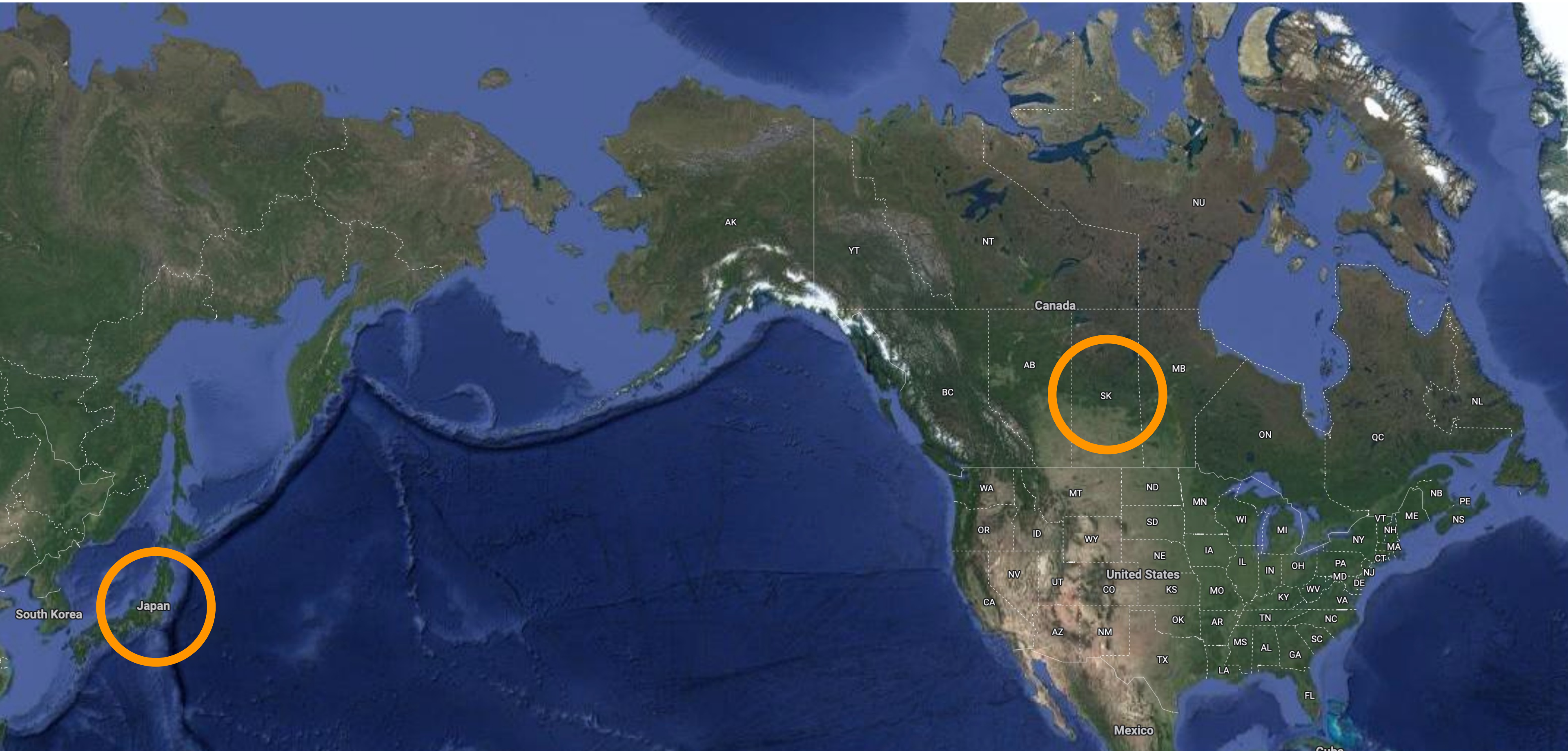
Who Is Hori?

Hori Participates in the Dual **Doctoral** Degree (DDD) Program

DDD between

1. Tokyo University of Science, Katsushika campus, Tohyama Lab.
2. University of Saskatchewan, Canada.

Hori has finished the dual **master** degree (DMD) program.



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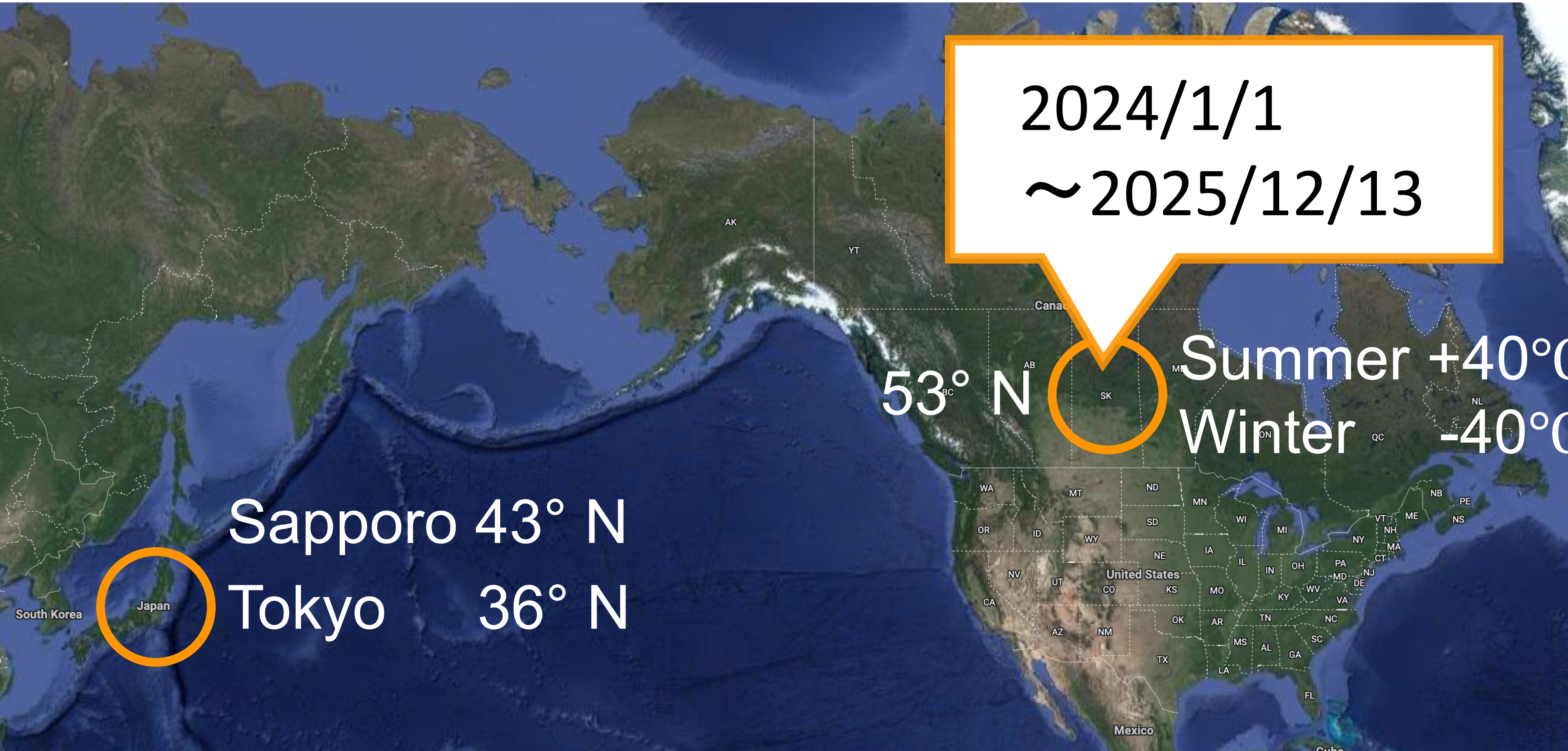


2024/1/1
～2025/12/13

53° N

Summer +40°
Winter -40°

Sapporo 43° N
Tokyo 36° N

Hori’s Research Topics

High school : Cell culture
B1 : Self-organization
B2 : Classical stochastic process
B3 : Green’s function
B4 : Critical exponent
M1 – now : Quasicrystals (QCs)

- **Topological superconductivity** [1-3]
- **Weyl superconductivity** [4]
- Haldane model and confined states [5]
- Multifractality & hyperuniformity in Bose-Hubbard model [6]

Today’s topics

[1] MH, サスカチュワン大学　修士論文 https://harvest.usask.ca/handle/10388/13865
[2] MH, R. Ghadimi, T. Sugimoto, T. Tohyama, and K. Tanaka, JPS Conf. Proc. **38**, 011062 (2023).
[3] MH, R. Ghadimi, T. Sugimoto, T. Tohyama, and K. Tanaka, JPS Conf. Proc. **38**, 011065 (2023).
[4] MH, R. Okugawa, K. Tanaka, and T. Tohyama, to be submitted.
[5] R. Ghadimi, MH, T. Sugimoto, and T. Tohyama, Phys. Rev. B **108**, 125104 (2023).
[6] MH, T. Sugimoto, Y. Hashizume, and T. Tohyama, to be submitted.

PART 1
Topological Superconductivity

PART 2
Weyl Superconductivity

PART 1

Topological Superconductivity

Al-Zn-Mg quasicrystal, $T_\mathrm{c} \cong 0.05\ \mathrm{K}$

Quasicrystals and Superconductivity

Show Bragg peaks & self-similarity

but NOT periodic

K. Kamiya *et al.*, Nat. Comm. **9**, 154 (2018).

Topology and Superconductivity

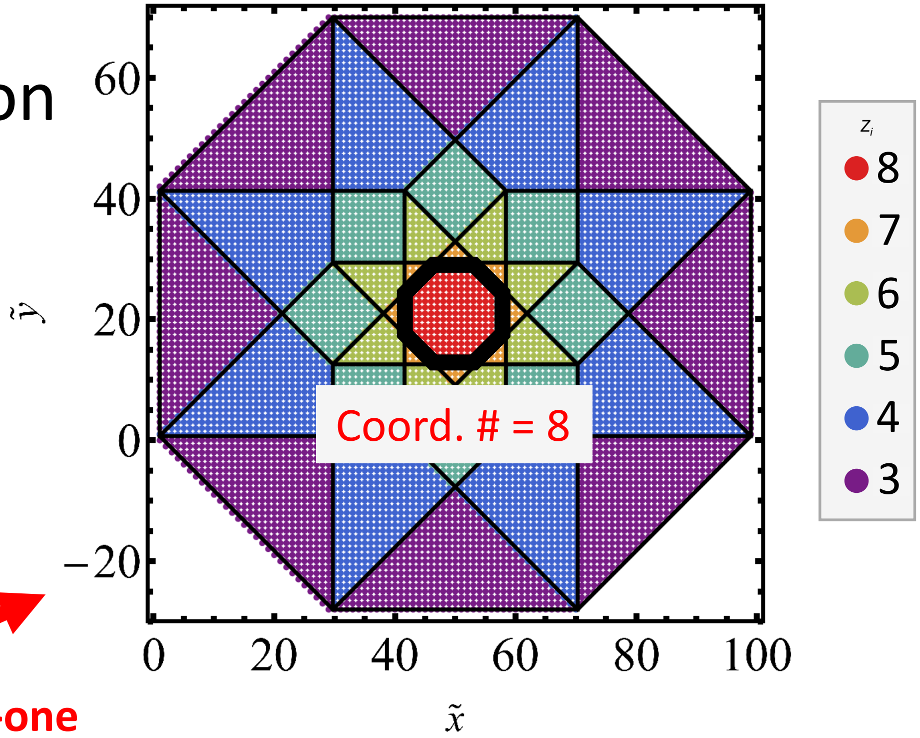
Topological Superconductivity in Quasicrystals

➡ Stably exists?

Perpendicular Space Representation

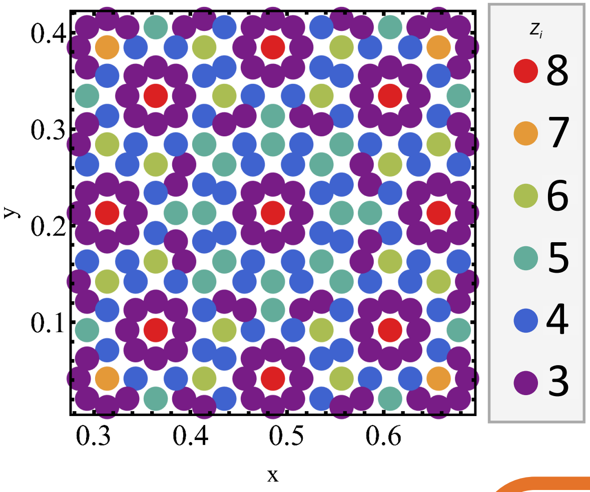
4-dimensional
hyper cubic lattice

Projection



Perpendicular space

Projection

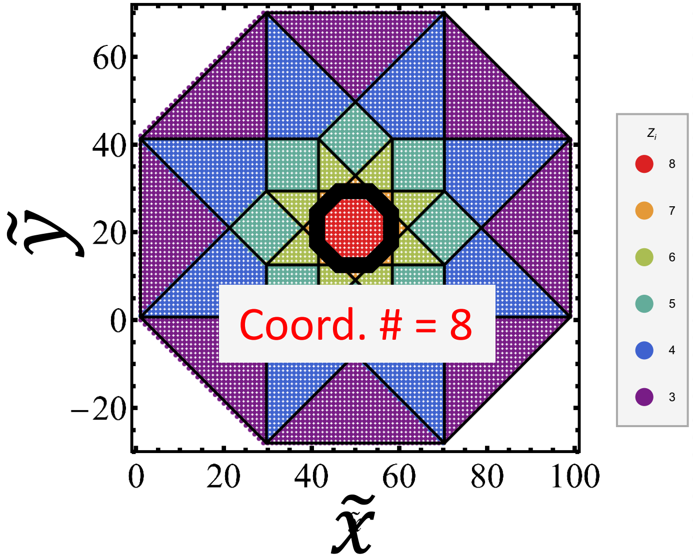


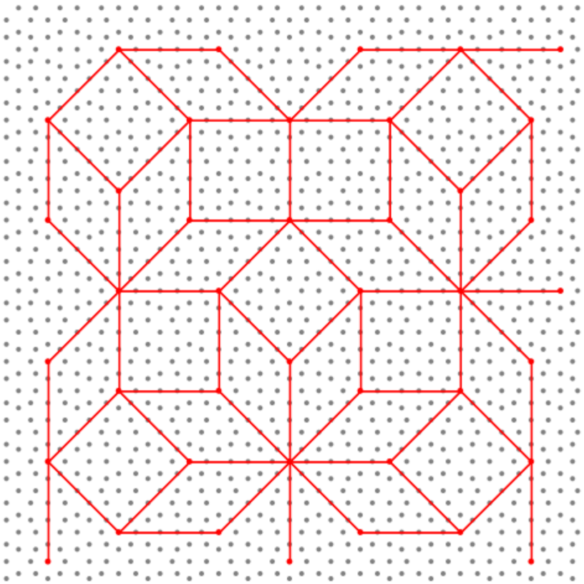
Real space

One-to-one correspondence

Points with the same coord. # gathers
→ points with the **similar local environments are placed closely** each other

Self-similarity in AB QCs

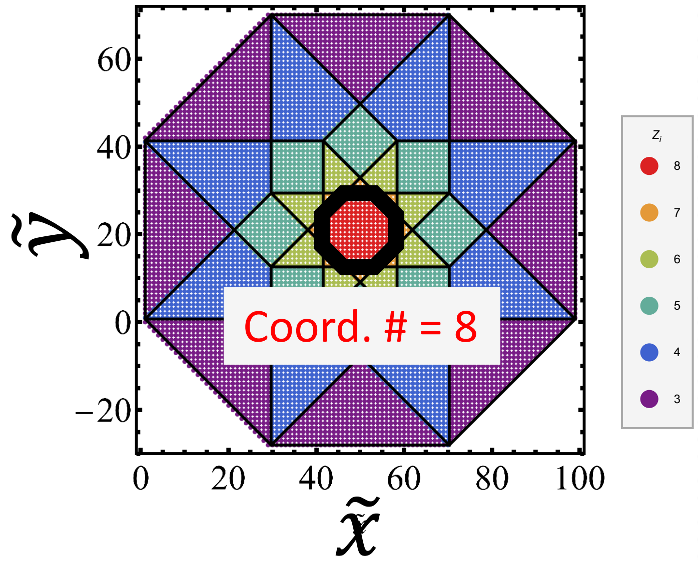


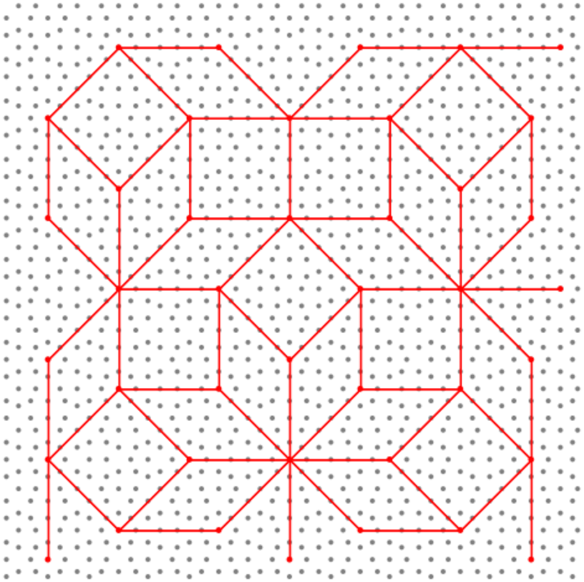


All points

Points with coord. # = 8

Self-similarity in AB QCs

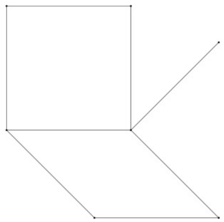




All points

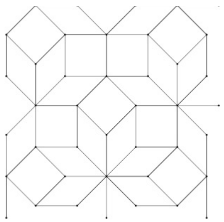
Points with coord. # = 8

N=7



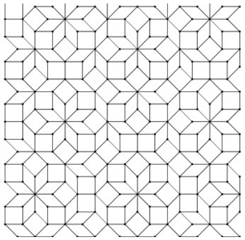
⊂

N=41



⊂

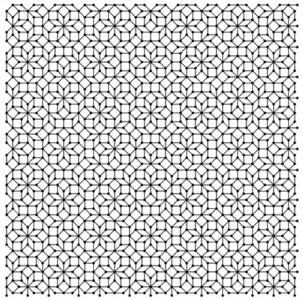
N=239



“Self-similar”

