JCDCG^3 2022

The 24th Japan Conference on Discrete and Computational Geometry, Graphs, and Games

9th September – 11th September 2022

Online held

The 24th Japan Conference on Discrete and Computational Geometry, Graphs, and Games

Conference date
September 9 – 11, 2022
Venue
Fully Online

Invited Plenary Speakers

Edy Tri Baskoro	(Institut Teknologi Bandung, Indonesia)
Erik D. Demaine	(MIT, USA)
Xueliang Li	(Nankai University, China)
Joseph O’Rourke	(Smith College, USA)
János Pach	(Rényi Institute, Hungary and IST, Austria)
Carol T. Zamfirescu	(Ghent University, Belgium)

Conference Chair

Jin Akiyama	(Tokyo University of Science, Japan)

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Outline of Program

9th Sep. (Fri.)

10:30-10:45	Opening Remark
10:45-11:45	Invited Talk 1 by Joseph O’Rourke
11:45-13:00	Lunch Break
13:00-14:00	Contributed Talks
14:15-15:35	Contributed Talks
16:00-17:00	Invited Talk 2 by Carol T. Zamfirescu

10th Sep. (Sat.)

09:30-10:30	Invited Talk 3 by Erik D. Demaine
10:45-11:45	Contributed Talks
11:45-13:00	Lunch Break
13:00-14:00	Invited Talk 4 by Xueliang Li
14:15-15:35	Contributed Talks
15:50-17:10	Contributed Talks

11th Sep. (Sun.)

09:30-10:30	Invited Talk 5 by Edy Tri Baskoro
10:45-12:05	Contributed Talks
12:05-13:15	Lunch Break
13:15-14:15	Invited Talk 6 by János Pach
14:30-16:10	Contributed Talks
16:10-16:25	Closing Remark

Table of Contents

Invited Talks

Contributed Talks

Reshaping Convex Polyhedra

Joseph O'Rourke*

Given two convex polyhedra P and Q, with Q inside P, one can reshape P to Q by repeated *vertex truncations*: slicing P with a plane that has a single vertex to one side. For example, this is one way to construct the truncated cube: see Fig. 1.

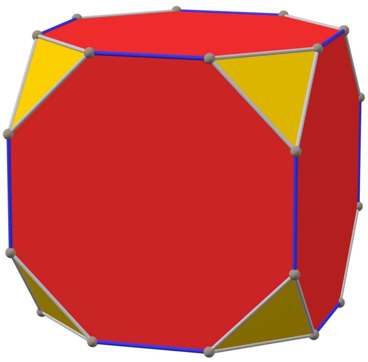

Figure 1: Truncated cube. [Image by Tilman Piesk, Wikipedia].

An alternative method of "snipping" off vertices we call *digon-tailoring*. Here we locate a *digon* containing a vertex v, a subset of P bounded by two equal-length geodesic segments γ_1, γ_2 that share endpoints. We excise the digon and suture closed its two sides by identifying γ_1 with γ_2. This produces a new convex polyhedron guaranteed by Alexandrov's Gluing Theorem.

The main result of this presentation is that any P can be reshaped to any $Q \subset P$ by a finite sequence of tailorings. This holds whether Q has fewer vertices than P or more vertices than P: see Fig. 2. For polyhedra of n vertices, the reshaping can be accomplished algorithmically in $O(n^4)$ time.

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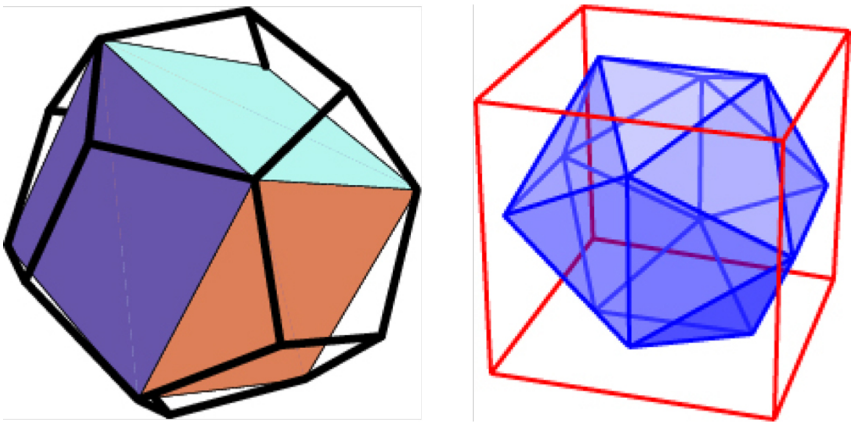

Figure 2: Cube inside dodecahedron. Icosahedron inside cube.

It is an easy corollary that, if S is the surface of any convex body inside P, then P may be tailored to approximate S as closely as desired. So P can be "whittled" to e.g., a sphere S.

I will also discuss consequences of this theorem, including "unfolding" Q onto P, and continuously folding P onto Q.

This work is joint with Costin Vîlcu [OV21].

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Counting cycles in regular and planar graphs

Carol T. Zamfirescu

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Abstract

Motivated by an old and famous conjecture of Sheehan – namely that every hamiltonian 4-regular graph has at least two distinct hamiltonian cycles – and two recent conjectures of Haythorpe, we first discuss the problem of determining the minimum number of hamiltonian cycles occurring in hamiltonian regular graphs. One of our results is that there exists a constant $c > 0$ such that there are infinitely many 4-regular 3-connected planar graphs, each containing exactly c hamiltonian cycles. Note that by a recent theorem of Brinkmann and Van Cleemput, we cannot replace "3-connected" by "4-connected" in the preceding sentence. We then investigate several variations of Sheehan's conjecture, one of which is as follows: we still impose the uniqueness of the cycle but instead of requiring it to necessarily be hamiltonian, we now simply ask for it to be a longest cycle.

Finally, we present a series of recent results on the enumeration of cycles in planar triangulations, for instance that every n-vertex triangulation with at most one separating triangle contains $\Omega(n)$ many k-cycles for every $k \in \{3, \ldots, n\}$; this last part of the talk is based on joint work with On-Hei Solomon Lo.

Pushing Block Puzzles

Erik D. Demaine[1]

[1]Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139, USA; edemaine@mit.edu (E. Demaine)

Abstract

Sokoban[*] is perhaps the most famous puzzle video game involving pushing blocks, where a 1×1 agent must push various 1×1 blocks (at most one at a time) into designated locations, while avoiding 1×1 fixed obstacle blocks. Figure 1 shows an example puzzle. Many other video games — such as the *Legend of Zelda*, *Tomb Raider*, and *Half-Life* series — feature similar pushing block puzzles, sometimes allowing pulling blocks in addition to or instead of pushing. The computational geometry and games community have since analyzed a wide range of pushing block puzzles, characterized by the following parameters (among others):

1. Whether the agent can push blocks (as in Sokoban), pull blocks, or both. These variations are referred to as Push, Pull, and PushPull, respectively.
2. When moving away from a block, whether the agent must pull that block. This variation is referred to as Pull!, whereas the optional pulling model is Pull?.
3. How many blocks the agent can push or pull at once. In Sokoban and variations such as Push-1, this number is 1.
4. Whether there are fixed blocks that the agent cannot push, as in Sokoban and variations such as Push-1F.
5. Whether pushed blocks (and optionally the agent) automatically slide until they hit a block.
6. Whether blocks and/or the agent fall according to gravity.
7. Whether the goal is to store blocks in specified locations (as in Sokoban) or just for the agent to reach a particular location (as in Push, Pull, and PushPull variations).

The goal is to characterize the complexity of each puzzle variation. While most variations are easy to show NP-hard, the more challenging question is which variations are PSPACE-complete. Sokoban has been known to be PSPACE-complete since 1998 [Cul98]. A brand new result is the PSPACE-completeness of Push-1F, which is essentially the same as Sokoban but where the goal is for the agent to just reach a particular location (and blocks do not have designated locations) [ACD+22]. Figure 2 shows a simple Push-1F puzzle. We will survey these and many other historical results, including pulling-block results from JCDCGGG 2019 [AAD+20].

Pushing/pulling block puzzles have served as a testing ground for developing hardness proof infrastructure. Many of the past PSPACE-hardness results use Nondeterministic Constraint Logic [HD09] or the recent motion-planning-through-gadgets framework that has developed over the past few years [DGLR18,DHL20,ABD+20,Lyn20,Hen21,AAD+20,ADG+21,ADHoL22,ADD+22,DHHL22]. Our latest analysis of Push-1F introduces a powerful new tool for the gadgets framework called *checkable gadgets*, which lets the gadget designer assume that certain traversals remain possible, effectively preventing the player from breaking gadgets in previously difficult-to-prevent ways.

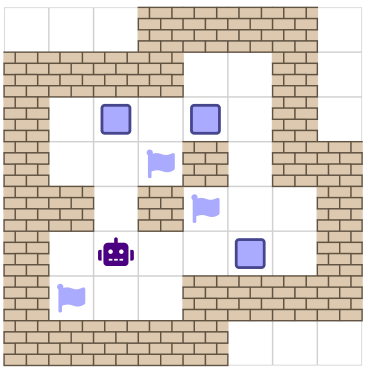

Figure 1: Sokoban level "Minicosmos 16" designed by Peloux in 2001 [dP01]. Can you push the three blocks onto the three flags? Playable at https://sokoban.info/?4_16.

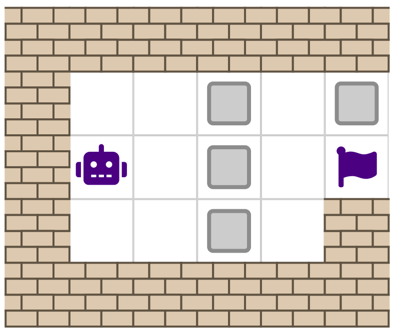

Figure 2: Simple Push-1F puzzle from [ACD+22, Figure 1a]. Can you push blocks to reach the flag position?

[*]Sokoban was created by Hiroyuki Imabayashi in 1980, and published by Thinking Rabbit in 1982.

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Rainbow cycles in edge-colored graphs

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Abstract

A cycle in an edge-colored graph is called rainbow if any two edges of the cycle have different colors. In this talk we will survey the results on rainbow cycles in edge-colored graphs. The results are mainly on short and long rainbow cycles, rainbow Hamiltonian cycles and (vertex- and edge-)disjoint rainbow cycles, as well as the number of rainbow triangles in edge-colored graphs. Some open problems are also presented for further study. (Joint work with Dr. Xiaozheng Chen and Dr. Luyi Li)

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On the existence of almost Moore digraphs

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2010 Mathematics Subject Classification : 05C35

Let d and k be two natural numbers. An *almost Moore digraph* is defined as a diregular digraph of degree d, diameter k and order $d + d^2 + \ldots + d^k$, and it is denoted by (d,k)-digraph. The question of the existence of (d,k)-digraphs has received a lot of attention. Fiol, Allegre and Yebra (1983) showed that $(d,2)$-digraph exists for any $d \geq 2$, and one of them is the line digraph $L(K_{d+1})$ of the complete digraph K_{d+1} on $d+1$ vertices. Miller and Fris (1988) showed that there are exactly three $(2,2)$-digraphs and Gimbert (2001) proved the uniqueness of $(d,2)$-digraphs for any $d \geq 3$. But, a (d,k)-digraph does not exist for $d = 2,3$ and $k \geq 3$, and for $d \geq 2$ and $k = 3,4$. For the remaining cases, namely for $d \geq 4$ and $k \geq 5$, the existence of (d,k)-digraphs is still open. The structural study on these digraphs (if they exist) was initiated by the work of Mirka Miller by introducing a repeat function. In this talk, we will discuss the beauty of repeat function used to explore the possibility of the existence of almost Moore digraphs for $d \geq 4$ and $k \geq 5$. Open problems related to this study are also presented.

Enumeration of intersection graphs of geometric objects

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Abstract

Given a collection C of n geometric objects, define their intersection graph $G(C)$ as follows. Let the vertex set of $G(C)$ be C, and connect two elements of C by an edge if and only if they have a point in common. The total number of graphs on n labeled vertices is $2^{\binom{n}{2}}$. How many of them are intersection graphs of connected arcs ("strings") in the plane? Pach and Tóth proved that the answer is $2^{(3/4+o(1))\binom{n}{2}}$.

If we restrict our attention to intersection graphs of strings, any pair of which intersect at most k times, for a fixed k, then the number becomes $2^{o(n^2)}$. On the other hand, it was shown by Pach and Solymosi that the number of segment intersection graphs on n vertices is $2^{(4+o(1))n \log n}$. After giving a whirlwind tour of enumeration results and methods of this kind, we present some recent results by Jacob Fox, Andrew Suk, and the speaker.

Extended Abstract

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Regular polygon on the equatorial section of regular n-simplex

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Regular polytopes are generated by the symmetry operations of an orthogonal reflection. Analogously by using unitary reflection, regular complex polytopes are generated in complex space [1-3]. As an application of the unitary reflection, Coxeter [4] constructed a regular pentagon on the equatorial section of a regular 4-dimensional simplex in real space. By extending this approach, we can construct an equatorial regular (n+1)-gon of the regular n-simplex. The regular n-simplex with n+1 vertices is projected just like a K_{n+1} graph onto 2-plane. Moreover, as dimension 'n' increase, the size of the projected regular (n+1)-gon is convergent to a half size of the K_{n+1} graph.

Background of the problem

Identify and connect four points - one on each triangular face - of a regular tetrahedron, subject to two constraints: (a) the four points form vertices of a square on a 2-plane, and (b) this square passes through the center of the tetrahedron (equatorial plane). In this study, we ask this simple question: is it possible to make such an arrangement of points? The answer could be one of the following:

(0) impossible, (1) uniquely determined, or (∞) infinitely many

Majority of answers is (0) or (1). (∞) is minority, but contra-intuitively (∞) is correct. Since seeing is believing, a picture of Petrie bisection will be a help to your understanding.

As mentioned above, Coxeter [4] constructed a regular pentagon on the equatorial section of a regular 4-simplex. Such a problem can be extended a hyper-simplex in hyperspace as follows; identify a points on each facet of a regular n-simplex and then connect all n+1 points. Is it possible to make a regular (n+1)-gon on the equatorial section of the n-simplex? The answer is 'yes, we can', but the more interesting question is 'how we should do it'. As an introductory example, we will illustrate how to construct an equatorial heptagon from a regular 6-simplex (Fig. 1).

The regular 6-simplex and its equatorial heptagon

Generally speaking, the coordinates of n+1 vertices of a regular n-simplex are irrational. Easiest way to avoid irrationality is to choose n+1 unit-points after raising up one more dimension. In the regular 6-simplex, seven vertices are represented as (1,0,0,0,0,0,0), (0,1,0,0,0,0,0),$\cdots$, (0,0,0,0,0,0,1) in the hyperplane $x_1+x_2+x_3+x_4+x_5+x_6+x_7=1$. Its center is (1/7,1/7,$\cdots$,1/7,1/7) and the edge length is $\sqrt{2}$. By the standard orthogonal projection, the regular 6-simplex with 7 vertices is projected onto the 2-plane, just like a K_7 graph.

[1] Typical vertex of an equatorial regular polygon

By choosing a temporary point $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ on the 4-face, we can take $x_6=x_7=0$. The other 6 points correspond to its cyclic permutations. Without loss of generality, we can let $x_0=0$, $x_1=1$ and $x_2=1+2\cos(2\pi/7)=X$, where, X denotes the diagonal length parallel to an edge of a regular heptagon with unit edge length. Then, x_3, x_4 and x_5 are determined as generalized Tribonacci sequence.

$x_3=X(x_2-x_1)+x_0=X^2-X$,

$x_4=X(x_3-x_2)+x_1=X^3-2X+1=X$,

$x_5=X(x_4-x_3)+x_2=X^4-3X^3+x^2+2X=1$

After the scale transformation by $y_i=x_i/\Sigma x_i$, $\Sigma x_i=X^4-2X^3+2X+2=X^2+X+2$, a typical vertex of the heptagon on the equatorial section of the regular 6-simplex is given by $(y_1,y_2,y_3,y_4,y_5,y_6,y_7)$, where $\Sigma y_i=1$. We can confirm that each of vertices are situated on the 2-plane as evidenced by the rank of the matrix (=2).

[2] Projection of equatorial polygon

By the projection into the unit circle, $(y_1,y_2,y_3,y_4,y_5,y_6,y_7)$ is projected on (ξ,η), whose components are given as a form of finite Fourier series, namely; $\xi=\Sigma y_i\cdot\cos(i-1)\theta$ and $\eta=\Sigma y_i\cdot\sin(i-1)\theta$, where $\theta=2\alpha=2\pi/7$. Therefore, the projected distance ‘d’ and Euclidean distance ‘D’ between the vertex and the center of heptagon are calculated as follows:

$$d^2=\xi^2+\eta^2=7(X^2-X+1)/(X^2+X+2)^2$$

$$D^2=\Sigma(y_i-1/7)^2=2(X^2-X+1)/(X^2+X+2)^2.$$

Theorems

If we choose $(x_1,x_2,x_3,x_4,x_5,x_6,x_7)$ on the facet $(x_7=0)$ or on the 3-face $(x_5=x_6=x_7=0)$, the simultaneous equation becomes indeterminate or inconsistent, respectively. The following theorem is derived directly from the above-mentioned procedure.