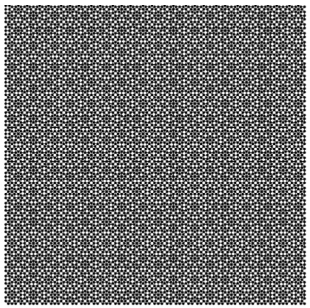
⊂

N=1393



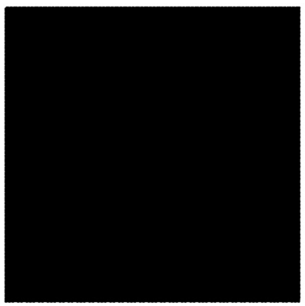
⊂

N=8119



⊂

N=47321

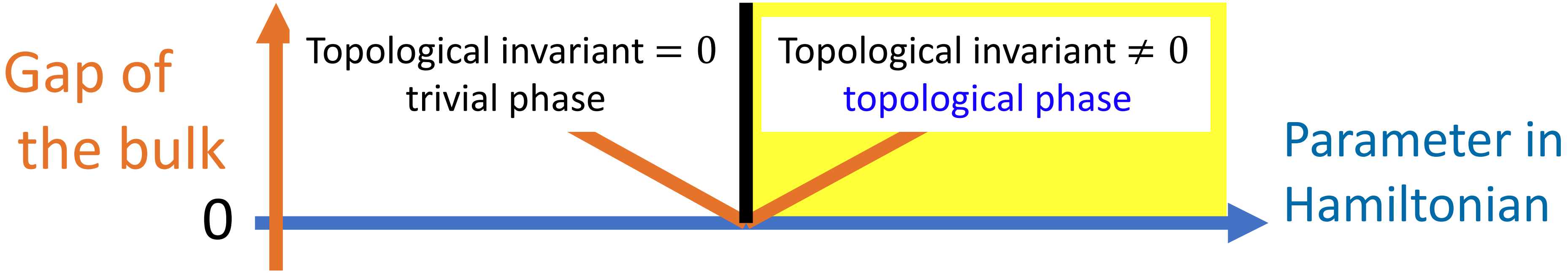


Topological Superconductivity

Definition of Topological Superconductivtiy (TSC):
“$|\Delta| > 0$ and Topological invariant $\neq 0$“

topological phase

Topological invariant changes ONLY when the gap of the bulk closes



Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{ij\sigma\sigma'} (c^{\dagger}_{i\sigma} \quad c_{i\sigma}) H \begin{pmatrix} c_{j\sigma'} \\ c^{\dagger}_{j\sigma'} \end{pmatrix} \qquad H = \begin{pmatrix} \mathbf{h} & \mathbf{\Delta} \\ \mathbf{\Delta}^{\dagger} & -\mathbf{h}^{*} \end{pmatrix}$$

$\mathbf{\Delta}$:pairing operator

$$[\mathbf{h}]_{i\alpha,j\beta} = [\underbrace{(t_{ij} - \mu\delta_{ij})\sigma_0}_{\text{Hopping} \quad \text{Chemical pot.}} + \underbrace{h_z\delta_{ij}\sigma_3}_{\text{Zeeman}} + \underbrace{\imath V_{ij}\vec{e}_z \cdot \vec{\sigma} \times \hat{R}_{ij}}_{\text{Rashba}}]_{\alpha\beta}$$

Sato *et al.*, PRB **82**, 134521 (2010).

Topological Superconductivity

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→ topological phase

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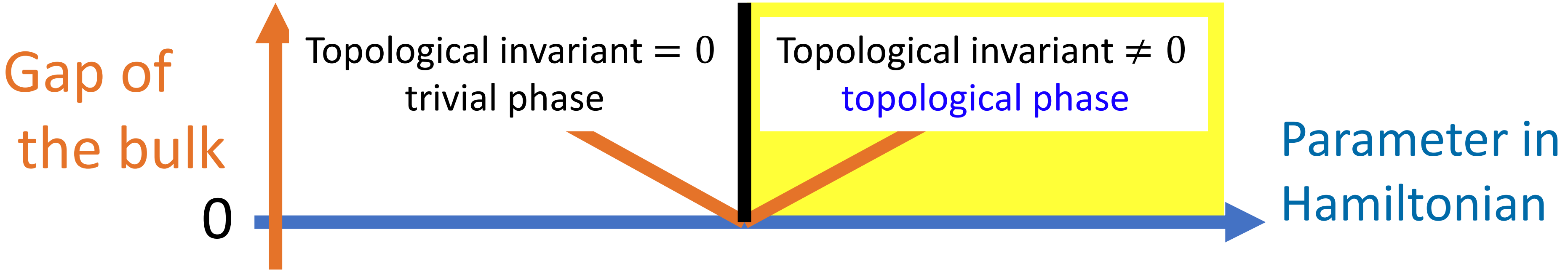
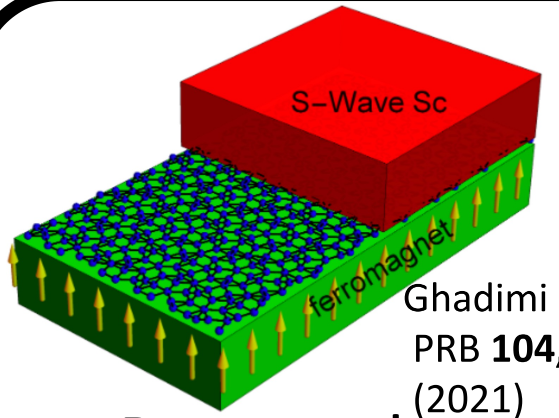


1. Rashba spin-orbit
2. Zeeman magnetic

$$\left.\begin{array}{l}\text{3. Superconducting order}\\ \text{4. Kinetic energy}\end{array}\right\}\text{BCS}$$



Ghadimi *et al.*, PRB **104**, 144511 (2021)

Proposed setup

Topological Invariant for QCs: Bott Index

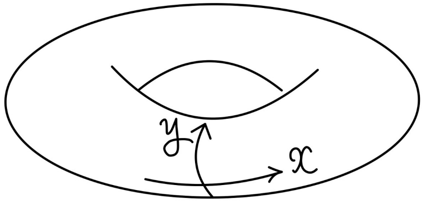
$$P = \sum_{E_n<0} |\phi_n\rangle \langle\phi_n| \,.$$

Projection operator (below Fermi level)

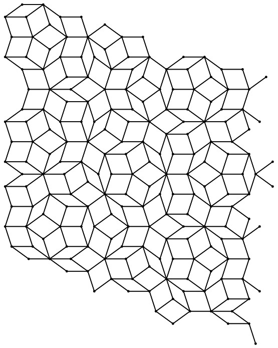
$$Q = I - P,$$

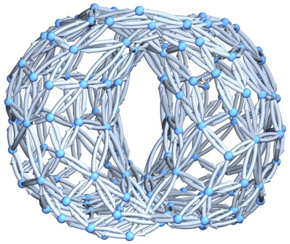
$x_i, y_i \in [0, 1)$

Position of $\imath$th vertex of QCs = (x_i, y_i)



$$X = \mathrm{Diag}\left(x_1, x_1, \ldots, x_N, x_N, x_1, x_1, \ldots, x_N, x_N\right),$$





$$U_X = P e^{\imath 2\pi X} P + Q, \quad U_Y = P e^{\imath 2\pi Y} P + Q$$

Bott index

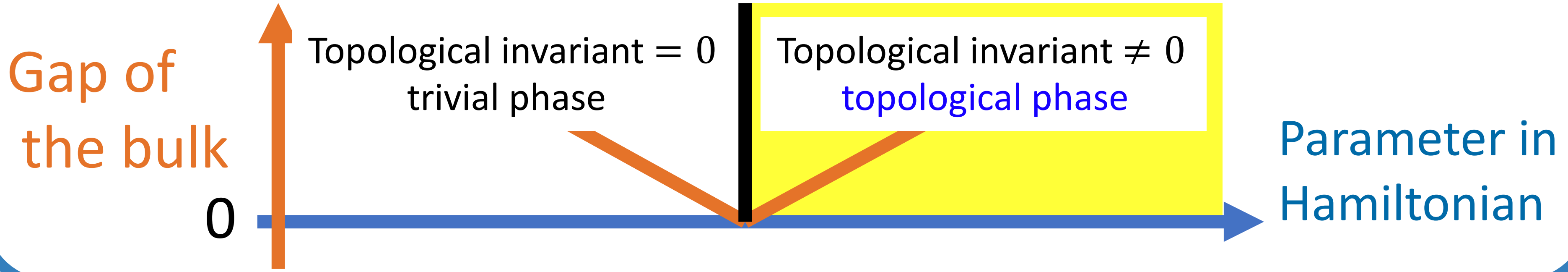
$$B = \frac{1}{2\pi} \mathrm{Im}\left(\mathrm{Tr}\left[\log(U_Y U_X U_Y^\dagger U_X^\dagger)\right]\right)$$

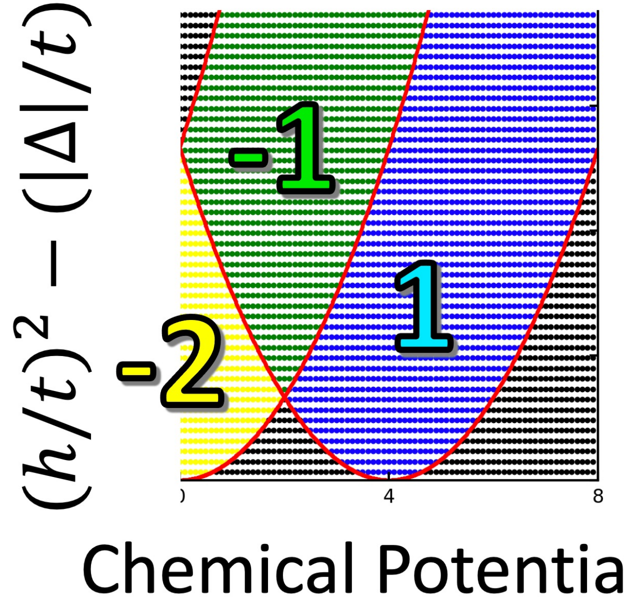
$$\in \mathbb{Z}$$

T. A. Loring and M. B. Hastings, EPL **92**, 67004 (2010).

Topological Phase Diagram

Topological invariant changes ONLY when the gap of the bulk closes

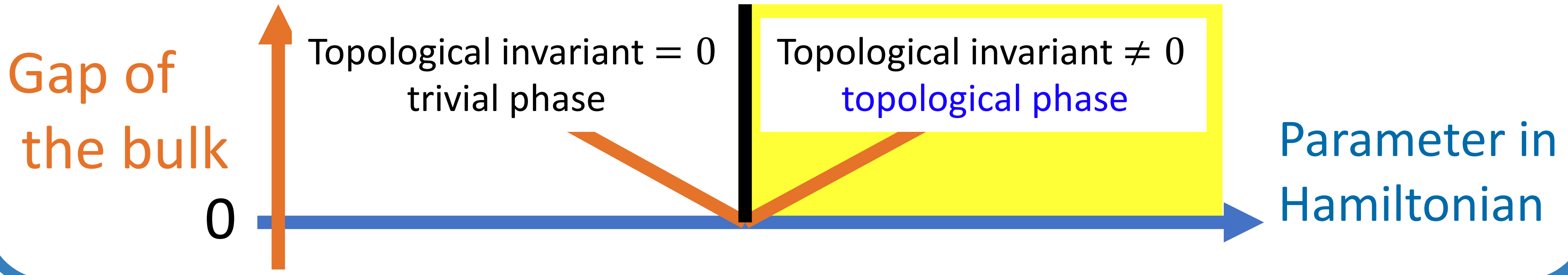



Topological invariant (Square Lattice)

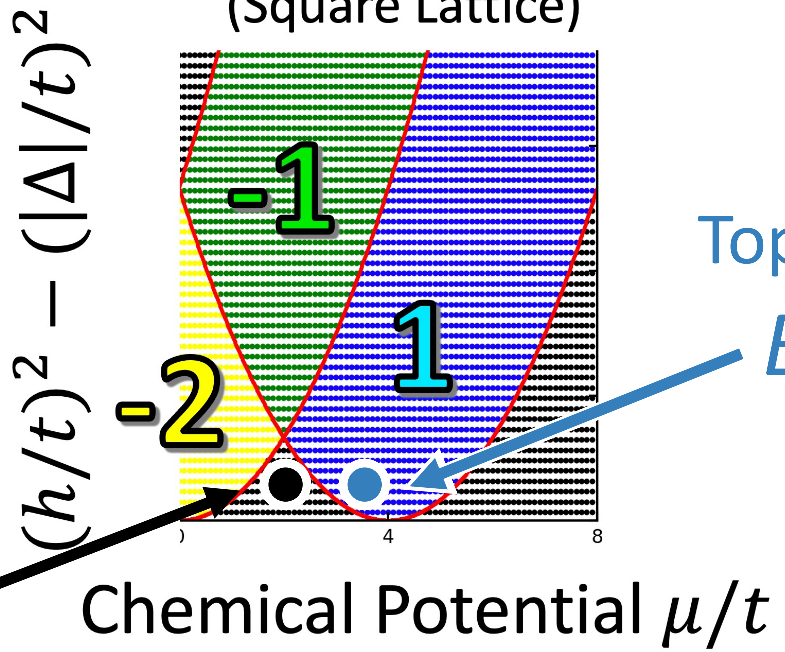
Sato *et al.*,
PRB **82**, 134521
(2010)

Topological Phase Diagram

Topological invariant changes ONLY when the gap of the bulk closes



Topological invariant
(Square Lattice)



Topological phase

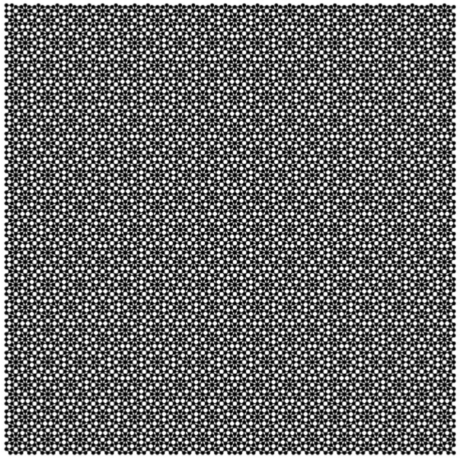
B=1

Trivial phase

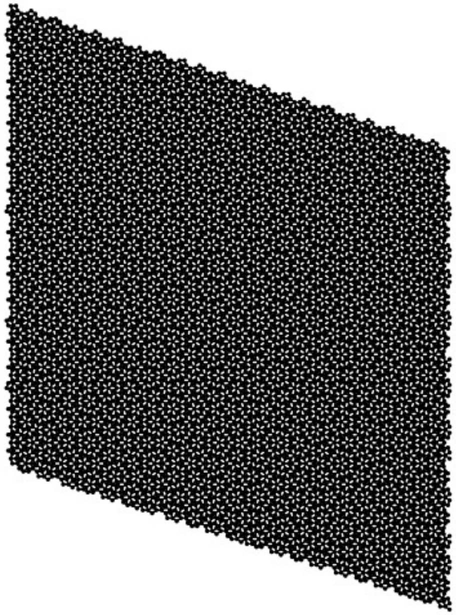
B=0

Sato *et al.*,
PRB **82**, 134521
(2010)

Order parameters are site-dependent



AB QC (8119-site)



Penrose QC (9349-site)

AB QC (8119-site)
and Penrose QC(9349-site)

→ Order params. are
site-dependent

→ Mean-field approximation
+ self-consistent calculation

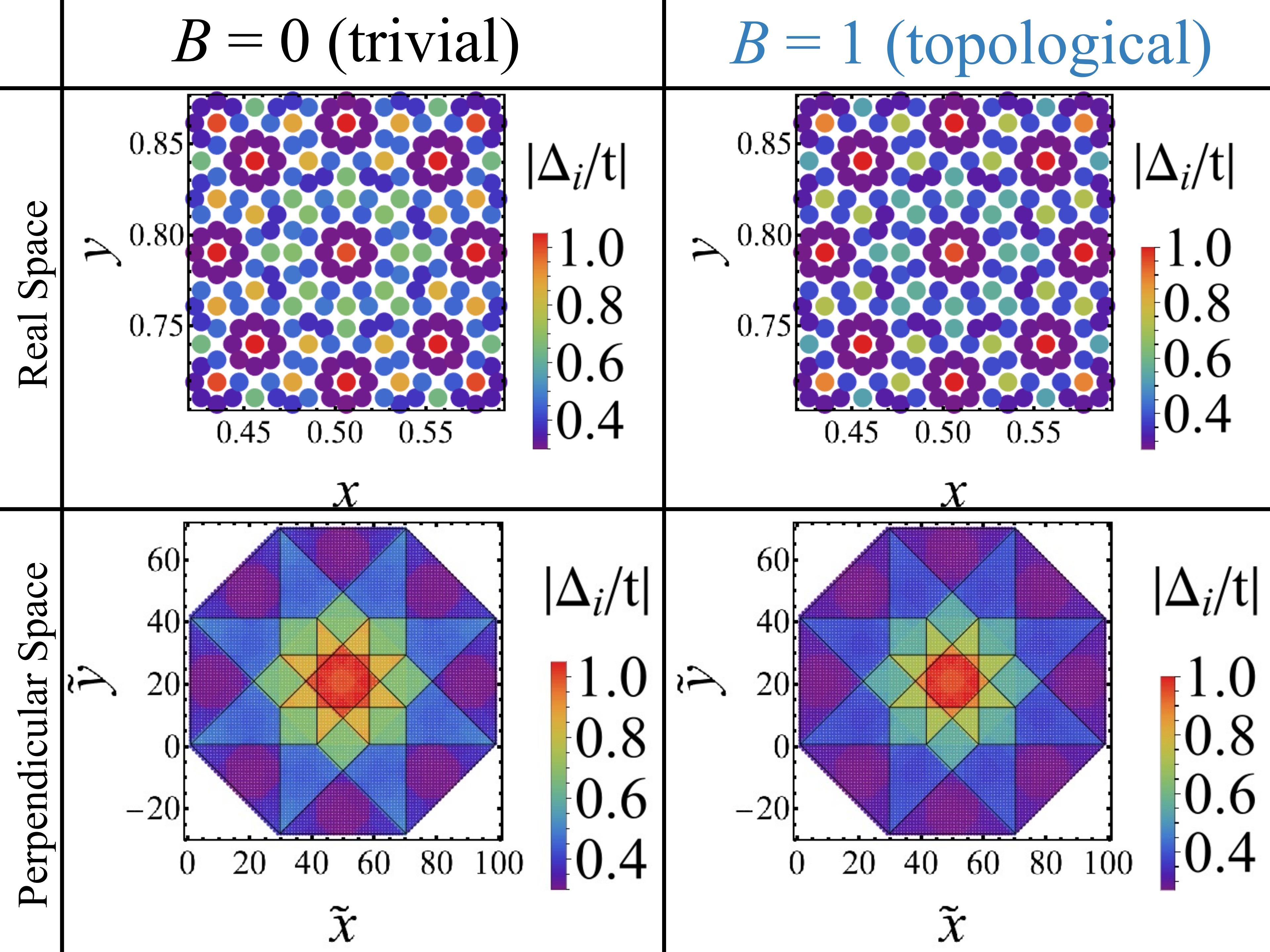
$$\left.\begin{aligned} \Delta_i &= U\langle c_{i\downarrow} c_{i\uparrow}\rangle, \\ V^{(H)}_{i\sigma} &= U\langle c^\dagger_{i\sigma} c_{i\sigma}\rangle \end{aligned}\right\}$$