[1] By appropriately selecting a point on each of the facets of a regular n-simplex, we can construct infinitely many numbers of regular (n+1)-gon on the equatorial section. If n+1 points are chosen on the (n-2)-faces, we can construct the equatorial regular (n+1)-gon of maximum size. If chosen on the (n-3)-faces, it is impossible to construct the equatorial regular (n+1)-gon of any size.

[2] $d^2=1$ (n=2), $d^2=1/2$ (n=3), $d^2=1/\tau^2$ (n=4), $d^2=1/3$ (n=5),$\cdots$, $d^2=1/(2\cos\alpha)^2$, where $\alpha=\pi/(n+1)$.
As dimension ‘n’ increase, d^2 decreases monotonously and converges to 1/4.

[3] $D^2=2/3$ (n=2), $D^2=1/4$ (n=3), $D^2=2/5\tau^2$ (n=4), $D^2=1/9$ (n=5),$\cdots$, $D^2=2d^2/(n+1)$.
This means d^2 is magnified (n+1)/2 times that of D^2. As dimension ‘n’ increase, D^2 decreases monotonously and converges to 0.

Points of Proofs

[a] One version of the proof roughly consists of lengthy and complicated trigonometric calculations, which eventually in results the following.

$$x_i=\sin(i\alpha)\sin((i+1)\alpha)/\sin\alpha\sin2\alpha,\ \alpha=\pi/(n+1).$$

$$(\xi,\eta)=(-\cos3\alpha\sin\alpha/\sin2\alpha,\ \sin3\alpha\sin\alpha/\sin2\alpha)\ \rightarrow\ d^2=\xi^2+\eta^2=1/(2\cos\alpha)^2\ \rightarrow\ 1/4$$

[b] Another version of the proof is based on concepts in elementary geometry.
In odd dimension, (ξ,η) situates on the intersection of two diagonals of K_{n+1}. For an example, the regular 7-simplex and its equatorial octagon are shown in Fig. 2. As $n\rightarrow\infty$, $X=1+2\cos(2\pi/(n+1))\rightarrow3$. By the ratio of two isosceles triangles, it divides a diameter in the ratio 3:1. No calculation necessary.

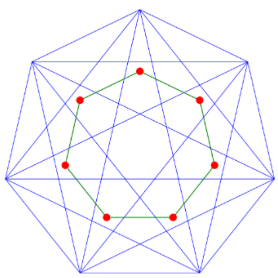

Fig. 1 6-simplex and equatorial heptagon

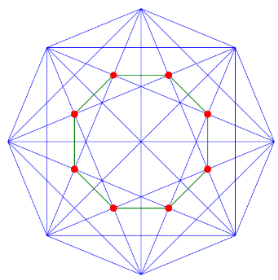

Fig. 2 7-simplex and equatorial octagon

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Universal Convex Covering Allowing Discrete Rotations

Mook Kwon Jung* Sang Duk Yoon† Hee-Kap Ahn‡ Takeshi Tokuyama§

Universal covering of objects

Given a set S of planar objects and a group G of geometric transformations, a G-covering K of S is a region such that every object in S can be contained in K by transforming the object with a suitable transformation $g \in G$. Equivalently, every object of S is covered by $g^{-1}K$ for a suitable transformation $g \in G$. That is,

$$\forall \gamma \in S,\ \exists g \in G \text{ such that } g\gamma \subseteq K.$$

We denote the group of planar translation by T and that of planar translation and rotation by TR. Mathematically, $TR = T \rtimes R$ is the semidirect product of T and the rotation group $R = SO(2, \mathbb{R})$. We often call coverings for G-coverings if G is known from the context. We are mainly interested in the case where G is generated by translation and discrete rotations in this paper.

The problem of finding a smallest-area covering is a classical problem in mathematics, and such a covering is often called a *universal covering.* In the literature, the cases where $G = T$ or $G = TR$ have been widely studied.

History

The universal covering problem has attracted many mathematicians. Henri Lebesgue (in his letter to J. Pál in 1914) proposed a problem to find the smallest-area convex TR-covering of all objects of unit diameter (see [7, 4, 10] for its history). Soichi Kakeya considered in 1917 the T-covering of the set S_{seg} of all unit line segments (called needles) [13]. Precisely, his formulation is to find the smallest-area region in which a unit-length needle can be turned round, but it is equivalent to the covering problem if the covering is convex [3]. Originally, Kakeya considered the convex covering, and Fujiwara conjectured that the equilateral triangle of height 1 is the solution. The conjecture was affirmatively solved by Pál in 1920 [16]. Moreover, the equilateral triangle can contain all unit-length curves. For the nonconvex covering, Besicovitch [5] gave a construction such that the area can be arbitrarily small.

Generalizing Pál's result, for any set of n segments, there is a triangle to be the smallest-area convex T-covering of the set, and the triangle can be computed efficiently in $O(n \log n)$ time [1]. It is further conjectured that the smallest-area convex TR-covering of a family of triangles is a triangle, which is shown to be true for some families [18].

The problem of finding the smallest-area covering and convex covering of the set of all curves of unit length and $G = TR$ was given by Leo Moser as an open problem in 1966 [14]. The problem is still unsolved, and the best lower bound of the smallest area of the convex covering is slightly larger than 0.23, while the best upper bound was about 0.27 for long time [8, 7]. Wetzel informally conjectured (formally published in [19]) in 1970 that the 30° circular fan of unit radius, which has an area $\pi/12 \approx 0.2618$, is a convex TR-covering of all unit-length curves, and it is recently proved by Paraksa and Wichiramala [17].

There are many variants of Moser's problem, and they are called *Moser's worm problem.* The history of progress on the topic can be found in an article [15] by William Moser (Leo's younger brother), in Chapter D18 in [8], and in Chapter 11.4 in [7]. It is interesting to find a new case of Moser's worm problem with a clean mathematical solution.

Universal covering of closed curves

From now on, we focus on the convex G-covering of the set S_{c} of all closed curves γ of length 2. Here, we follow the tradition of previous works on this problem to consider length 2 instead of 1, since a unit line segment can be considered as a degenerate convex closed curve of length 2. Since the boundary curve of the convex hull $C(\gamma)$ of any closed curve γ is not longer than γ, it suffices to consider only convex curves.

This problem is known to be an interesting but hard variant of Moser's worm problem, and remains unsolved for T and TR despite of substantial efforts in the literature [9, 19, 20, 8, 7]. As far as the authors know, the smallest-area convex TR-covering known so far is a hexagon obtained by clipping two corners of a rectangle given by Wichiramala, and its area is slightly less than 0.441 [22]. It is also shown that the smallest area is at least 0.39 [11],

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which has been recently improved to 0.4 [12] with help of computer programs. The smallest area of the convex T-covering is known to be between 0.620 and 0.657 [7].

There are some works on restricted shapes of covering. Especially, if we consider triangular coverings, Wetzel [20, 21] gave a complete description, and it is shown that an acute triangle with side lengths a, b, c and area X becomes a T-covering (resp. TR-covering) of S_{c} if and only if $2 \leq \frac{8X^2}{abc}$ (resp. $2 \leq \frac{2\pi X}{a+b+c}$). As a consequence, the equilateral triangle of side length 4/3 (resp. $\frac{2\sqrt{3}}{\pi}$) is the smallest triangular T-covering (resp. TR-covering) of S_{c}. Unfortunately, their areas are larger than those of the known smallest-area convex coverings.

Our results

If H is a subgroup of G, an H-covering is a G-covering. Since $T \subset TR$, it is quite reasonable to consider groups G lying between them, that is $T \subset G \subset TR$. The group $R = SO(2, \mathbb{R})$ is an abelian group, and its finite subgroups are $Z_k = \{e^{2i\pi\sqrt{-1}/k} \mid 0 \leq i \leq k-1\}$ for $k = 1, 2, \ldots,$ where $e^{\theta\sqrt{-1}}$ means the rotation of angle θ. We consider the coverings under the action of the group $G_k = T \rtimes Z_k$. To the authors' knowledge, there is no previous work on the universal covering problem under G_k.