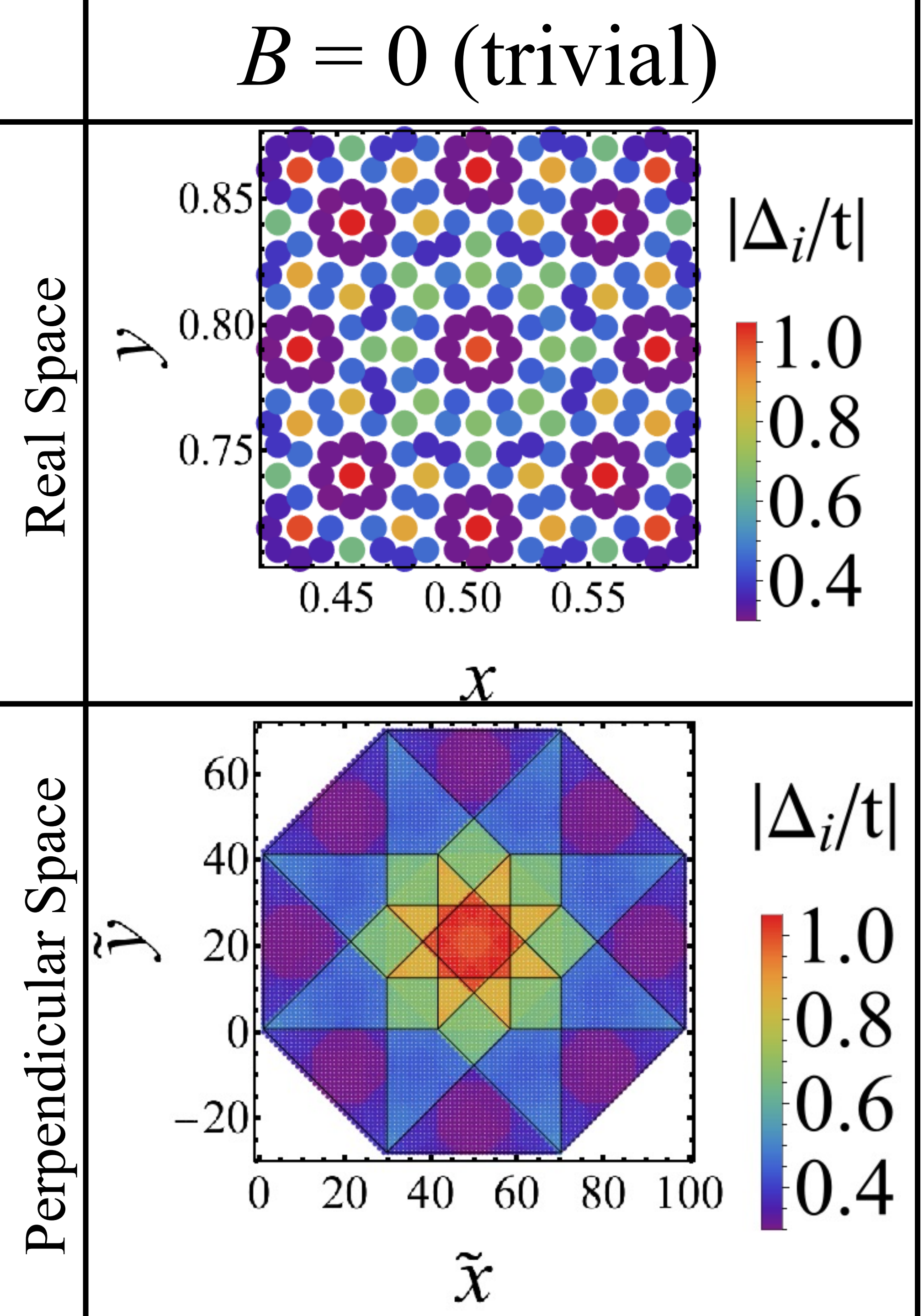
Self-consistently obtain

U: pairing strength (assume uniform)

Goertzen *et al.*,
PRB **95**, 064509
(2017).

AB QCs


$B = 0$ (trivial)
$B = 1$ (topological)
Real Space
Perpendicular Space
y
0.85
0.80
0.75
0.45
0.50
0.55
x
$|\Delta_i/\mathrm{t}|$
1.0
0.8
0.6
0.4
$\tilde{y}$
60
40
20
0
−20
0
20
40
60
80
100
$\tilde{x}$


$B = 0$ (trivial)
Real Space
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0.85
0.80
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60
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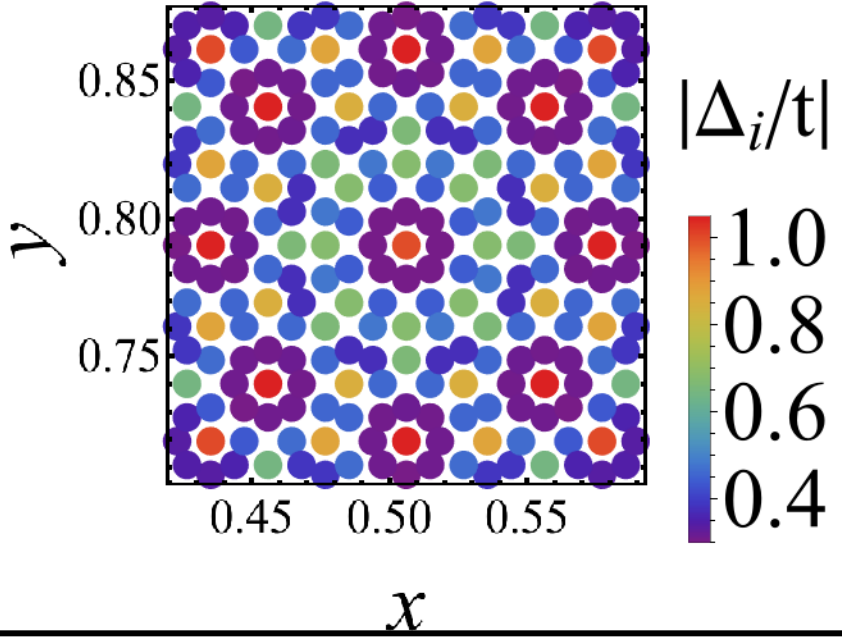
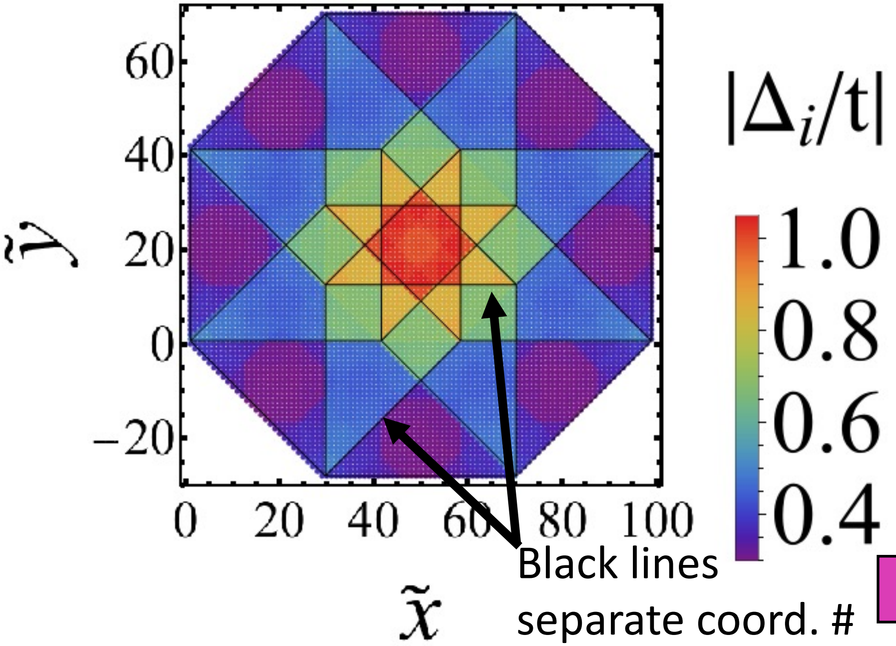
$\tilde{x}$
$|\Delta_i/\mathrm{t}|$
1.0
0.8
0.6
0.4

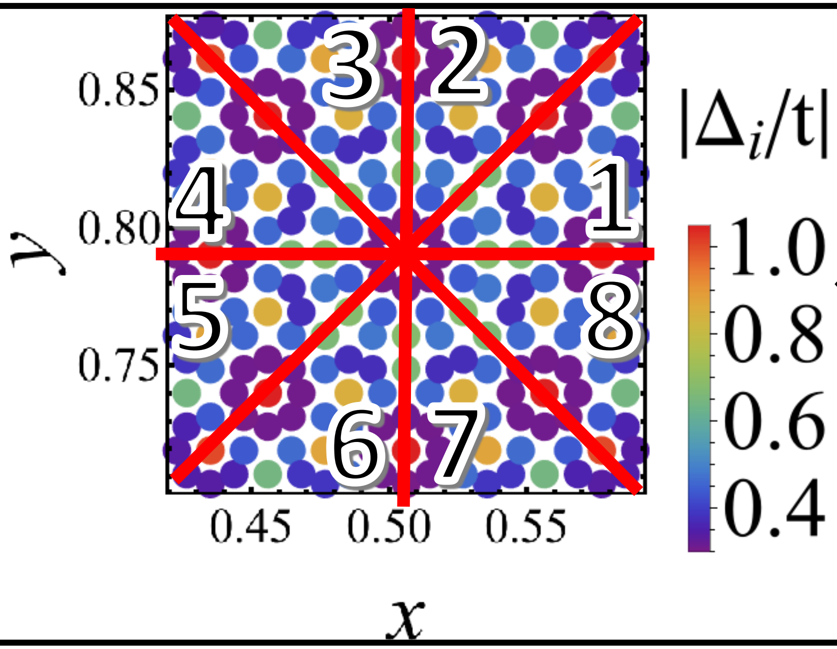
	$B = 0$ (trivial)
Real Space	
Perpendicular Space	

Black lines separate coord. #

The most dominant factor governing $|\Delta_i/t|$ is coord. #

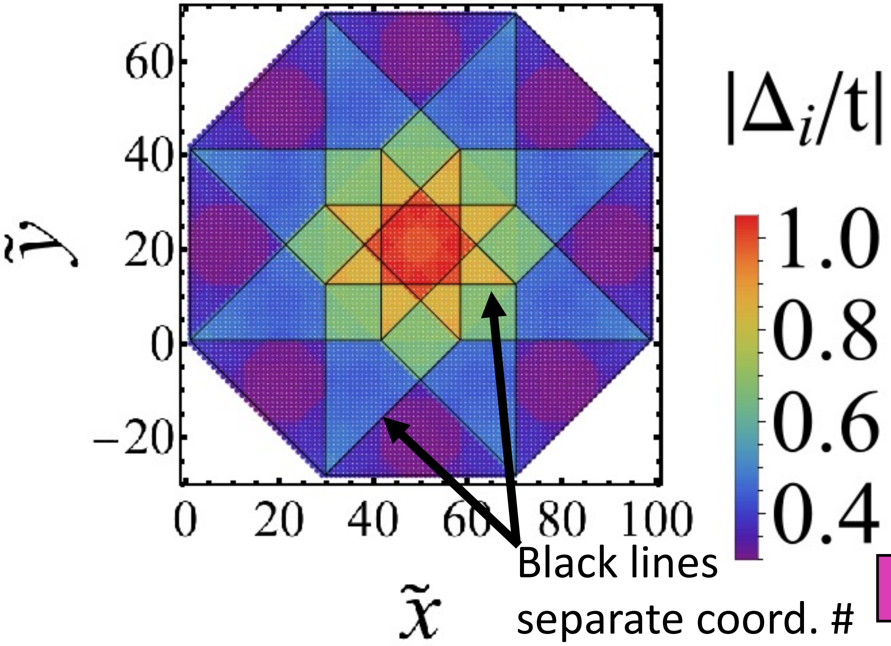
$B = 0$ (trivial)

Real Space



8-fold rotational symmetry
mirror symmetry
(AB QC shows)

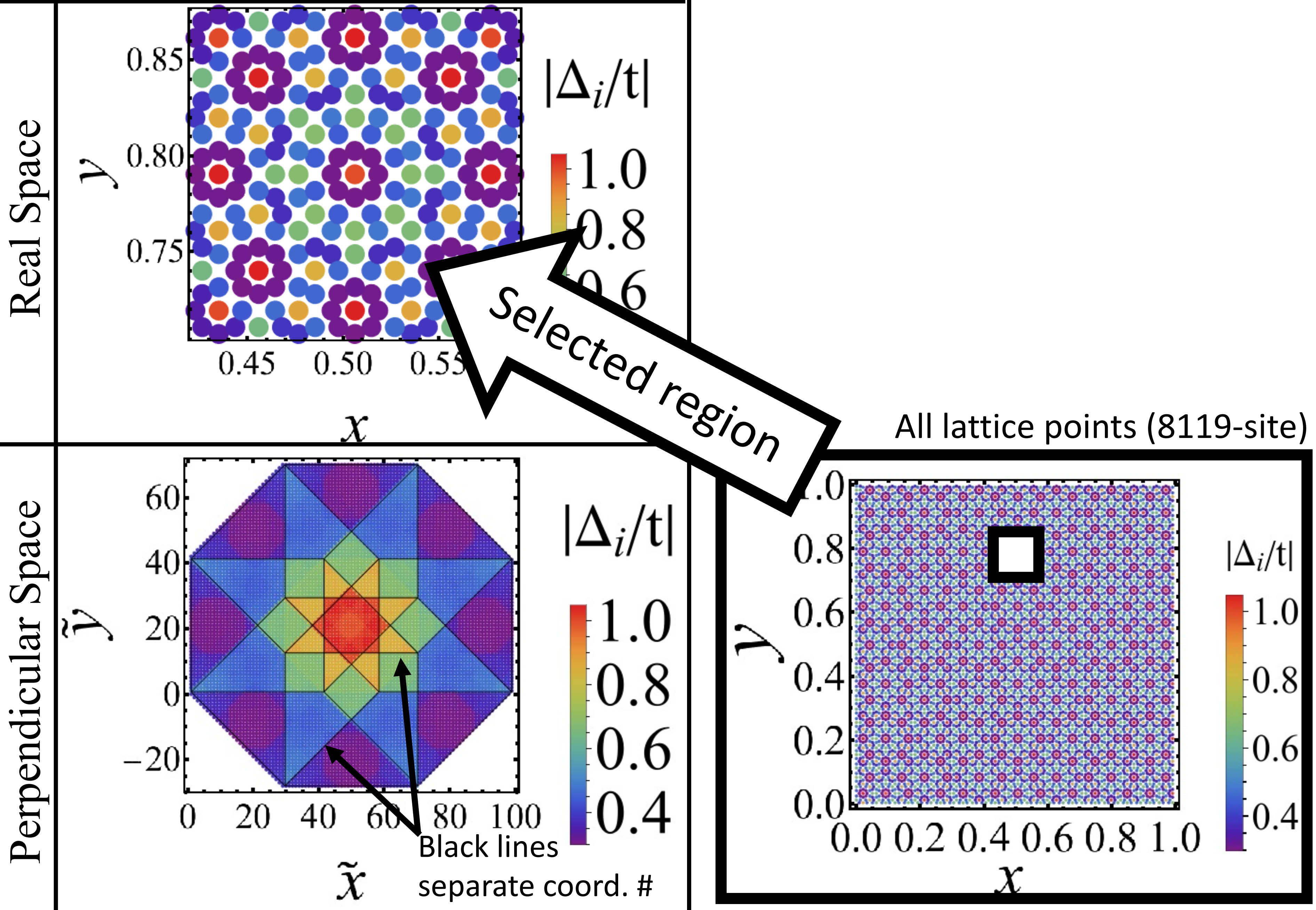
Perpendicular Space



Black lines
separate coord. #

The most dominant factor
governing $|\Delta_i/t|$ is coord. #

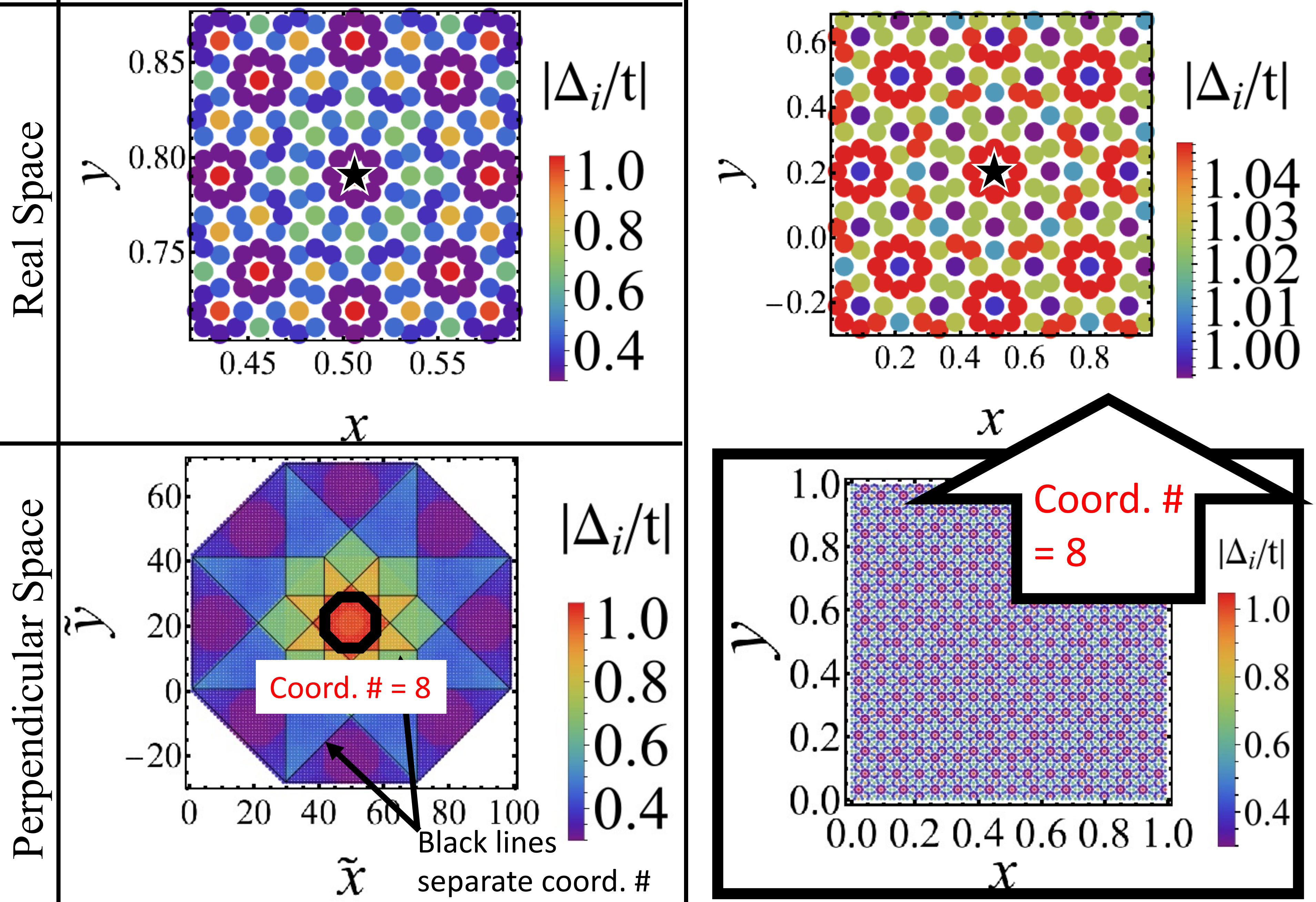
$B = 0$ (trivial)



Selected region

All lattice points (8119-site)

Black lines separate coord. #



$B = 0$ (trivial)

Coord. # = 8

Real Space

Perpendicular Space

Coord. # = 8

Coord. # = 8

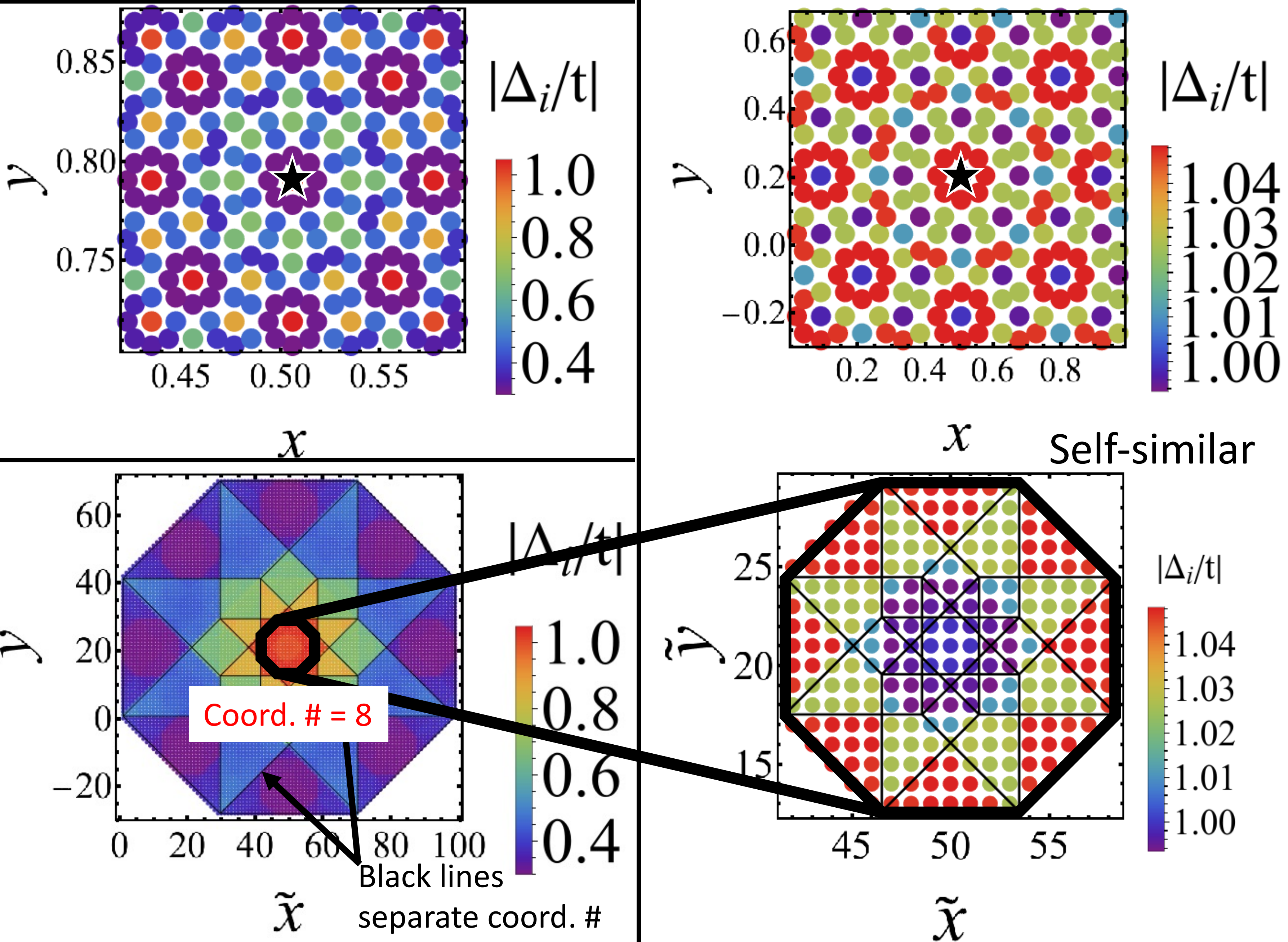
Black lines separate coord. #

$B = 0$ (trivial)

Coord. # = 8

Real Space

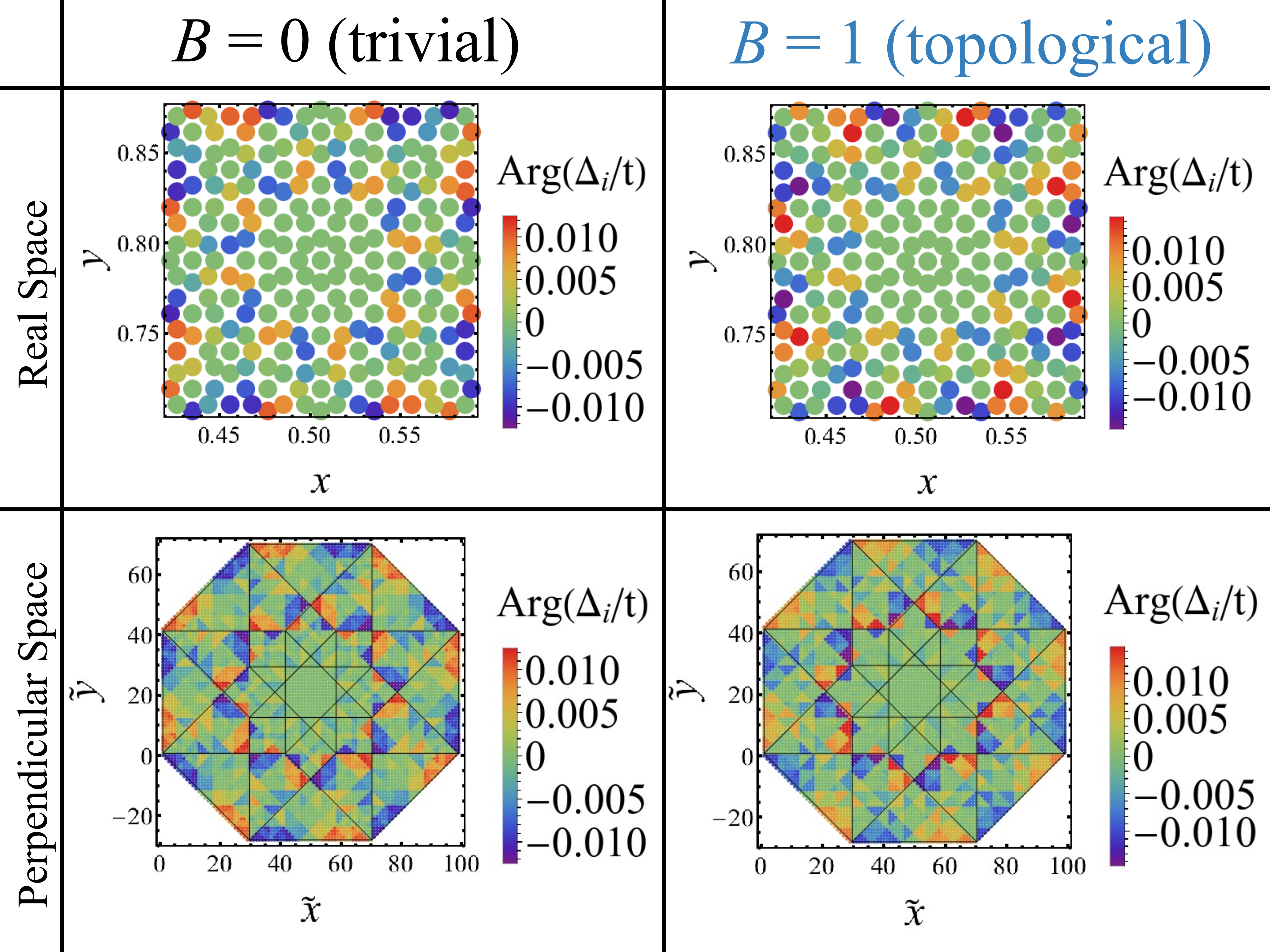
Perpendicular Space



Self-similar

Coord. # = 8

Black lines separate coord. #

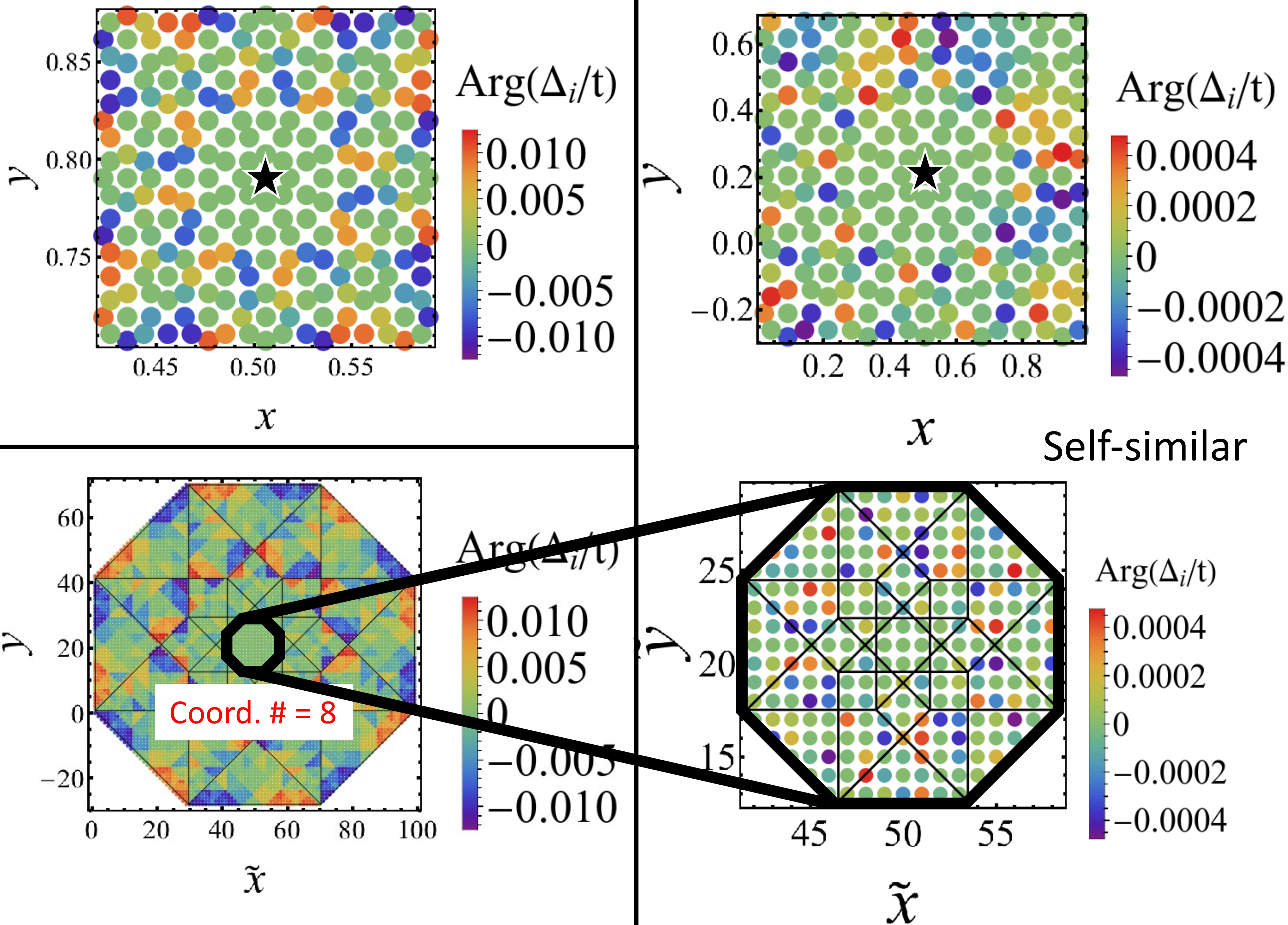

$B = 0$ (trivial)
$B = 1$ (topological)
Real Space
Perpendicular Space
y
x
0.85
0.80
0.75
0.45
0.50
0.55
Arg(Δ_i/t)
0.010
0.005
0
−0.005
−0.010
$\tilde{y}$
$\tilde{x}$
60
40
20
0
−20
0
20
40
60
80
100

$B = 0$ (trivial)

Coord. # = 8

Real Space

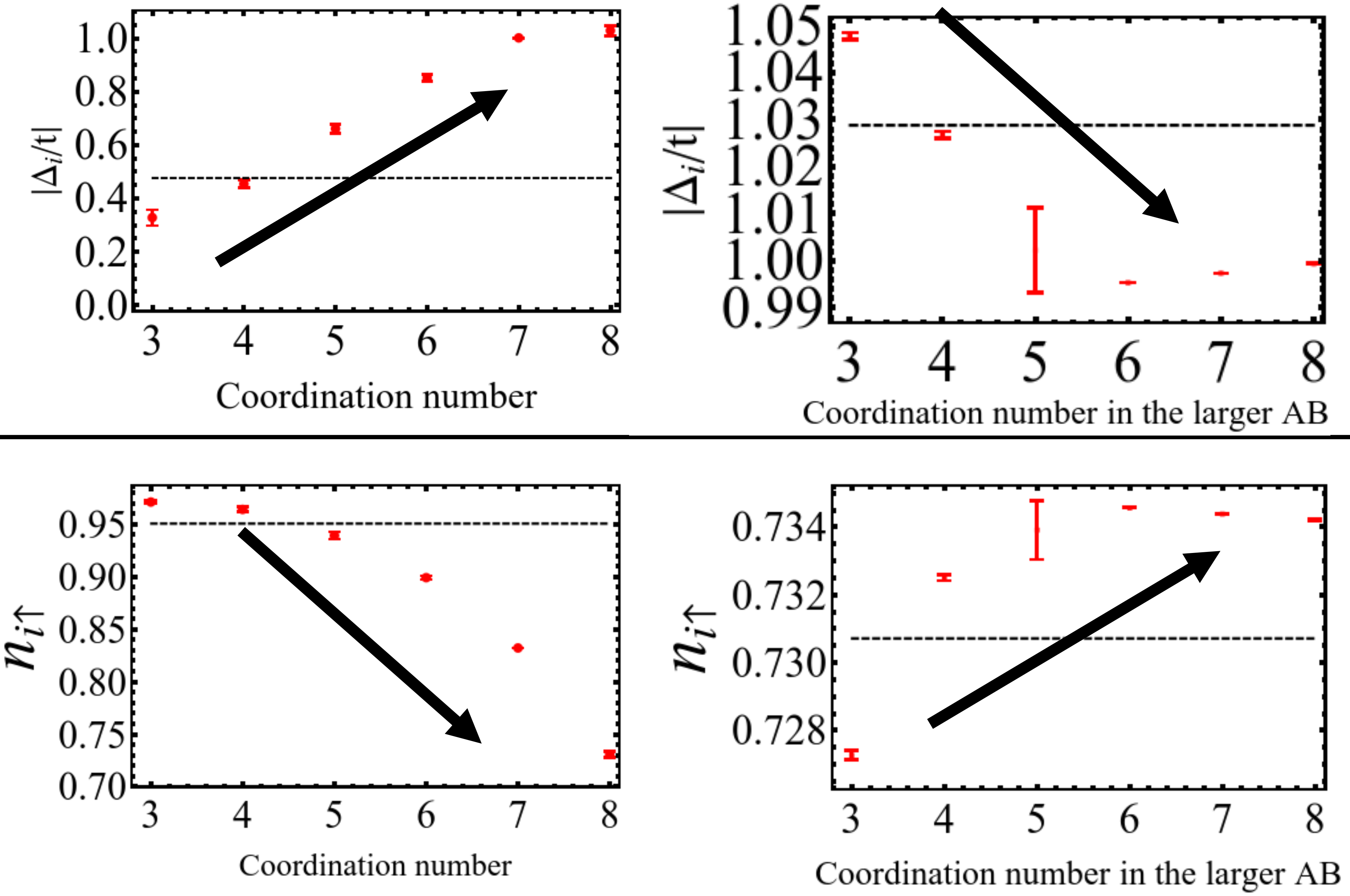
Perpendicular Space



All points

$B = 0$ (trivial)

Coord. # = 8

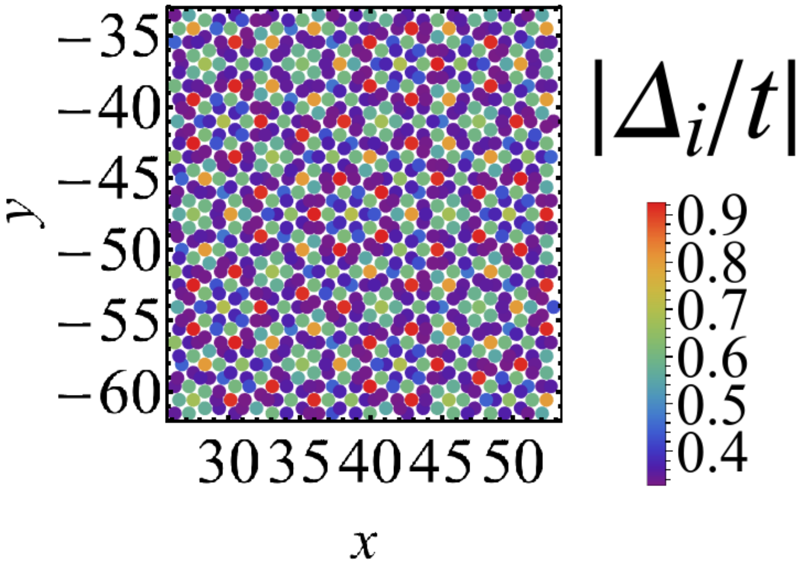


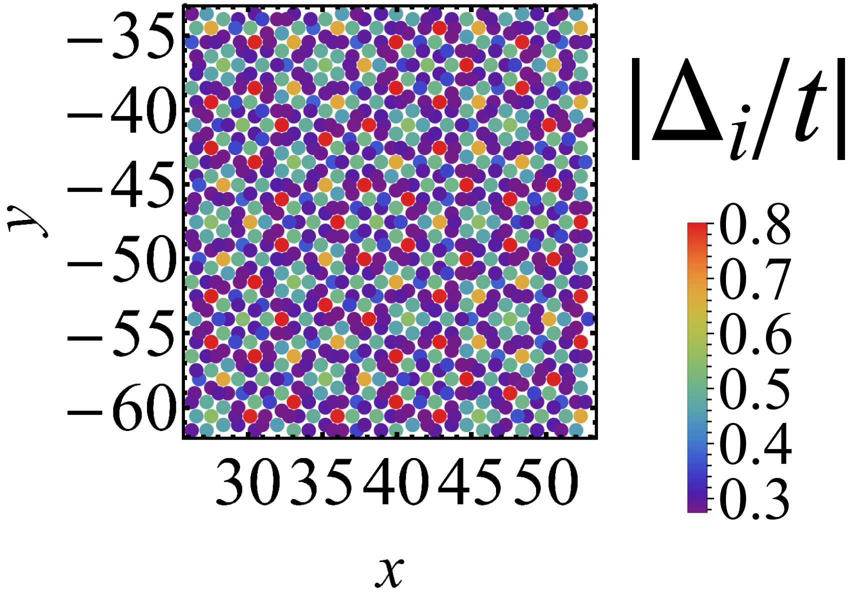
Penrose QCs

$B = 0$ (trivial)