We show that the smallest-area convex G_2-covering of S_{c} is the equilateral triangle $\triangle_1$ of height 1, whose area is $\frac{\sqrt{3}}{3} \approx 0.577$. A nice feature is that the proof is purely geometric and elementary. As a corollary, $\triangle_1$ is the smallest T-contaier of the set of all centrary-symmetric closed curve of length 2. This implies and includes the aforementioned fact that $\triangle_1$ is the smallest-area convex T-covering of unit-length curves, since a unit-length curve and its rotated copy by π can be concatinated to be a centrary-symmetric closed curve of length 2.

Then, we show that the equilateral triangle with height $\beta = \cos(\pi/12) \approx 0.966$ is a G_4-covering of S_{c}. Its area is $\frac{2\sqrt{3}+3}{12} \approx 0.538675$, and we conjecture that it is the smallest-area convex G_4-covering. It is a pleasant surprise that the equilateral triangular covering becomes the optimal convex covering if we consider rotation by π (G_2-covering), and it also seems to be true if we consider rotations by $\pi/2$ (G_4-covering).

However, the minimum area convex G_3-covering is no longer an equilateral triangle. We give a convex G_3-covering of S_{c} which has area not larger than 0.568. Unlike the G_2- and G_4-coverings, the G_3-covering is not *regular* under rotation. Moreover we can shave its portion of an area 0.005 to have a smaller convex G_3-covering of triangles of perimeter 2, which is conjectured to be also a G_3-covering of S_{c}.

We also determine the set of all smallest-area convex G_k-coverings of S_{seg}, which are all triangles.

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A Family of Convex Polyhedra Generated by Regular Polylinks

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Abstract

We introduce a family of polyhedra generated by regular polylinks such that the edges of a polyhedron are edges of regular polygons, and the vertices of the polyhedron are joints between those polygons. We enumerate all possible polyhedra and introduce a criterion to verify whether a given polyhedron satisfies the assumptions.

1 Introduction

Inspired by the traditional lantern in northern Thailand (Kome), the lantern's shape is recognized as a rhombicuboctahedron. Based on the manual for composing this kind of lantern, the rhombicuboctahedron is composed of the (congruent regular) octagons by arranging the octagons in different directions such that the edges of the rhombicuboctahderon are derived from octagons, and vertices of rhombicuboctahedron are joints of octagon vertices as shown in Figure 1.

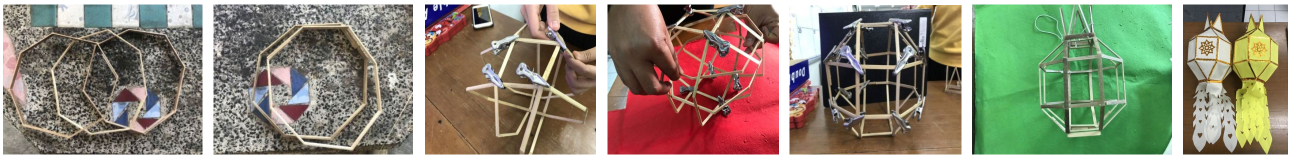

Figure 1: The processes to compose a traditional lantern (The figures are captured from the manual published by the social welfare division of Lamphun municipal office, Thailand.)

From the mathematical viewpoint, it is interesting to investigate the existence, necessary and sufficient conditions of convex polyhedra, which faces are regular polygons, and a polyhedron is composed of congruent regular polygons. Holden [1] considered a similar problem by investigating the polylinks, but the condition for composing to be a polyhedron has not been proposed yet. The studies of [5] proposed models of polyhedral links considered by the method of cross-curve and single-line covering, which shows some results on Archimedean polyhedra.

In this study, we focus on the enumeration problem of vertices and edges of polyhedra to consider all possible candidates. After that, we introduce a criterion to verify whether a given polyhedron satisfies the assumptions.

2 Modeling Assumptions

We focus on a polyhedron $\mathcal{P}$ whose faces are regular polygons, which are not necessary to be the same. Based on the study of Johnson [1], [2], and the summarization in [4], there are no more than 92 convex polyhedra with regular faces, in addition to Platonic solids, Archimedean solids, prisms, and antiprisms. In this investigation, we consider the following assumptions:

1. P is composed by k n-gons;
2. each vertex of $\mathcal{P}$ is joint by vertices of P;
3. all edges of $\mathcal{P}$ are all edges of P;
4. each edge is not overlapped except its endpoint, and no edge is laid inside a polyhedron;
5. those n-gons cannot be faces of $\mathcal{P}$.

3 Enumeration on the number of vertices and edges of polyhedra

Assumptions (1), and (2) implies that each vertex of a polyhedron $\mathcal{P}$ has an even degree of at least 4. Also, the number of edges E of a polyhedron $\mathcal{P}$ is equal to kn; i.e., $E = kn$.

For the base case, the octahedron is the only polyhedron that can be composed of three squares. Suppose that the first square is oriented on the plane. To compose other squares with the first oriented square, the diameter of other squares should be laid on the diameters of the first square only. Otherwise, the edges of squares cannot be the edges of the octahedron. In this case, $n = 4$ and $k = 3$.

Let V and F be the number of vertices and faces of polyhedra, respectively. Using the Euler theorem on polyhedra, $V - E + F = 2$, we obtain the simple condition $kn = V + F - 2$ with assumptions $n > 4$ and $k \geq 3$. In addition, the summation of the degree of vertices, $\sum \deg$ is at least $4V$, i.e. $\sum \deg \geq 4V$. Using handshaking lemma with the conditions of polygon edges, we obtain the inequality

$$\frac{19}{2} \leq \frac{kn}{2} + 2 \leq F.$$

Based on the information of polyhedra in [1] with the considered conditions, there are 29 polyhedra each vertex of which has an even degree, and the number of edges is a composite number. Remark that the polyhedra satisfying the assumptions (1) – (5) have vertices of degree 4.

4 A criterion to verify whether a polyhedron satisfies those properties

We would verify whether a given polyhedron $\mathcal{P}$ enumerated in Section 3 satisfies the assumptions mentioned in Section 2. Since the information on polyhedra in Section 3, such as the coordinate of vertices and the connection between polyhedron vertices, is clearly considered, we assume to use that information in our verification.

Let v be an arbitrary vertex of a given polyhedron $\mathcal{P}$ such that v connects $v_1, ..., v_4$ and $v_1, ...v_4$ are arranged in clockwise (with respect to a projection). We choose two set of vertices v_1, v, v_3 and v_2, v, v_4. Hence, there are two planes passing through v_1, v, v_3 and v_2, v, v_4. (The choice of those group of points are guaranteed by assumption (5).) We then compute the angle $\angle v_1vv_3$ and $\angle v_2vv_4$. If $\angle v_1vv_3 \neq \angle v_2vv_4$, then the candidate is fail. Otherwise, assume $\angle v_1vv_3 = \theta = \angle v_2vv_4$. We firstly check the consistent from the equation $E = kn$ such that $n = 360°/(180° - \theta)$ with the information of [1]. The candidate fails if it does not satisfy the condition $E = kn$. If it satisfies the condition, we will find a regular polylink that specifies the plane of the polygon by considering the next vertex in the direction of the previously chosen vertex and checking the coplanarity of those points. Therefore, we finally obtain polylinks of the given polyhedron by continuing to check the remaining vertices of the polyhedron.

Figure 2 shows examples of the polyhedra satisfying these conditions: octahedron ($n = 4, k = 3$), cuboctahedron ($n = 6, k = 4$), rhombicuboctahedron ($n = 8, k = 6$), and icosidodecahedron ($n = 10, k = 5$). The different colors of edges shows the polygons in the different direction.

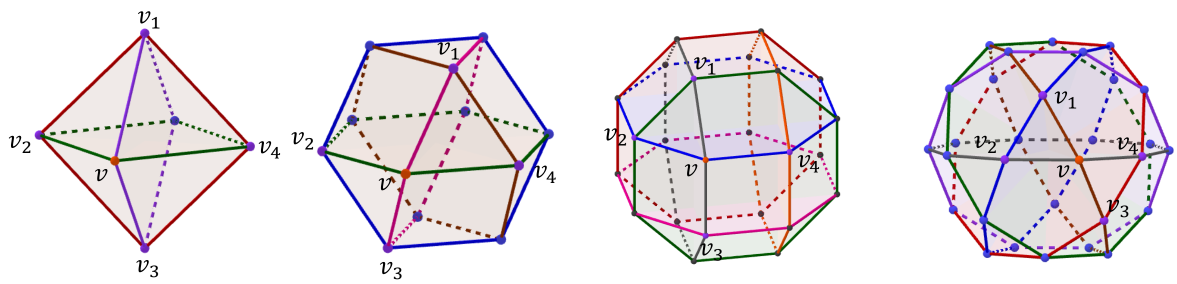

Figure 2: Examples of the polyhedra satisfying the assumptions by a proposed criterion: octahedron, cuboctahedron, rhombicuboctahedron, and icosidodecahedron

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Degree factors with red-blue coloring of regular graphs

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We consider graphs which have neither loops nor multiple edges. A graph G is called an r-regular graph if every vertex of G has degree r. A 3-regular graph is often called a *cubic graph*. A graph that has no induced subgraph isomorphic to $K_{1,3}$ is called a *claw-free graph*. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of a subgraph H of G, we denote by $\deg_H(v)$ the degree of v in H. For a set $\mathbb{S}$ of integers, a spanning subgraph F of G is called an $\mathbb{S}$-*factor* of G if $\deg_F(v) \in \mathbb{S}$ for every vertex v of G. In particular, an $\{a, b\}$-factor F of G satisfies $\deg_F(v) = a$ or b for every vertex v of G. A $\{k\}$-factor is briefly called a k-*factor*.