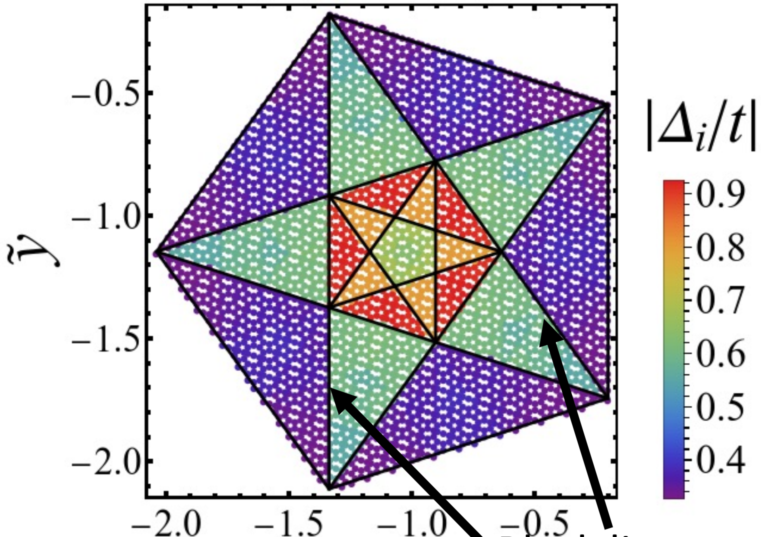
$B = 1$ (topological)

Real Space

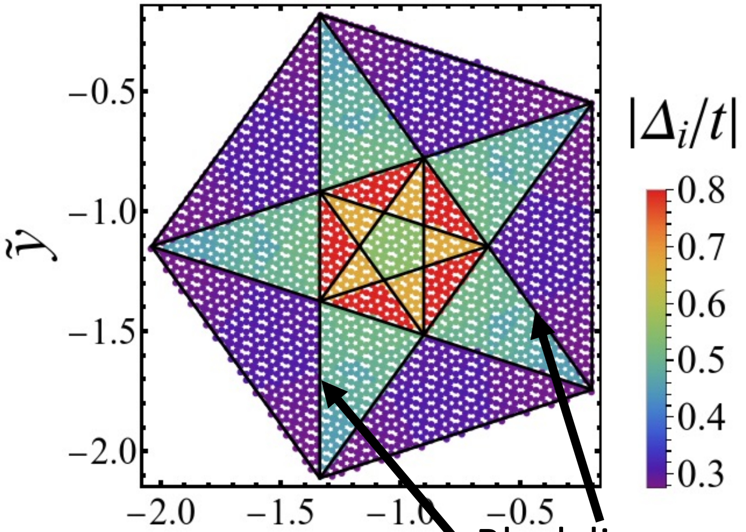




Perpendicular Space



Black lines separate coord. #



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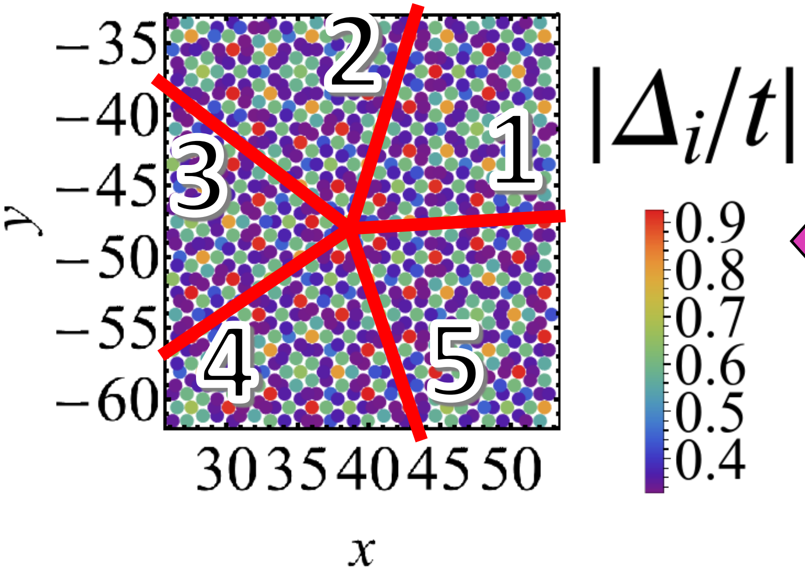
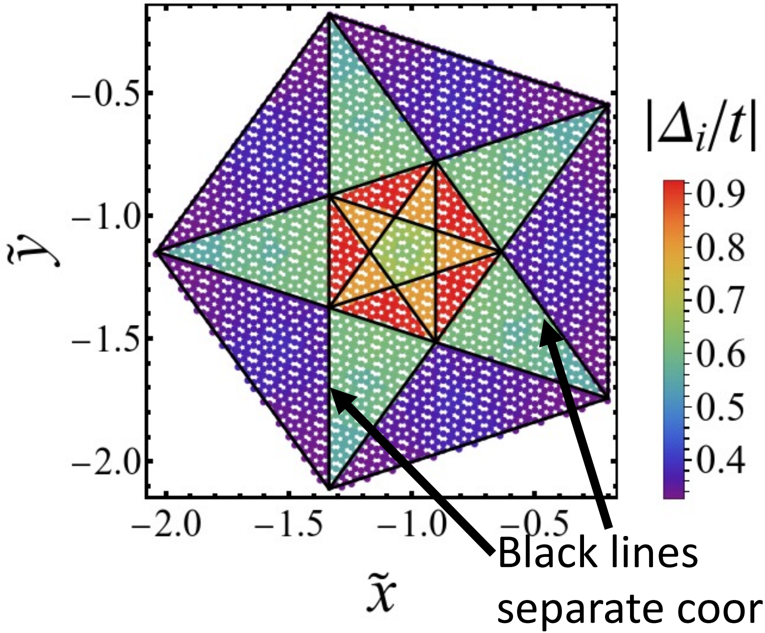
$B = 0$ (trivial)

Real Space



5-fold rotational symmetry
mirror symmetry
(Penrose QC shows)

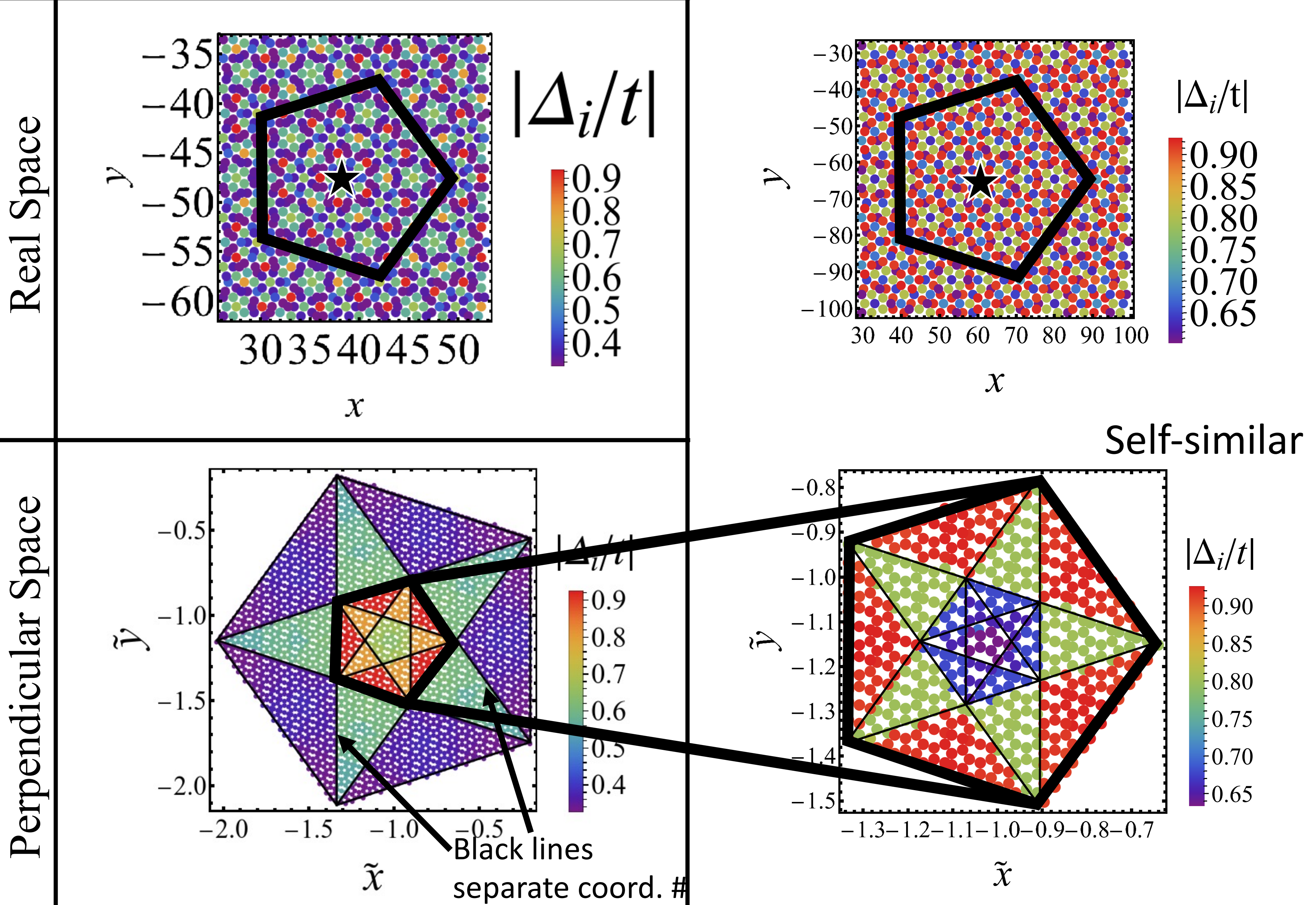
Perpendicular Space



Black lines
separate coord. #

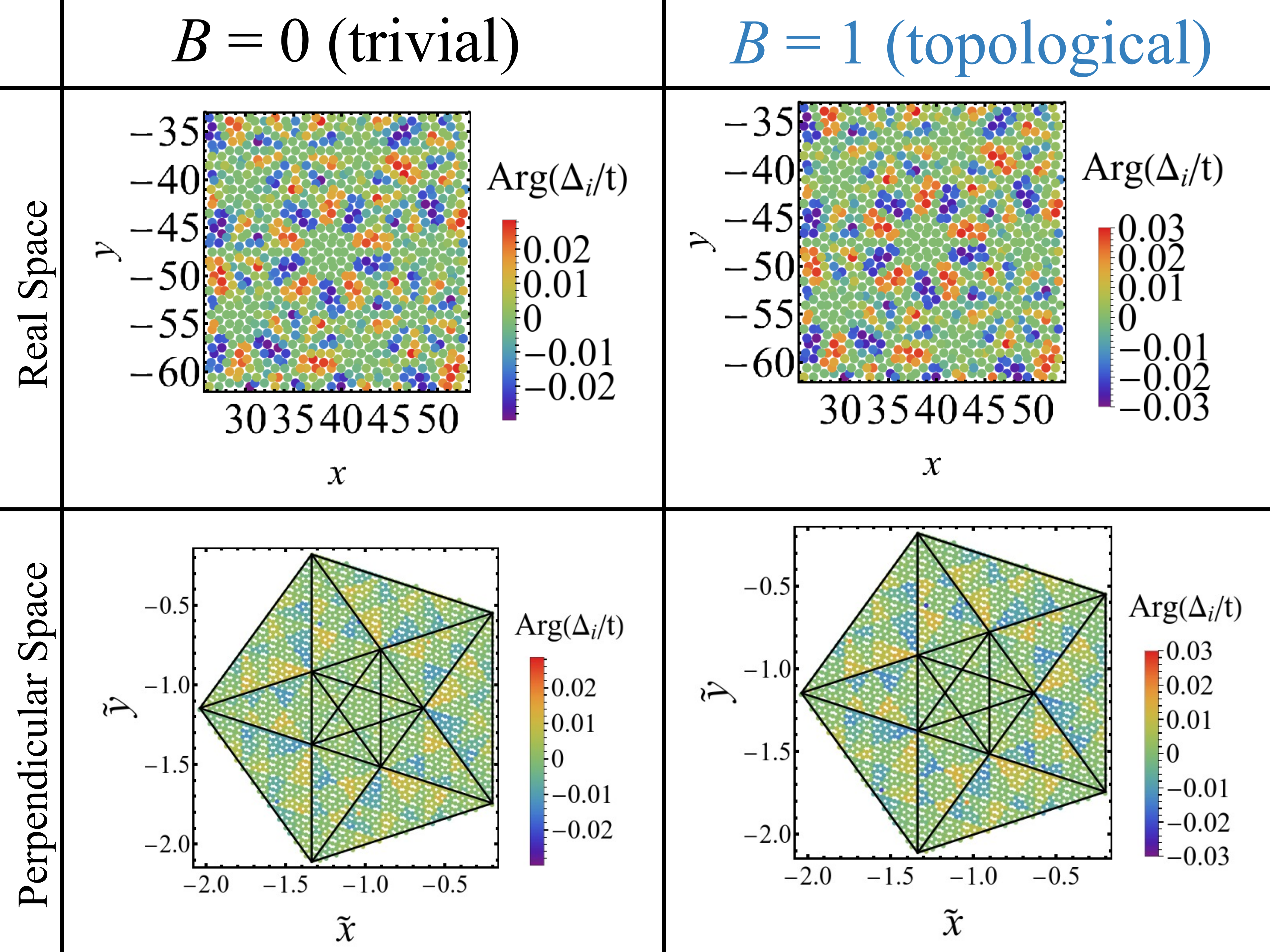
$B = 0$ (trivial)

Inside small pentagons

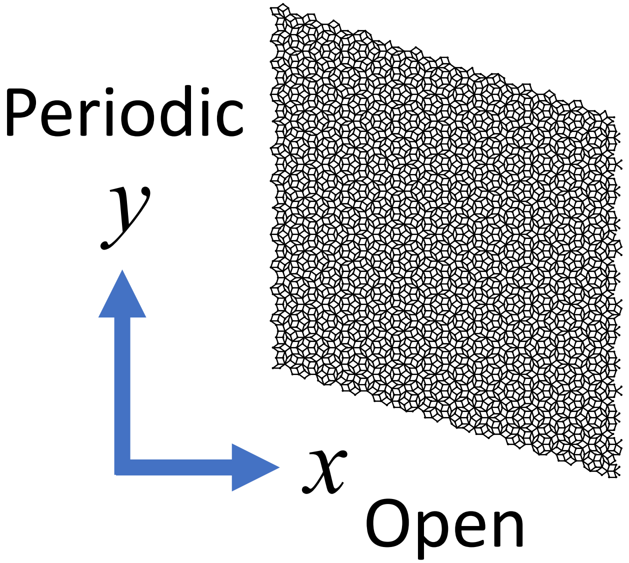


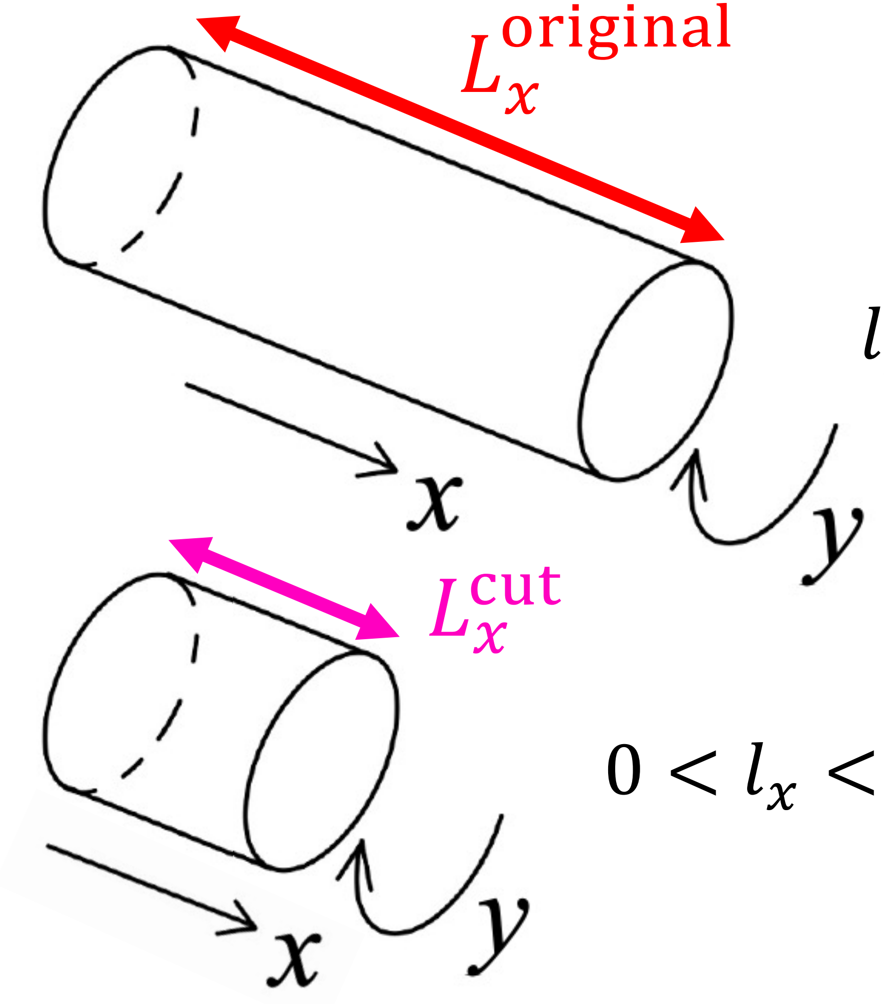
Self-similar

Black lines separate coord. #


$B = 0$ (trivial)
$B = 1$ (topological)
Real Space
Perpendicular Space
y
x
−35
−40
−45
−50
−55
−60
30 35 40 45 50
Arg(Δ_i/t)
0.03
0.02
0.01
0
−0.01
−0.02
−0.03
$\tilde{y}$
$\tilde{x}$
−0.5
−1.0
−1.5
−2.0
−2.0 −1.5 −1.0 −0.5

Distance Between the Two Edges is Changed





$l_x = 1$

$0 < l_x < 1$

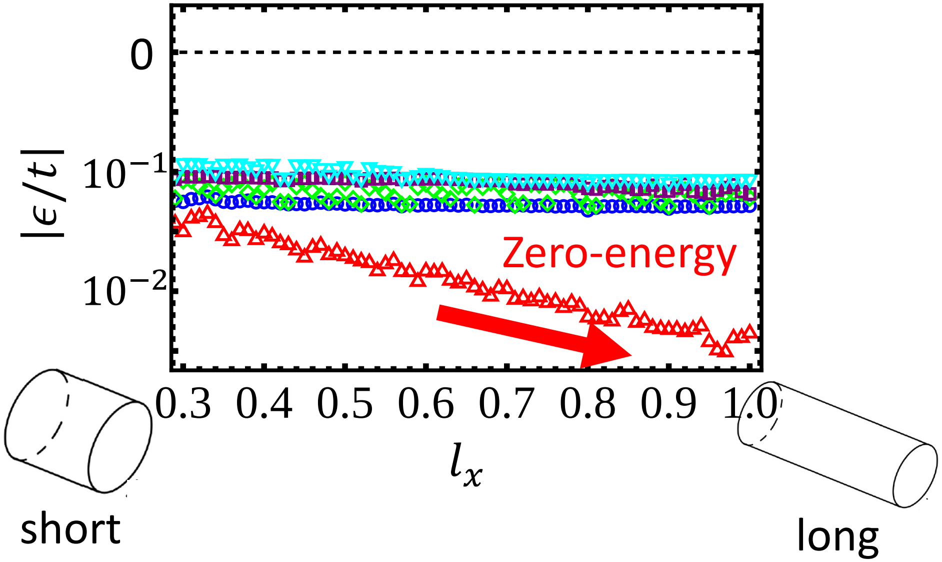
$$l_x := \frac{L_x^{\text{cut}}}{L_x^{\text{original}}}$$

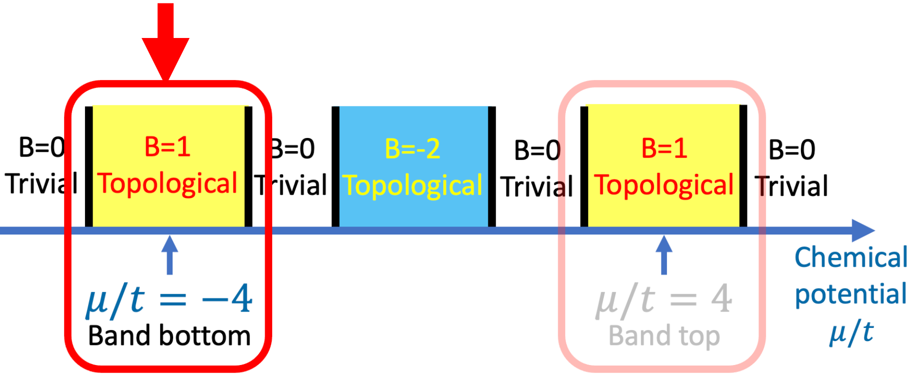
l_x dependence of the energy of the edge mode in TSC?

Zero-energy Edge Mode Exists When $l_x \to \infty$

$B = 1$

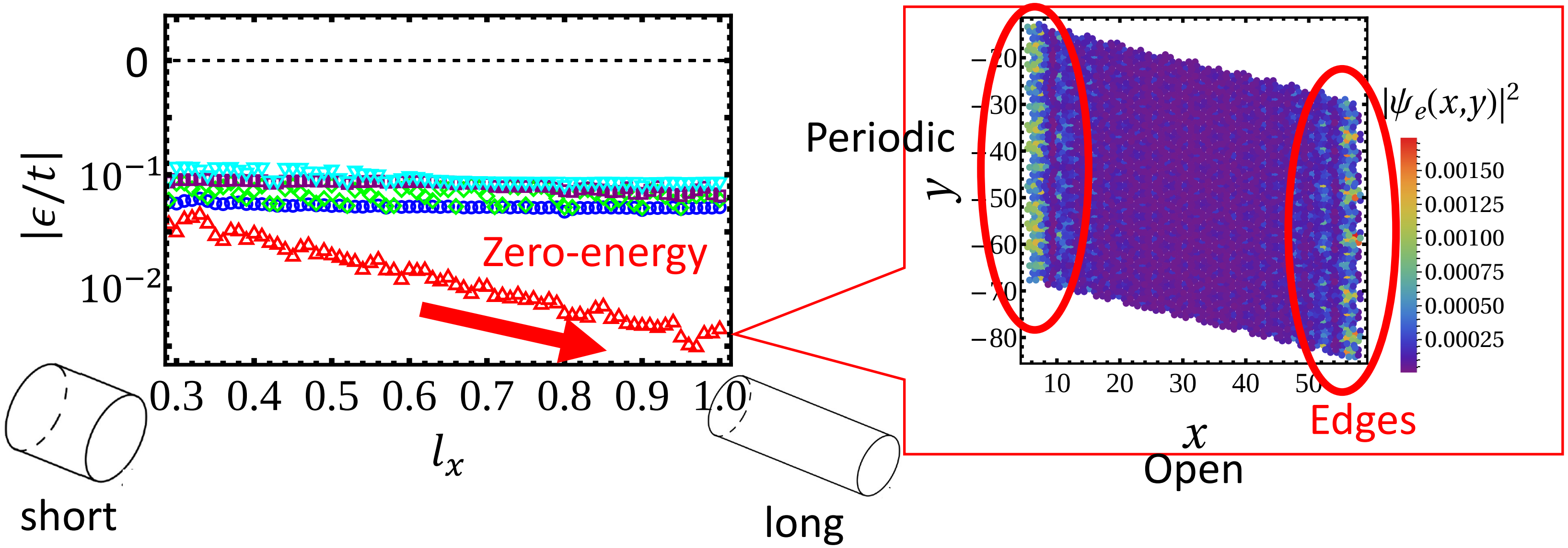
Band bottom

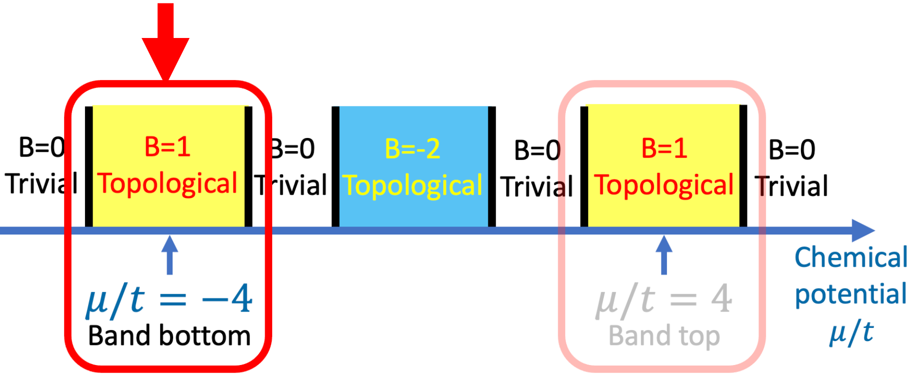




Zero-energy Edge Mode Exists When $l_x \to \infty$

$B = 1$
Band bottom





B=0
Trivial

B=1
Topological

B=0
Trivial

B=−2
Topological

B=0
Trivial

B=1
Topological

B=0
Trivial

$\mu/t = -4$
Band bottom

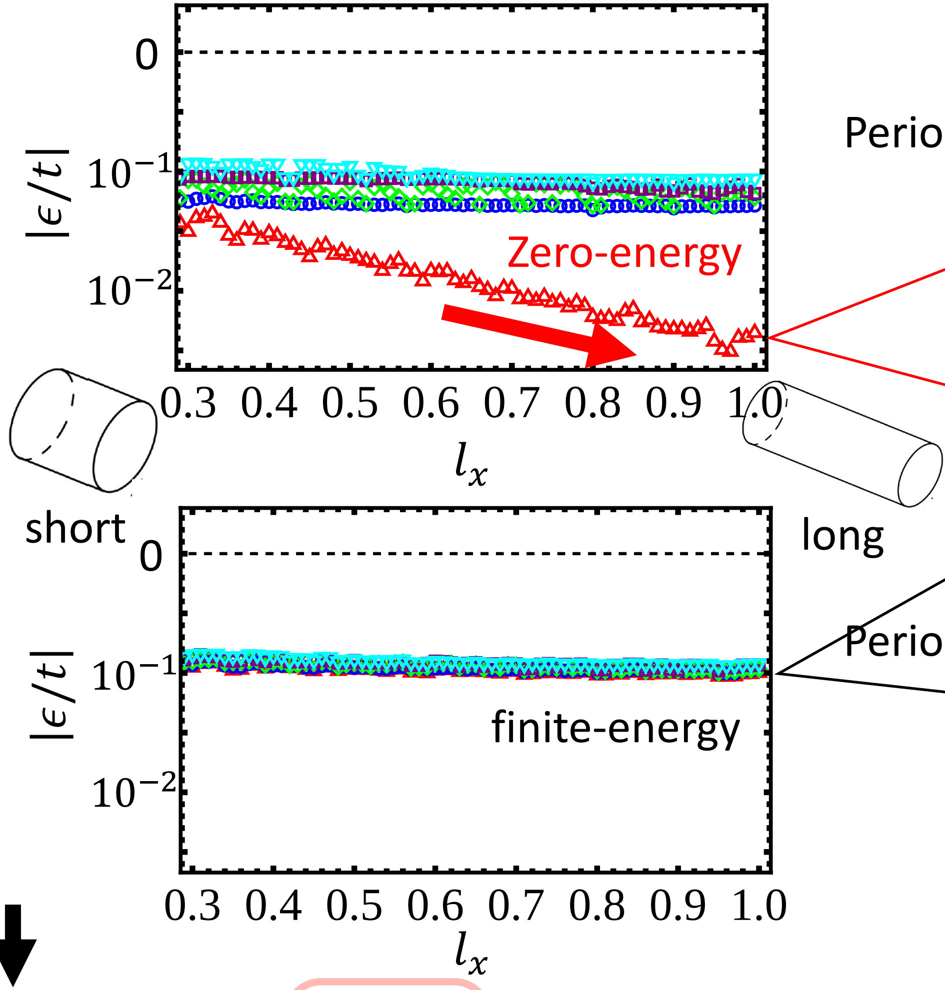
$\mu/t = 4$
Band top

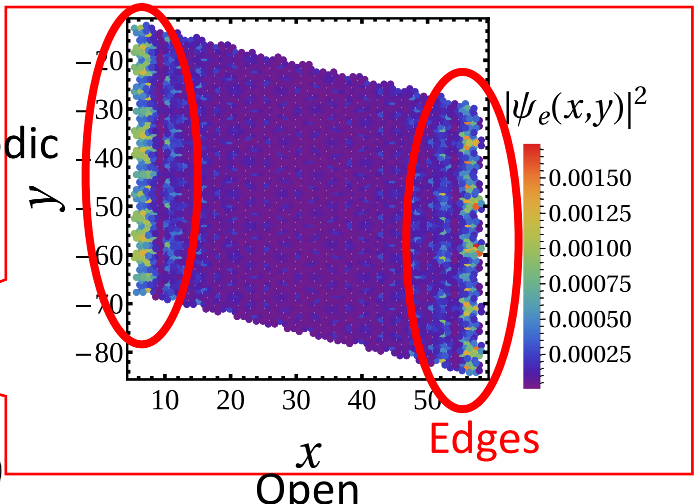
Chemical
potential
μ/t

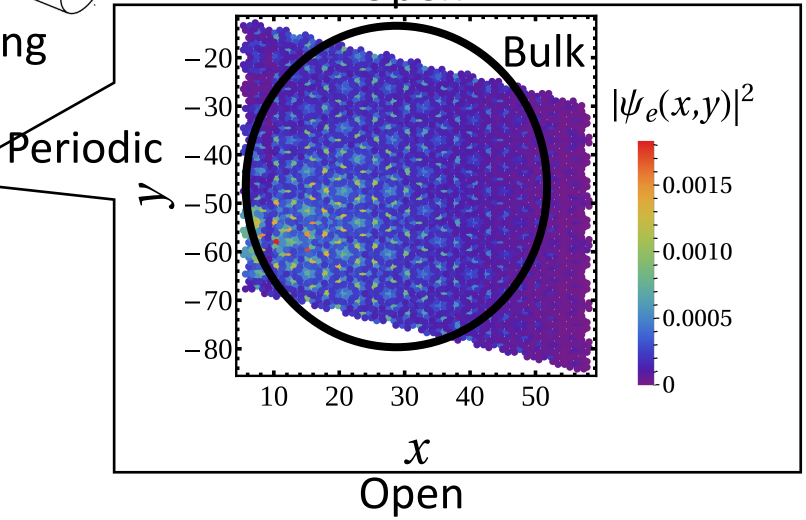
Zero-energy Edge Mode Exists When $l_x \to \infty$

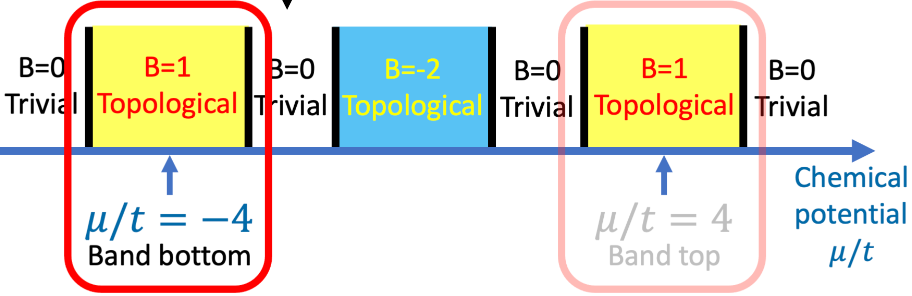
$B = 1$
Band bottom

$B = 0$
Trivial



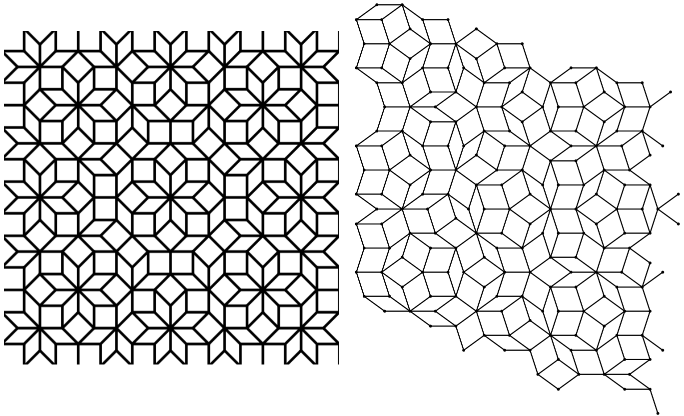






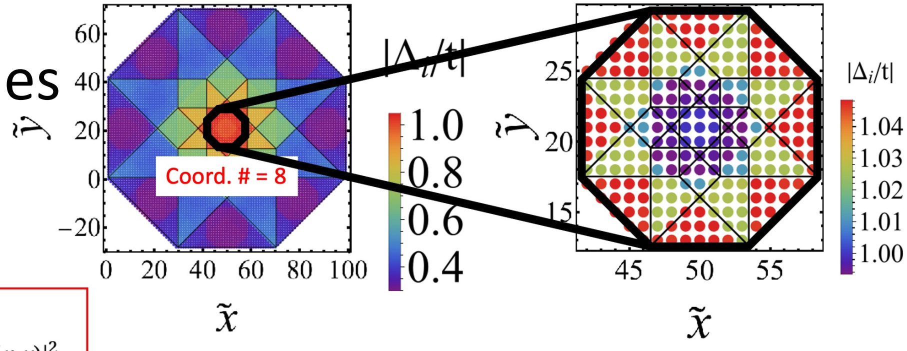
Summary 1

- Two-dimensional QCs (AB/Penrose QCs)
- Self-consistent calculation

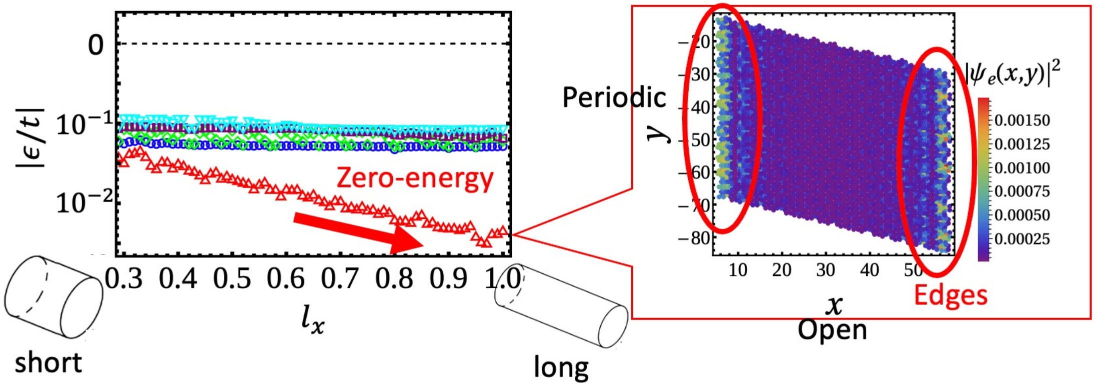


In AB/Penrose QCs without periodicity,
TSC exists stably ($\Delta_i/t \in \mathbb{C}$)

→We find the **self-similar** distribution of superconducting order parameter associated with the **self-similarity** in AB/Penrose QCs



Also, we find zero-energy edge modes



PART 2

Weyl Superconductivity

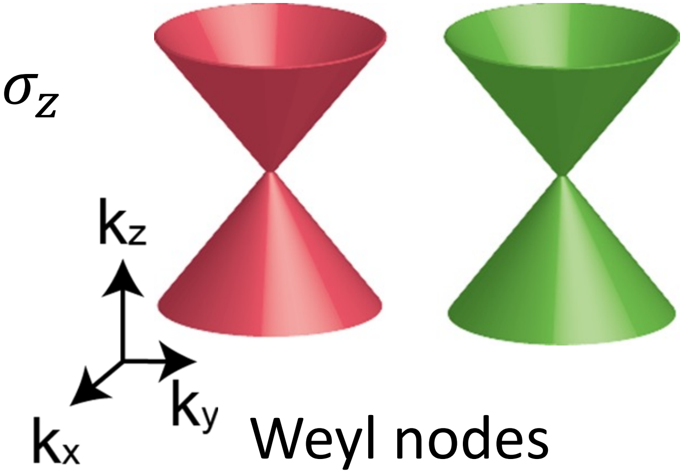
(Crystalline) Weyl Superconductor

Weyl superconductors (WSC) host point nodes due to an accidental band touching, which are described as the Weyl equation.

$$H_{\mathrm{Weyl}}(\boldsymbol{k}) = v_x k_x \sigma_x + v_y k_y \sigma_y + v_z k_z \sigma_z$$

Weyl nodes…

- require **translational symmetry**.
- are protected topologically.
- yield Majorana surface states.