We begin with some known results on degree-factors of regular graphs.

Theorem 1 (Petersen, Theorem 3.4 of [1]) *Let r be a positive integer, and let G be a $2r$-regular graph. Then $E(G)$ can be decomposed into r 2-factors of G. In particular, for every integer $k, 1 \le k < r$, G has a $2k$-factor.*

Theorem 2 *Let r and k be positive integers such that $1 \le k \le 2r + 1$, and let G be a connected $(2r + 1)$-regular graph. Then the following statements hold.*

(i) G has a $\{k - 1, k\}$-factor (Tutte [6], Theorem 3.4 of [1]).

(ii) For a maximal independent vertex set W of G, G has a $\{k-1, k\}$-factor F such that $\deg_F(x) = k-1$ for all $x \in V(G)-W$, as well as a $\{k-1, k\}$-factor H such that $\deg_H(y) = k$ for all $y \in V(G)-W$ (Egawa and Kano [2], Theorem 4.23 of [1]).

(iii) If $1 \le k \le 2(2r + 1)/3$, then G has a $\{k - 1, k\}$-factor, each of whose components is regular (Kano [5], Theorem 4.37 of [1]).

Let $\mathbb{S}_R$ and $\mathbb{S}_B$ be two sets of integers. We color every vertex of G red or blue. Then a spanning subgraph F of G is called a *two-tone* $(\mathbb{S}_R, \mathbb{S}_B)$-*factor* of G if $\deg_F(x) \in \mathbb{S}_R$ for every red vertex x and $\deg_F(y) \in \mathbb{S}_B$ for every blue vertex y.

In this paper, we prove the following theorems.

Theorem 3 ([3]) *Let G be a connected claw-free graph. We arbitrarily color every vertex of G red or blue so that the number of red vertices is even. Then G has a two-tone $(\{1\}, \{0, 2\})$-factor.*

The above theorem can be proved by using the following theorem, where $\omega(G - S)$ denotes the number of components of $G - S$.

Theorem 4 (Lu and Kano [4]) *A connected graph G satisfies*

$$\omega(G - S) \leq |S| + 1 \quad \textit{for all} \quad S \subseteq V(G)$$

if and only if for any red-blue coloring of $V(G)$ such that the number of red vertices is even, G has a two-tone $(\{1\}, \{0, 2\})$-factor.

Note that a graph G has a two-tone $(\{1\}, \{0, 2\})$-factor if and only if G has vertex-disjoint paths whose end-vertices are exactly the same as the red vertices of G, in particular, all the inner-vertices of the paths are blue.

Theorem 5 ([3]) *Let G be a 3-edge connected claw-free cubic graph. We arbitrarily color every vertex of G red or blue so that the number of red vertices is even and the distance between any two red vertices is at least 3. Then G has a two-tone $(\{1\}, \{2\})$-factor and a two-tone $(\{2\}, \{1\})$-factor.*

Theorem 6 *Let $\lambda \geq 2, k \geq 1$ and $r \geq 3$ be integers, and let G be an λ-edge connected r-regular graph. Assume that $r/\lambda \leq k \leq r(1 - 1/\lambda)$. We arbitrarily color every vertex of G red or blue so that the number of red vertices is even if k is odd, otherwise the number of blue vertices is odd. Then G has a two-tone $(\{k\}, \{k - 1, k + 1\})$-factor.*

Theorem 7 *Let a, c and r be positive integers such that $1 \leq a \leq r$ and $1 \leq c \leq r - 1$. Let G be a connected r-regular graph. We arbitrarily color every vertex red or blue so that the distance between any two red vertices is at least 3. Then G has a two-tone $(\{a\}, \{c, c + 1\})$-factor.*

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Algorithms for Burning Schedule Reconfiguration Problem on Path Forests

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Abstract. Graph burning is a mathematical model that represents the spread of an influence. In each round, the fire spreads to the neighbors of the burned vertices in the previous rounds, and one additional unburned vertex is selected and burned. The burning schedule problem is a decision problem whether the given graph is burned only using the given subset of vertices as burning sources. In this paper, we study the reconfiguration problem for burning schedule problems that is, given a graph, a set of burning sources, and two feasible solutions of the burning schedule problem, whether one of the solutions can be transformed into the other. We show that the burning schedule reconfiguration problem on path forests can be solved in polynomial time if there is one burning source in each path and also solved if there are two burning sources in fixed number of paths.

Keywords: Graph algorithm · Combinatorial reconfiguration · Graph burning.

1 Introduction

To represent the spread of an influence with the discrete passage of time, graph burning is introduced by Bonato et al [2]. Graph burning proceeds as follows. As an input, a simple undirected graph G is given. Time is expressed in integer rounds of 0 or greater. At the initial state, all vertices are unburned. In each subsequent round, the fire spreads to the neighbors of the burned vertices in the previous rounds, and one additional unburned vertex is selected and burned. If a vertex is burned, it remains burned until all vertices are burned. With the above setting, the minimum number of rounds required for all vertices on the graph to burn has been studied in various graph classes such as graph products [6] and path forest [3]. The decision problem whether the given graph is burned less than or equal number of rounds to the given integer is NP-complete when restricted to trees of maximum degree three [1], path forest or spider graph. On the other hand, when the number of arms in the spider graph or the number of components and the order of each component of path forest are fixed, there are polynomial time algorithms. [1].

The burning schedule problem (BSP) is the decision problem, that is, given a graph G and a set $B \subseteq V(G)$ as input, the output is whether the given graph is burned within $|B|$ rounds using the vertices in B as burning sources. BSP is known as NP-hard [7].

The reconfiguration problem for a problem is given the input of the problem and two feasible solutions as input and determines whether the feasible solutions can traverse to each other on some adjacent relation. Various reconfiguration problems have been studied, such as the independent set reconfiguration problem or the satisfiability reconfiguration[4, 5].

Now we consider the burning schedule reconfiguration problem (BSRP) as below. BSRP is given a graph G, the burning sources $B \subseteq V(G)$, and the feasible solutions s, t for BSP and determines whether s can traverse to t by the operation of swapping the order of two vertices in the schedule. In this setting, we show that BSRP is PSPACE-complete using a polynomial-time reduction from the 3-SAT Reconfiguration problem. This reduction is almost the same way as the NP-hardness proof of BSP[7].

In this paper, we study the computational complexity of BSRP with some restrictions. We show that BSRP can be solved in polynomial time if there is a burning source in each path and there are two burning sources in fixed number of paths.

Theorem 1. *Let G be a path forest and $B \subseteq V(G)$ be a set of burning sources that there is one burning source in each path. Given G, B and two feasible solutions of BSP, they are reconfigurable.*

Theorem 2. *Let G be a path forest and $B \subseteq V(G)$ be a set of burning sources that there are two burning source in fixed number of path. Given G, B and two feasible solutions of BSP, BSRP can be solved in polynomial time.*

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The set multipartite Ramsey numbers for Paths versus wheels

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Abstract

Given two graphs G_1 and G_2, the set multipartite Ramsey number $M_j(G_1, G_2) = t$ is the smallest integer such that every factorization of graph $K_{t\times j} := F_1 \oplus F_2$ satisfies the following condition: either F_1 contains G_1 as a subgraph or F_2 contains G_2 as a subgraph, where $F_1 \oplus F_2$ means $E(K_{t\times j}) = E(F_1) \cup E(F_2)$ with $E(F_1) \cap E(F_2) = \emptyset$ and $V(F_1) = V(F_2)$. In this paper, we establish exact value of the set multipartite Ramsey number $M_j(P_n, W_s)$ for all integers $j \geq 2$ where P_n is a path on n vertices and W_s is a wheel of order $s + 1$ vertices with $s \geq 3$, and $3 \leq n \leq 4$.