Weyl nodes

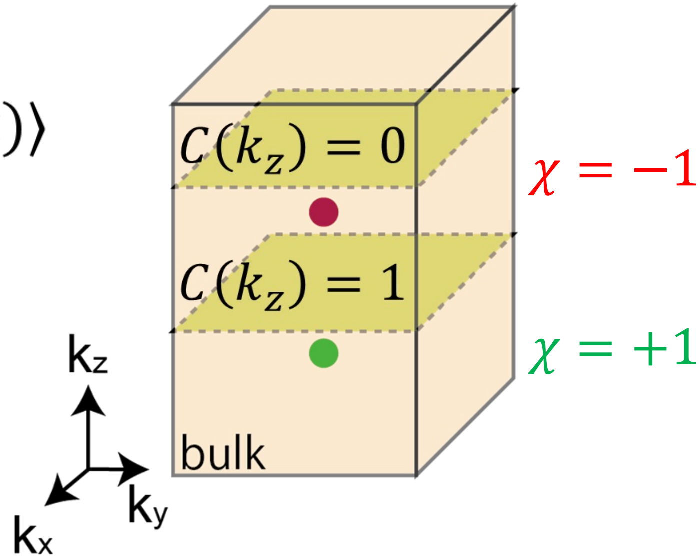
By definition, translational invariance is essential.

→Weyl superconductivity in QCs is impossible?

Band Topology of Weyl Nodes

Weyl nodes are topologically characterized by a change in the Chern number:

$$C(k_z) = \frac{1}{2\pi i} \sum_{E_n<0} \int_{2DBZ} d\boldsymbol{k}\, \langle \psi_n(\boldsymbol{k}) | \nabla_{\boldsymbol{k}} | \psi_n(\boldsymbol{k}) \rangle$$



Chirality for a Weyl node at $k_z = k_0$

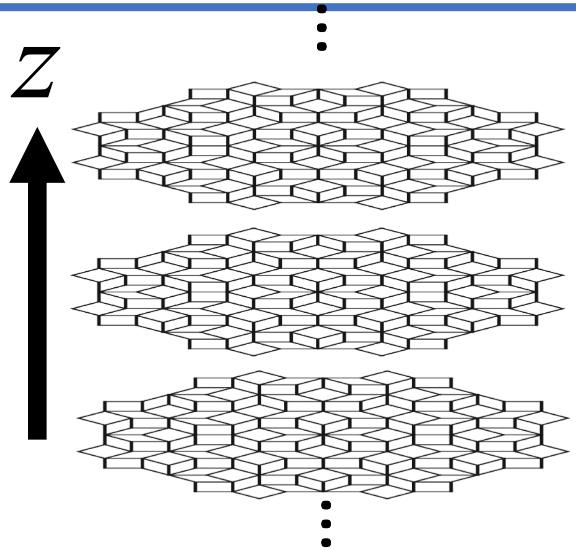
$$\chi(k_0) = \lim_{\delta \to 0^+} [C(k_0 + \delta) - C(k_0 - \delta)]$$

For Weyl superconductivity in QCs,

1. Chern number in QCs?
2. Chirality in QCs?

2. Chirality in QCs?
→ Weyl Superconductivity in Multilayered QCs

We consider stacking 2D superconducting QCs without time-reversal symmetry periodically.



Vertical momentum $\boldsymbol{k_z}$
in the stacking direction *z*.

The $\boldsymbol{k_z}$-dependent Bott index $\boldsymbol{B(k_z)}$ and chirality

$$\chi(k_0) = \lim_{\delta \to 0^+} [B(k_0 + \delta) - B(k_0 - \delta)]$$

Weyl nodes can be defined as gapless points to change the Bott index in the $\boldsymbol{k_z}$ space.

Cf. Periodically stacked QCs (Ta-Te) show superconductivity.

Y. Tokumoto *et al.*, arXiv:2307.10679.

Topological Superconductivity in 2D AB QCs

$$H = \frac{1}{2}\sum_{rr'\sigma\sigma'} (c^\dagger_{r\sigma}\ c_{r\sigma}) \begin{pmatrix} \mathcal{H}_0 & \Delta \\ \Delta^\dagger & -\mathcal{H}_0^* \end{pmatrix} \begin{pmatrix} c_{r'\sigma'} \\ c^\dagger_{r'\sigma'} \end{pmatrix}$$

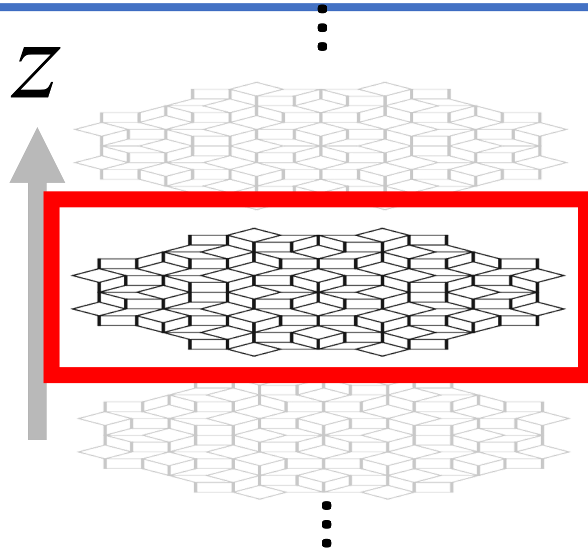
$$[\mathcal{H}_0]_{r\sigma,r'\sigma'}(\mu)$$

$$= [\underbrace{(t_{rr'} - \mu\delta_{rr'})\sigma_0}_{\text{Hopping} \quad \text{Chemical pot.}} + \underbrace{h_z\delta_{rr'}\sigma_z}_{\text{Zeeman}} + \underbrace{i\lambda_{rr'}\boldsymbol{e}_z \cdot \boldsymbol{\sigma} \times \hat{R}_{rr'}}_{\text{Rashba}}]_{\sigma\sigma'}$$

$$[\Delta]_{r\sigma,r'\sigma'} = [\delta_{rr'} i\Delta_0\sigma_y]_{\sigma\sigma'}$$

$t_{rr'} = -t$: the nearest neighbor hopping μ: chemical potential

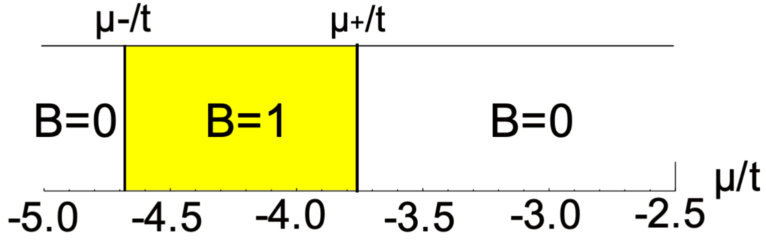
h_z: external magnetic field $\lambda_{rr'} = \lambda$: Rashba paramter σ, σ':spin

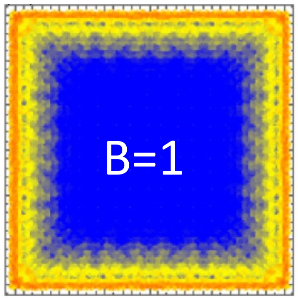


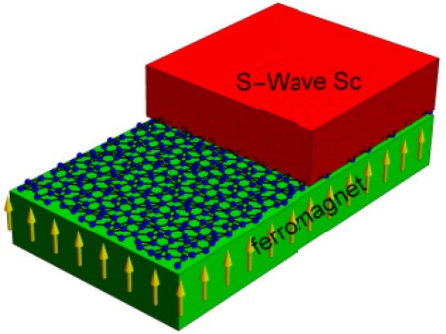
R. Ghadimi, T. Sugimoto, K. Tanaka, and T. Tohyama, PRB **104**, 144511 (2021).

Phase diagram ($\lambda = h_z = 0.5t, \Delta_0 = 0.2t$)

$\mu_\pm$: critical points







→ Stack AB QCs periodically.

Model for Multilayered AB QCs

Introduce nearest-neighbor interlayer hopping t_z

$$H_z = \sum_{k_z} (2t_z \cos k_z c) c^\dagger_{k_z r\sigma} c_{k_z r\sigma}$$

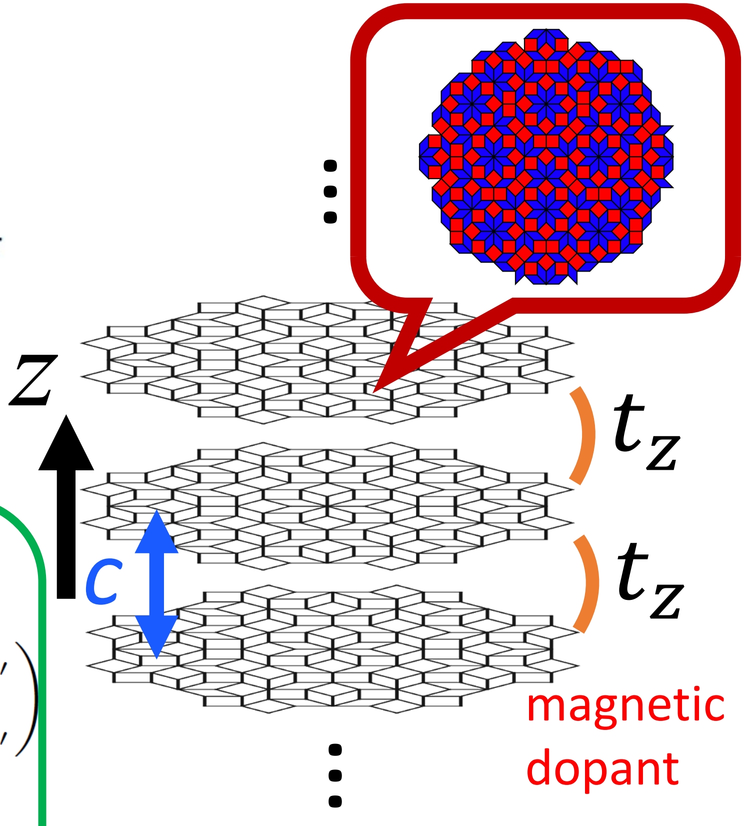
c : period

The 3D Hamiltonian

$$H_{\rm 3D} = \frac{1}{2} \sum_{k_z} \sum_{rr'\sigma\sigma'} (c^\dagger_{k_z r\sigma} \; c_{k_z r\sigma}) \begin{pmatrix} \mathcal{H}_0^{\rm 3D} & \Delta \\ \Delta^\dagger & -\mathcal{H}_0^{\rm 3D*} \end{pmatrix} \begin{pmatrix} c_{k_z r'\sigma'} \\ c^\dagger_{k_z r'\sigma'} \end{pmatrix}$$

$$\mathcal{H}_0^{\rm 3D}(k_z) = \mathcal{H}_0(\underline{\mu - 2t_z \cos k_z c})$$

Shift of chemical potential



multilayered model

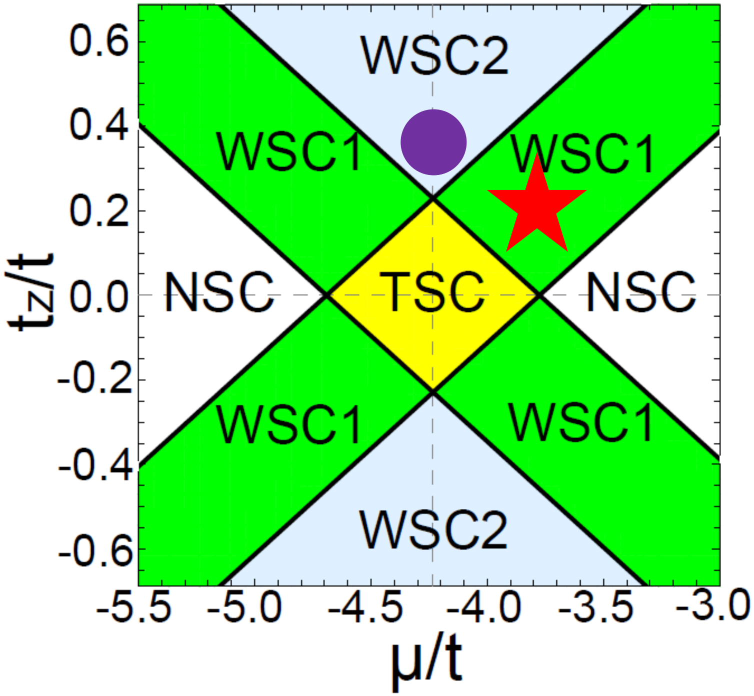
→ Weyl superconductivity in QCs?

Weyl Superconductivity in Multilayered AB QCs

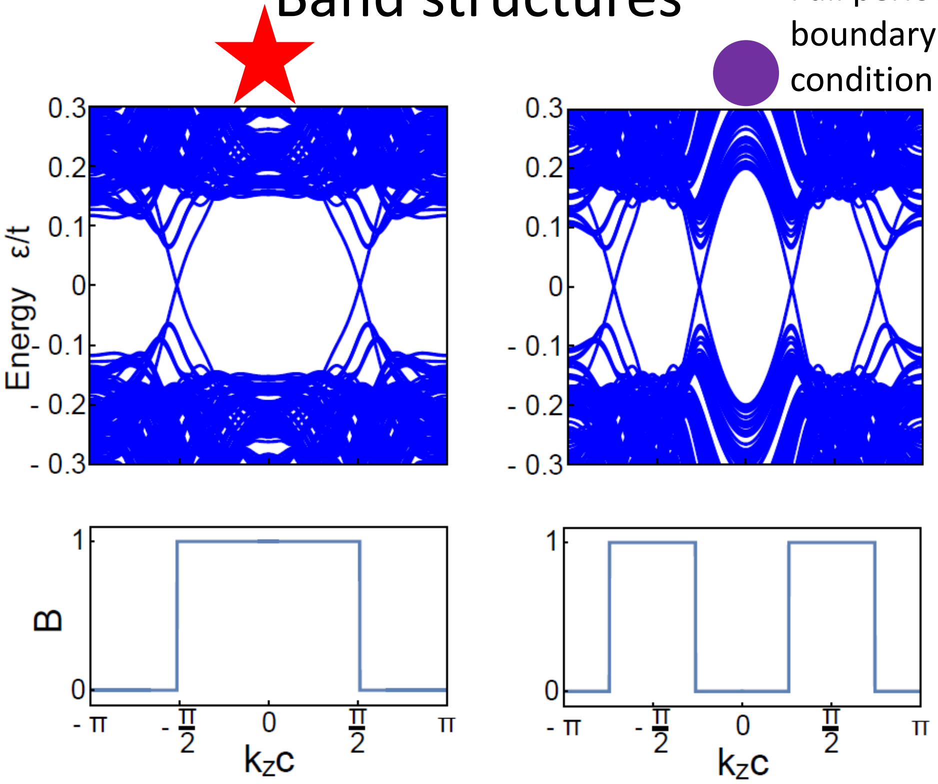
$$H_{\rm 3D}=\frac{1}{2}\sum_{k_z}\sum_{rr'\sigma\sigma'}\left(c^\dagger_{k_z r\sigma}\;c_{k_z r\sigma}\right)\begin{pmatrix}\mathcal{H}_0^{\rm 3D} & \Delta\\ \Delta^\dagger & -\mathcal{H}_0^{\rm 3D*}\end{pmatrix}\begin{pmatrix}c_{k_z r'\sigma'}\\ c^\dagger_{k_z r'\sigma'}\end{pmatrix}$$

$$\mathcal{H}_0^{\rm 3D}(k_z)=\mathcal{H}_0(\mu-2t_z\cos k_z c)$$

Phase diagram



Band structures



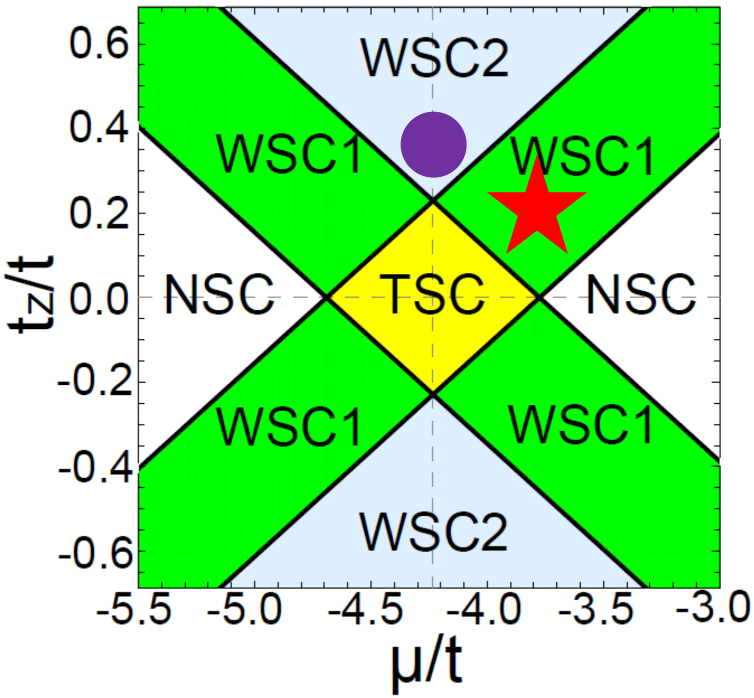
Full periodic boundary condition (PBC)

Gapless points (quasicrystalline Weyl nodes) appear and $B(k_z)$ change.