For any natural number $j \geq 2$, clearly $K_{2\times j}$ contains no W_s for each integer $s \geq 3$. The following theorem determines the exact values of $M_2(P_3, W_n)$ for $n \geq 3$,

- $M_2(P_3, W_n) = \begin{cases} 4 & \text{for } n = 3, \\ \lceil \frac{n+3}{2} \rceil & \text{for } n \geq 4. \end{cases}$
- $M_2(P_4, W_n) = \begin{cases} 5 & \text{for } 3 \leq n \leq 7, \\ \frac{n}{2} + 2 & \text{for } n \geq 8 \text{ even}, \\ \lceil \frac{n}{2} \rceil + 2 & \text{for } n \geq 9 \text{ odd and } n \equiv 0 \mod 3, \\ \frac{n+1}{2} + 1 & \text{for } n \geq 11 \text{ odd and } n \not\equiv 0 \mod 3. \end{cases}$

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Upper Bounds on the Minimum Number of Pieces for Anti-slide Packing

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Abstract. Given a box of some specified size and a number of pieces of some specified shape, the anti-slide problem considers how to pack the pieces such that none of the pieces in the box can slide. In this work, we focus on the sparsest anti-slide packing for $2 \times 2 \times 1$ pieces and L-tricube pieces. Firstly, we show that there is an anti-slide packing of $2 \times 2 \times 1$ pieces with volume density 0.48 when a box is torus. Secondly, we give a new construction of an anti-slide packing of L-tricubes.

1 Introduction

"*Anti-slide*" [2] is a puzzle that asks us to pack $2 \times 2 \times 1$ pieces into $4 \times 4 \times 4$ box in such a way that none of the pieces can slide. The objective is to find a packing with the smallest number of pieces. The puzzle was originally invented by William Strijbos in 1994. An optimal packing uses 12 pieces (Fig. 1).

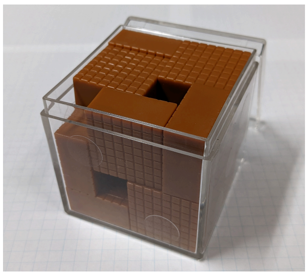

Fig. 1: A sparsest anti-slide packing for a $4 \times 4 \times 4$ box.

In this work, we assume that each piece is aligned so that every corner of the piece matches the coordinate of integers, and that each piece slides toward an orthogonal direction, i.e., parallel to x, y, or z-axes.

Knuth [3] gave a construction of an anti-slide packing of $2 \times 2 \times 1$ pieces whose volume density is $11/16 = 0.6875$ when the size of a box goes to infinity. Amano et al. [1] used an IP approach to show that the minimum density is between 0.288 and 2/3. In this work, we show that there is an anti-slide packing of density $12/25 = 0.48$ when a box is *torus*. We say that a packing for a torus is anti-slide if none of the pieces in the box can slide and all pieces of the packing move together as one block. We believe that the same bound can be achieved for a usual three-dimensional box. We will describe our construction in Sec. 3.

It is also quite natural to consider a general version of the problem where the pieces are not rectangular.

In [4], we investigated the asymptotic behavior of the minimum number of pieces in an anti-slide packing for an $n \times n \times n$ box for various shapes of *polycubes* (Fig. 2) and observed that these could be categorized into three groups. Namely, each polycube in Group A and B has a packing with $O(n)$ and $O(n^2)$ pieces, respectively. Note that a non-zero density lower bound shown for a piece in Group C implies that it needs $\Theta(n^3)$ pieces to pack without sliding. We *believe* that a piece in Group B needs $\Omega(n^2)$ pieces to pack.

In this work, we focus on the L-tricube, which would be the easiest shape to get a quadratic lower bound. We analyze anti-slide packings of L-tricubes in terms of projection. We also obtain a new construction of an anti-slide packing of L-tricubes. We will discuss these in Sec. 4.

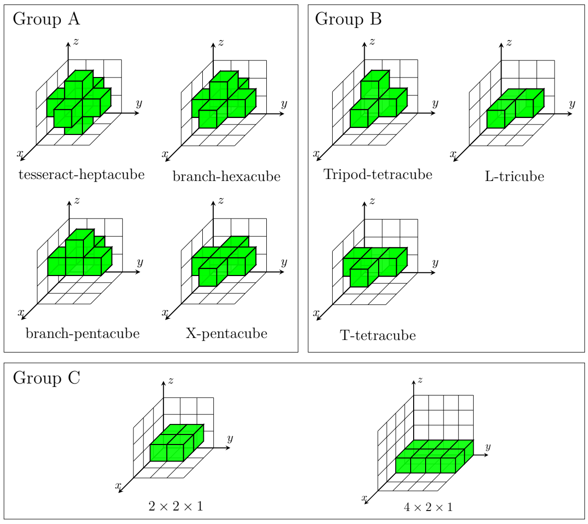

Fig. 2: Several shapes of pieces analyzed.

See e.g., [5, 6, 7] for some related works on the anti-slide problem.

2 Notations

For a positive integer n, $[n]$ denotes the set $\{0, 1, \ldots, n-1\}$. We view the $l \times m \times n$ box as a three-dimensional array of cells of unit size. Each cell is identified by $(i, j, k) \in [l] \times [m] \times [n]$. A piece is said to be placed at (i, j, k) if some specified cell of the piece is placed at the cell (i, j, k). By such a placement, several cells are occupied by the piece.

3 Density for $2 \times 2 \times 1$ Pieces

In this section, we assume that a box is torus.

We say that a three-dimensional array is a torus with size $l \times m \times n$ if $(i, j, k) \in \mathbb{N}^3$ is equivalent to $(i \bmod l, j \bmod m, k \bmod n)$ for every $(i, j, k) \in \mathbb{N}^3$. For example, if a $2 \times 2 \times 1$ piece is placed in the $4 \times 4 \times 4$ torus at $(3, 3, 3)$ horizontally, then the cells $(3, 3, 3)$, $(0, 3, 3)$, $(3, 0, 3)$, and $(0, 0, 3)$ are occupied by the piece. Let $R(n)$ denote the minimum ratio of the volume occupied by pieces in an anti-slide packing of $2 \times 2 \times 1$ pieces for the three-demensional torus of side length n.

We conducted a computer search for a sparse packing of a torus of small size. We use an IP solver based on the IP formulation developed in [1]. The sparsest packing obtained so far is for the $5\times5\times5$ torus and it uses 15 pieces (Fig. 3). The volume density is $4\cdot 15/5^3 = 12/25$. Since we can obtain an anti-slide packing for a larger torus by repeating its placement in the direction of x, y, and z-axes, we have the following bound on $R(n)$.

Theorem. $R(5k) \le 12/25$ *for every* $k \ge 1$.

We believe that we can remove the assumption that a box is torus without increasing the density by adjusting the placement around the boundary of the box using an extra, say $O(n^2)$ pieces.

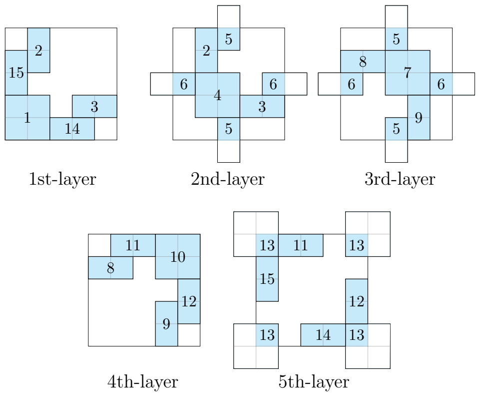

Fig. 3: A sparsest anti-slide packing of $2\times2\times1$ pieces for a $5\times5\times5$ torus. Each number represents a piece.

4 Minimum Number of L-tricubes

Let $F(n)$ be the minimum number of L-tricube pieces of an anti-slide packing for an $n\times n\times n$ box. We conducted computer searches based on the IP formulation developed in [1] to find a better upper bound on $F(n)$. We add some heuristically found constraints to the IP model, and obtain the values for $n \le 11$. The results are shown in Tab. 1, and the example of the packing for $n = 8$ is shown in Fig. 4. Our experimental results suggest that $F(n)$ tends to $n^2/2$ when n goes to infinity.