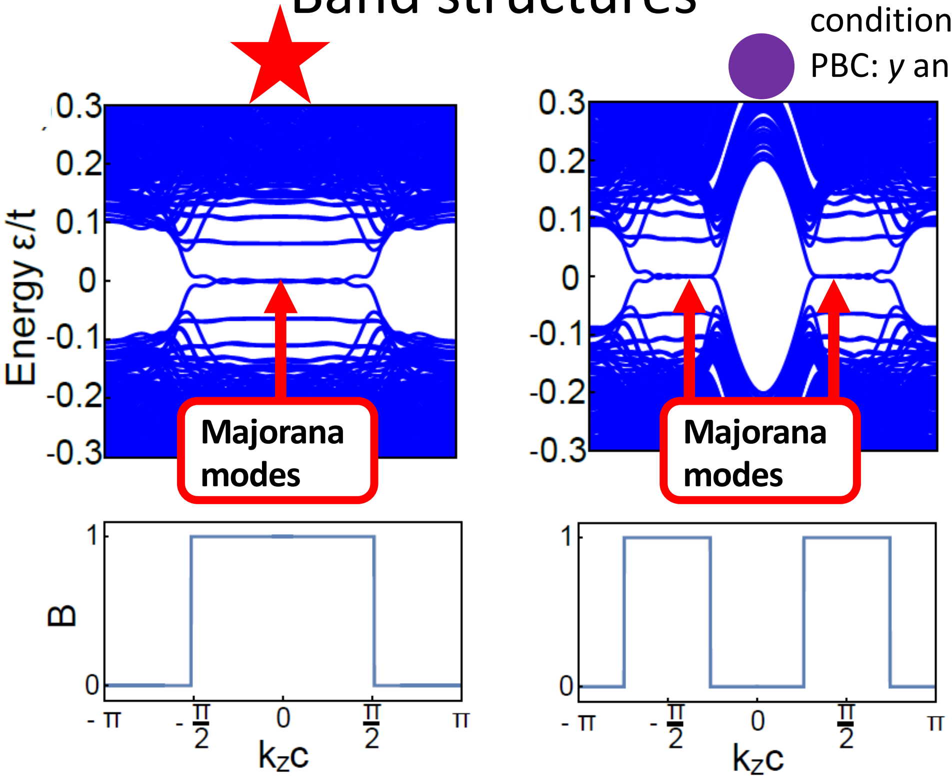
Majorana Surface States

Quasicrystalline WSCs host Majorana surface states between the surface projection of Weyl nodes because of the nonzero Bott index.

Phase diagram



Band structures



Open boundary condition : *x*
PBC: *y* and *z*

→ **Interesting properties of Majorana modes?**

Spectral Density of States (SDOS)

SDOS $$\rho(q_y,k_z,\varepsilon=0)=\sum_{r\in S_{\mathrm{b}}}\sum_{\alpha}\sum_{m}|\underline{\langle r,q_y,k_z,\alpha|\psi_m\rangle}|^2\underline{\delta(E_m)}$$

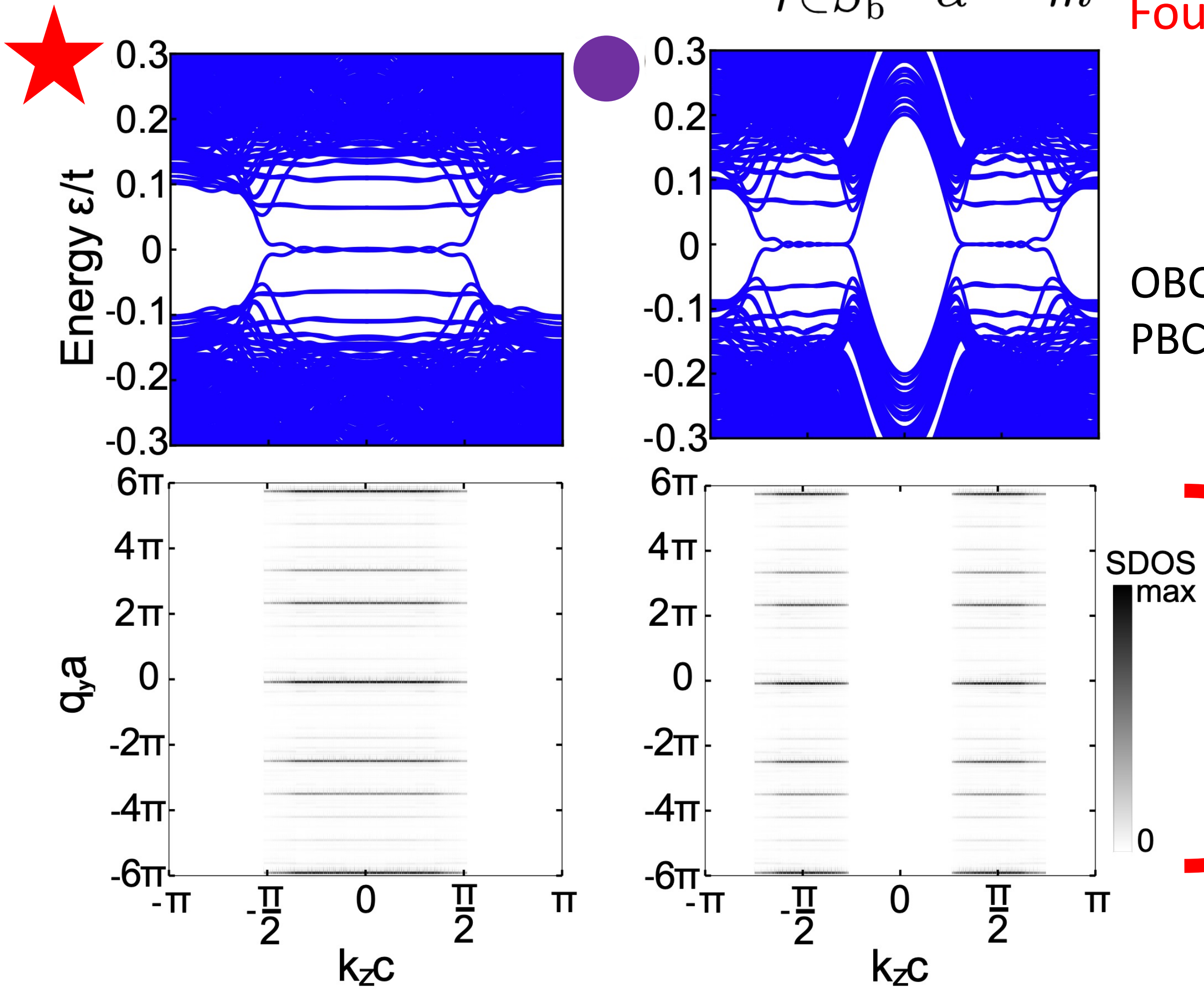
Fourier component

@0 energy

S_{b} : near the boundaries

E_m : eigenenergy

$|\psi_m\rangle$: eigenstate



OBC: *x*

PBC: *y* and *z*

Dence Majorana arcs found in QCs only

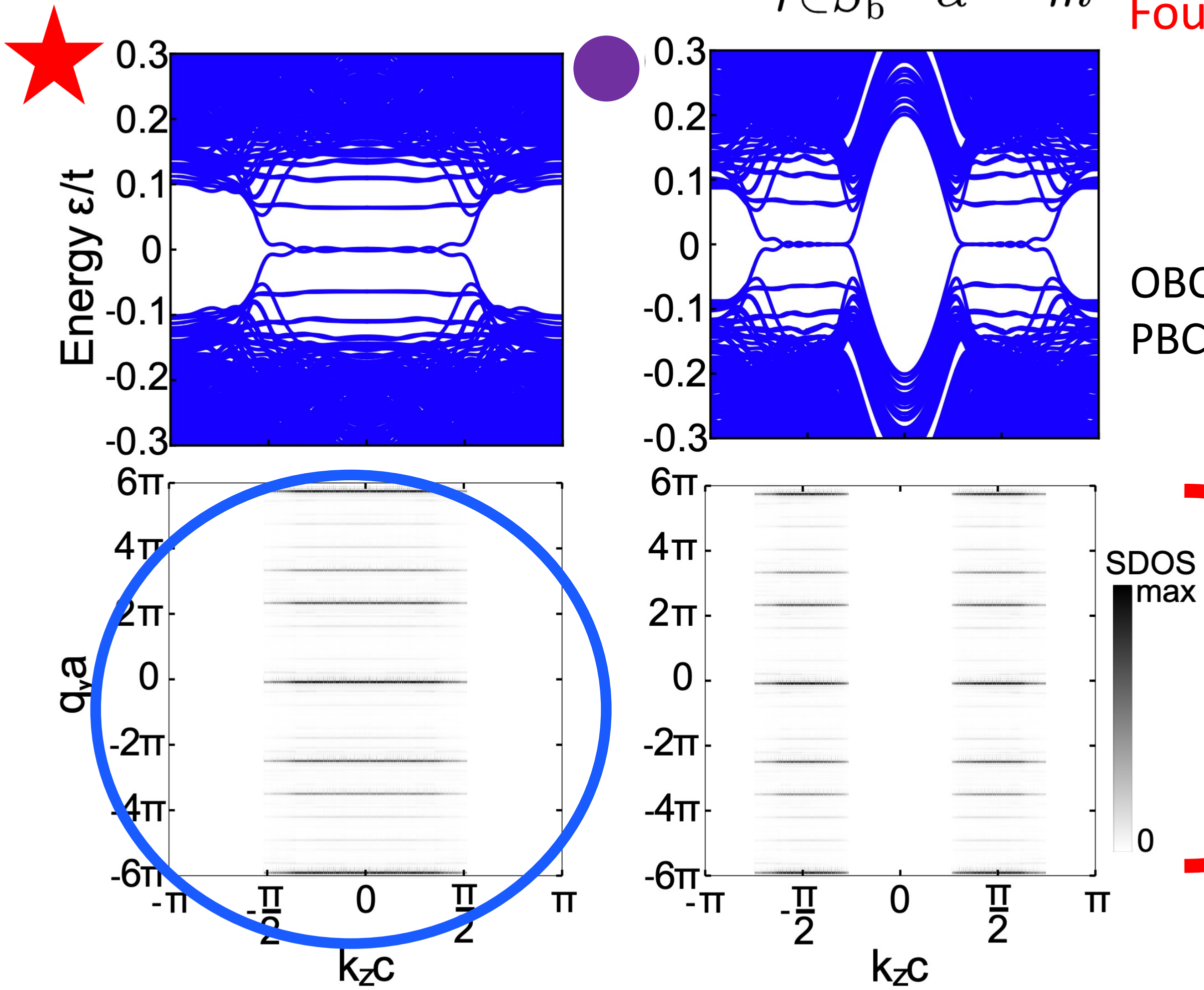
Spectral Density of States (SDOS)

SDOS $$\rho(q_y, k_z, \varepsilon = 0) = \sum_{r\in S_\mathrm{b}} \sum_{\alpha} \sum_{m} |\underbrace{\langle r, q_y, k_z, \alpha|}_{\text{Fourier component}}\psi_m\rangle|^2 \underbrace{\delta(E_m)}_{\text{@0 energy}}$$

S_b : near the boundaries

E_m : eigenenergy

$|\psi_m\rangle$: eigenstate

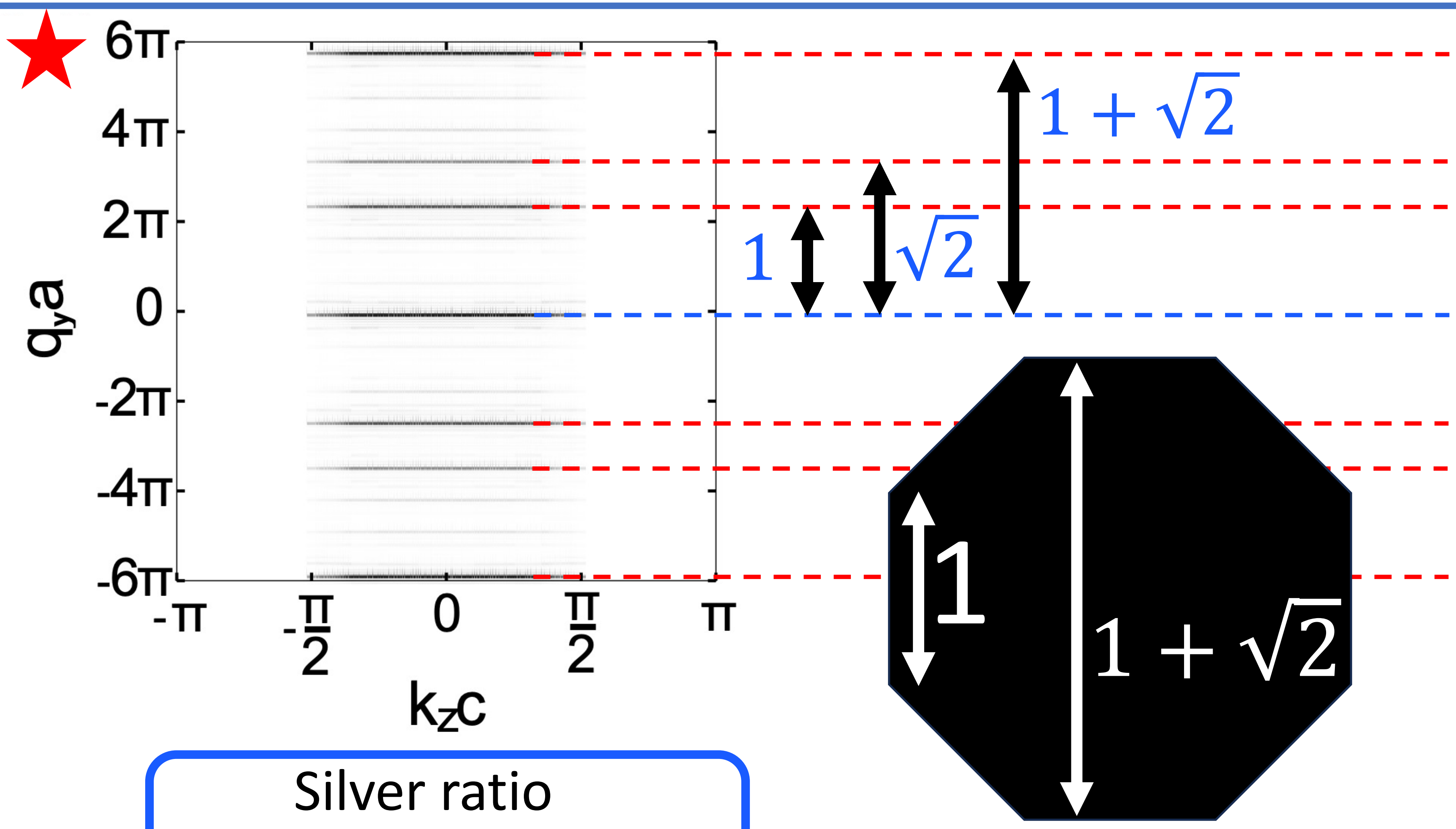


OBC: *x*

PBC: *y* and *z*

Dence Majorana arcs found in QCs only

Majorana Arcs Can Be Characterized by the Silver Ratio



Silver ratio

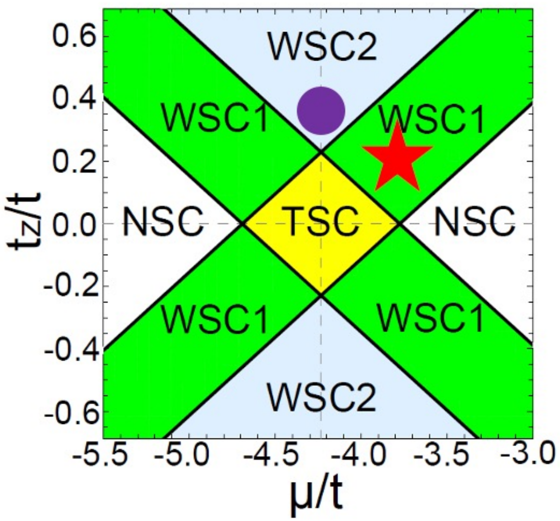
$1:1+\sqrt{2}$

octagon

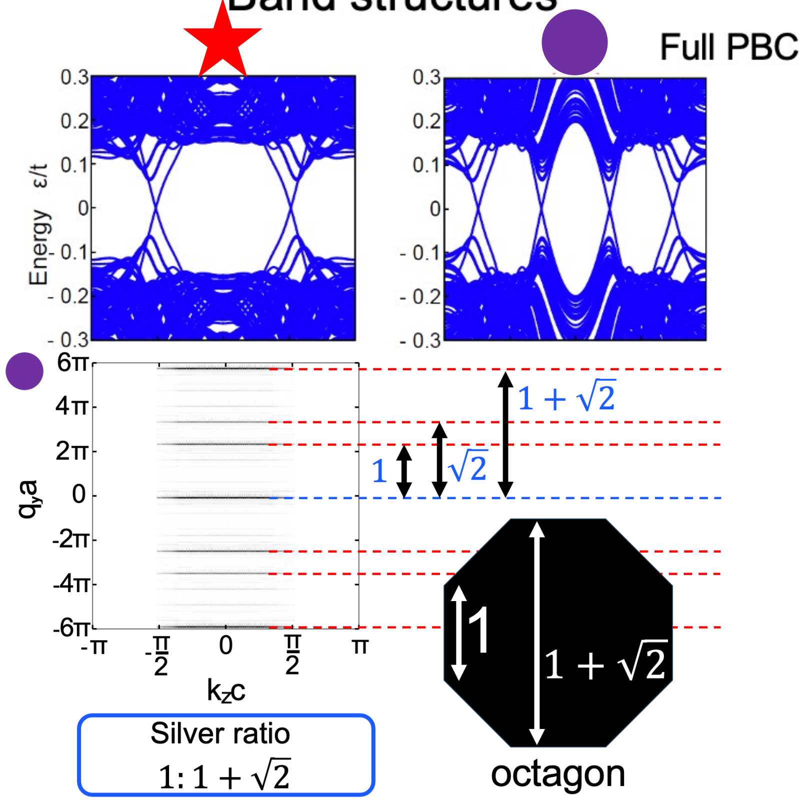
Summary 2

Proposal of Weyl superconductivity in QCs.

Phase diagram

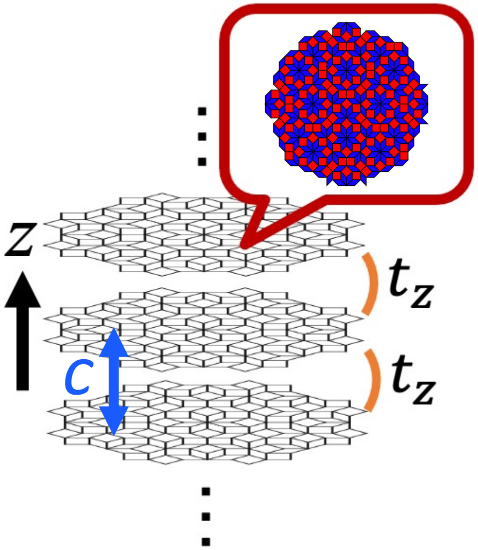


Band structures



Silver ratio

$1:1+\sqrt{2}$



Dence Majorana arcs are characterized by the silver ratio

最後に

・準結晶においてトポロジカル超伝導が（理論的には）実現します。
・積層系準結晶においてワイル超伝導が（理論的には）実現します。

堀は今後も準結晶の研究を続けます。
ただし、12月の準結晶研究会を最後に、
しばらく対面の学会でお会いすることはありません。

一方、準結晶の布教活動を行っています。
【共同研究例】
・名古屋大学　St研　水野航希くん
　フィボナッチ準結晶におけるグリーン関数（超伝導）
　→国際学会1件、国内学会1件で発表済

・大阪大学　波多野研　舟見優くん
　準周期系における電荷密度波の模型

今後ともよろしくお願いいたします。