Tab. 1: Values of the upper bound on $F(n)$ for $n \le 11$.

n	4	5	6	7	8	9	10	11
$F(n) \le$	9	15	23	31	40	50	61	72

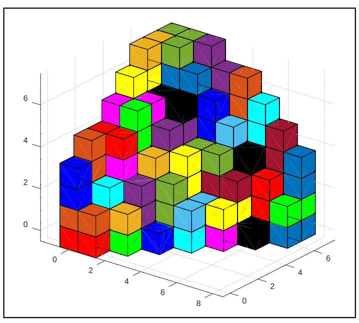

Fig. 4: A packing of L-tricube pieces for an $8\times8\times8$ box.

Next, we consider the lower bound on $F(n)$. For an anti-slide packing of L-tricubes, we say that a pixel on the xy-plane (resp., xz-plane and yz-plane) is *marked* if the orthogonal projection of the set of pieces in the packing onto the plane includes this pixel. Given an anti-slide packing, the *shadow size* of the packing is defined as the maximum number of marked pixels in a plane among xy, xz and yz -planes. Let $T(n)$ denote the minimum of a shadow size over all anti-slide packings of L-tricubes for an $n\times n\times n$ box. If one can show a quadratic lower bound on $T(n)$, then it would immediately imply $F(n) = \Omega(n^2)$.

We performed a computer experiment using an IP solver to find an anti-slide packing having a small value of $T(n)$. The results are shown in Tab. 2. The example of the packing for $n = 6$ is shown in Fig. 5 (a). Fig. 5 (b) shows the projection on the xz-plane given by this packing. The projections on the xy-plane and the yz-plane are similar. The result suggests that $T(n)$ would be quadratic in n. We conjecture that $F(n) = \Theta(n)$, or more ambitiously $F(n) \sim n^2/2$.

Tab. 2: Values of the upper bound on $T(n)$ for $n \le 8$.

n	4	5	6	7	8
$T(n) \le$	13	19	26	35	45

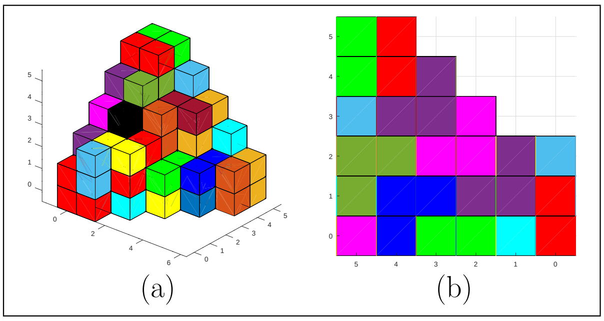

Fig. 5: A packing of L-tricube pieces for a $6\times6\times6$ box.

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Semi-restricted Rock, Paper, Scissors

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July 15, 2022

Abstract

Consider the following variant of Rock, Paper, Scissors (RPS) played by two players Rei and Norman. The game consists of $3n$ rounds of RPS, with the twist being that Rei (the restricted player) must use each of Rock, Paper, and Scissors exactly n times during the $3n$ rounds, while Norman is allowed to play normally without any restrictions. Answering a question of Spiro, we show that a certain greedy strategy is the unique optimal strategy for Rei in this game, and that Norman's expected score is $\Theta(\sqrt{n})$. Moreover, we study semi-restricted versions of general zero sum games and prove a number of results concerning their optimal strategies and expected scores, which in particular implies our results for semi-restricted RPS.

1 Introduction

The game Rock, Paper, Scissors, or RPS for short, consists of two players simultaneously selecting to play either Rock, Paper, or Scissors; where Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. If a player uses a move which beats their opponents move, then that player gains a point and their opponent loses a point, with the match resulting in a draw if both players select the same move.

RPS, while fun to play, is not particularly interesting from a mathematical perspective. As a symmetric zero-sum game, both players have an expected score of 0, and it is easy to show that the unique optimal strategy for both players is to choose each of the three options with uniform probability; see for example [3, Chapter 17]. In this paper we study a non-trivial variant of RPS, which is inspired by a game played in the manga Tobaku Mokushiroku: Kaiji [1] and of work by Spiro [2], called *semi-restricted RPS*.

Semi-restricted RPS is a perfect-information zero-sum game played by two players named Rei and Norman. The game consists of $3n$ rounds of RPS, with the twist being that Rei must use each of Rock, Paper, and Scissors exactly n times during the $3n$ rounds, while Norman is allowed to play normally without any restrictions. It is clear that Norman has an advantage in this game, since in particular he is guaranteed to win the last round of the game (assuming he is paying attention and has a good memory). However, it is unclear how much more Norman is expected to win over Rei when both play optimally, and it is also unclear what the optimal strategies are for either player.

One simple strategy that Rei can implement is what we call the *greedy strategy*. Under this strategy, if Rei can still play each of Rock, Paper, and Scissors, she selects each option with probability 1/3, regardless of how many actions remain of each option. If she can only play, say, Rock and Paper, then she chooses Paper with probability 2/3 and Rock with probability 1/3. More generally, if she has two options remaining, she will choose the option which beats the other with probability 2/3 and the other option with probability 1/3. And of course, if only one option remains, she plays this with probability 1.

This strategy is "greedy" because with this strategy, Rei minimizes her expected loss for any given round. Indeed, if she has all three options remaining, then playing each with probability 1/3 makes it so that, regardless of what Norman does, Rei is just as likely to win as she is to lose. Similarly if only Rock

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and Paper remains, then Norman should only ever play Paper or Scissors (since Rock is guaranteed not to win). If Rei follows the greedy strategy and Norman plays Paper, then Rei will lose with probability 1/3. If Norman plays Scissors, then Rei will lose with probability 2/3 but win with probability 1/3. Thus regardless of what Norman does, Rei will expect to lose 1/3 points each round under the greedy strategy, and one can show that this is best possible.

While the greedy strategy is optimal if Rei is only concerned about a given round, it is far from clear that this is a good strategy overall. Indeed, say the game reaches the point where Rei can play 100 Rocks, 100 Papers, and just 1 Scissors. Intuitively, in this scenario Rei should not play Scissors with high probability, as doing so will severely limit her remaining options for the rest of the game. As such the following result may come as a bit of a surprise.

Theorem 1.1. The greedy strategy is the unique optimal strategy for Rei in semi-restricted RPS. Moreover, if the game consists of $3n$ rounds with both players playing optimally, then Norman's expected score is $\Theta(\sqrt{n})$.

The problem of studying semi-restricted RPS was first proposed by Spiro [2]. In [2], a semi-restricted version of another classical zero-sum game, Matching Pennies, was studied, and again a certain greedy strategy turned out to be optimal. One can easily generalize these problems to study semi-restricted versions of arbitrary simultaneous zero-sum games. For simplicity, we will focus on games which come from digraphs, though many of our results hold in broader generality.

Given a digraph D, we define the *D-game* by having two players simultaneously select a vertex of D. If the selected vertices are u, v and $uv \in E(D)$, then the player who chose u gains a point and the player who chose v loses a point, and nothing happens if $uv, vu \notin E(D)$. For example, if D is the circuit of length 3, i.e. the 3-vertex digraph with arcs $1 \to 2 \to 3 \to 1$, then the D-game is equivalent to RPS. Observe that the D-game is always a symmetric zero-sum game.

We will say that a vector r is a *restriction vector* (with respect to a digraph D) if it is a vector of non-negative integers indexed by $V(D)$. Given a digraph D and restriction vector r, we define the *semi-restricted D-game with parameter r* by having two players Rei and Norman iteratively play the D-game for a total of $\sum_u r_u$ rounds such that Rei must select each $u \in V(D)$ exactly r_u times. We let $S_D(r)$ denote the expected score for Norman when both players play optimally in the semi-restricted D game with parameter r, and throughout we let $\mathbf{1}$ denote the all 1's vector of dimension $|V(D)|$. For example, Theorem 1.1 says $S_D(n \cdot \mathbf{1}) = \Theta(\sqrt{n})$ where D is the circuit of length 3.

Determining optimal strategies for semi-restricted D-games in full generality seems impossible. Nevertheless, we are able to obtain effective bounds on $S_D(r)$ for all D. For example, a basic observation is that for all vertices $v \in V(D)$, we have

$$S_D(n \cdot \mathbf{1}) \geq (d^+(v) - d^-(v))n.$$

Indeed, if Norman uses the deterministic strategy of playing v every round, then he will win exactly $d^+(v)n$ rounds and lose exactly $d^-(v)n$ rounds, so he can achieve an expected score of at least $(d^+(v) - d^-(v))n$ with this strategy. It turns out that this trivial lower bound is close to best possible.

Theorem 1.2. For all digraphs D and $n \geq 1$, we have

$$\max_v\{d^+(v) - d^-(v)\}n \leq S_D(n \cdot \mathbf{1}) \leq \max_v\{d^+(v) - d^-(v)\}n + O_D(\sqrt{n}).$$

More generally we can prove bounds for $S_D(r)$ with r arbitrary, as well as effective bounds when D is an Eulerian tournament by using techniques from linear algebra.

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Extended Abstract
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Discrete Richman-bidding Partizan Games

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Abstract

Combinatorial Game Theory is a branch of mathematics and theoretical computer science that studies sequential 2-player games with perfect information. Partizan subtraction games are combinatorial games where two players, say Left and Right, alternately remove a number of objects from a heap of objects, based on the different sets of subtraction sets given to them. The first player unable to move loses. These games were studied by Fraenkel and Kotzig in 1987 and their properties were further studied by Duchêne et al. in 2021. Here we generalize the alternating normal play to a discrete bidding scheme, similar to Develin et al. Several properties of such bidding games related to different subtraction sets of Left and Right are defined, namely *Strongly dominating*, *Weakly dominating* and *Fair*. As the main result, we prove that for all sufficiently large heap sizes, the equilibrium outcomes of bidding partizan games are eventually periodic.

Keywords: Discrete bidding, Combinatorial games, Partizan, Fairness.

1 Introduction

Partizan subtraction games were introduced by Fraenkel and Kotzig in 1987 [1]. There are two players, say Left and Right, who play combinatorial games on a heap of items. The players are given a different combination of subtraction sets. They remove items based on their subtraction sets and the player who removes the last object wins the game. In other words, the player who cannot move loses. A combinatorial game belongs to one of the following four outcome classes.

1. $\mathcal{L}$eft regardless of who moves first.
2. $\mathcal{R}$ight regardless of who moves first.
3. the $\mathcal{N}$ext player whether it is left or right.
4. the $\mathcal{P}$revious player whether it is left or right.

We call $\mathcal{L}$, $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{P}$ as the outcome classes. Games in the above classes are said to be in $\mathcal{L}$, $\mathcal{R}$, $\mathcal{N}$, and $\mathcal{P}$ positions. We have the following theorem for partizan subtraction games in Fraenkel and Kotzig [1]. The *outcome sequence* of G is the sequence of the outcomes for n = 0, 1, 2, 3, $\cdots$, i.e. $\widehat{o}(0), \widehat{o}(1), \cdots$.

Theorem 1.1. *The outcome sequence of any partizan subtraction game is ultimately periodic.*

Example 1.2. Consider the subtraction sets {1,4} and {2,3} for Left and Right, respectively. The outcome sequence of the game is:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	.	.
L	**N**	**R**	**L**	**N**	**R**	**P**	**N**	**R**	**R**	**N**	**R**	**R**	**R**	**R**	**R**

2 Our Game and Results

Here we introduce the bidding variant of the *Partizan* subtraction games. Let $\mathbb{N} = \{1, 2, \ldots\}$, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Discrete bidding here means that there is a given total budget of $\text{TB} \in \mathbb{N}_0$. One of the players has a budget $p \in \{0, \ldots, \text{TB}\}$, and the other player has the budget $q = \text{TB} - p$. At each turn, players submit a (closed) bid. The player with the higher bid gets to play, and pays their bid to the player with the lower bid. Thus, if player Left wins by bidding ℓ, she pays this bid to Right, and makes her desired move. The new budget partition becomes $(p - \ell, q + \ell)$. Ties are resolved using a *tie-breaking marker*: one of the players has the marker, and this player wins the turn in case of equal bids. They make their desired move, pay the other player the bidding amount, and pass them the marker. Similar to [3], we study pure bidding strategies, since this is closer to recreational play, in the tradition of combinatorial game studies.

Example 2.1. Consider the subtraction sets {1,4} and {2,3} for Left and Right, respectively. The outcome sequence of the Discrete Bidding partizan subtraction game for TB = 1 is shown in Table 1.

Definition 2.2. We say that a game converges at heap size x if, $\widehat{o}(x+i) = \widehat{o}(x+p+i)$, $\forall i = 0, 1, \cdots, p-1$, where p is defined as a period. Also, we say that the rows from x to $x+p-1$ form a convergence table.

We have a *Strongly Dominating Game* w.r.t. Left (Right) if the period of the game is reduced only to L (R). We call a game *Weakly Dominating* w.r.t. Left (Right) if the period of the game contains more L than R (more R than L). The game is *Fair* if the period contains an equal number of L and R.

We got some properties for Discrete Richman bidding partizan games for TB = 1:

Lemma 2.3. *The period of the Bidding partizan subtraction game for S_L = a and S_R = b is a+b.*

Lemma 2.4. *The ratio of L and R in the Bidding partizan subtraction game for S_L = a and S_R = b is the same as the ratio of b and a.*

Lemma 2.5. *The Bidding partizan subtraction game for S_L = a and S_R = b is always Weakly dominating w.r.t. Left if a < b, and vice versa.*

Above three lemmas are the full results for the case when Left's and Right's subtraction sets contain exactly one element. We looked for different combinations of subtraction sets when they contain more than one element. For example,

Lemma 2.6. *The Bidding partizan subtraction game for S_L = {a,b} and S_R = {b+1,c} where 'a < c' is always fair.*

Table 1: The equilibrium outcomes for TB = 1 and $S_L = \{1, 4\}$ and $S_R = \{2, 3\}$.

x, p	$\dot{L}1$	$\dot{L}0$	$\dot{R}1$	$\dot{R}0$
0	R	R	L	L
1	L	L	L	L
2	L	R	L	R
3	L	R	R	R
4	L	L	L	R
5	L	L	L	R
6	L	R	L	R
7	L	R	L	L
8	L	L	L	R
9	L	R	L	R
10	L	R	L	R
11	L	L	L	R
12	L	R	L	R
13	L	R	L	R
14	L	R	L	R
15	L	R	L	R

The main theorem of this work is the result regarding convergence. For any set of subtraction sets for the players, the equilibrium outcome always converges.

Theorem 2.7. *The outcome sequence of any Discrete richman bidding partizan games is ultimately periodic.*

3 Future Work

We hope to find the rate of convergence of the Discrete richman bidding partizan games. This work is carried out at the Multi-agent Laboratory, Kyushu University, Japan; which is headed by Prof. Makoto Yokoo.

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Extended Abstract

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Motion Bounded Maximum Volumes: An Optimization Problem Applied to Kaleidocycle Shapes

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Geometric optimization problems result in fascinating shapes appearing in science, technology, and design. Classical examples are minimal surfaces which are useful as well as esthetically pleasing and are used in architectural design today. Shape optimization in engineering is another example where shapes are derived based on given boundary conditions (eg., structural loads) and constraints (eg., limited supply of material).

We propose a new shape optimization problem where the only input is the motion of a collection of rigid bodies. The objective is to find the shapes with the maximum volume still allowing the bodies to perform the prescribed motion without collision. We will present first results of a currently developed optimization algorithm and will also address interesting aspects of the problem, such as uniqueness, local maxima, convexity, and additional boundary conditions.

Shapes bounded only by motion

We prescribe the time evolution (motion) of the reference points and orientations of N rigid bodies $B_1, \ldots, B_N$ of *identical* (initially unknown) shape $\mathcal{S}$. The objective is to find the shape $\mathcal{S}_{\mathsf{max}}$ with the maximum volume V_{max}, such that no pair of bodies $P_{ij} = \{B_i, B_j\},\ i, j = 1, \ldots, N,\ i \neq j$ overlaps at any time. No rolling or gearing condition is prescribed, the surfaces of any pair P_{ij} may contact and glide freely on each other.

For V_{max} to be bounded, the prescribed motion is subject to certain conditions which are problem specific and hard to define in general. But we can give one example of sufficient (but not necessary) conditions as follows. The bodies $B_1, \ldots, B_N$ with reference points $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ must form a ring such that each body i has two neighbors $i \pm 1 \pmod N$ whose distance is bounded $|\boldsymbol{r}_i - \boldsymbol{r}_{i\pm1}| < C$ with some positive constant C. Further, the body orientations must describe a periodic everting motion, meaning that the ring turns continuously inside out. The everting motion ensures that every body has to “pass through” the finitely sized ring and is therefore bounded. The motion of a kaleidocycle in the example below fulfills the described conditions.

Example: optimizing the volume of a six-hinged kaleidocycle

A classical six-hinged kaleidocycle (K6) is a ring made of six identical tetrahedra with opposing edges used as hinges. Each tetrahedron corresponds to the identical shape $\mathcal{S}$. A K6 has one internal degree of freedom, it can perform one specific internal motion. For a K6 to be able to evert (turn inside out continuously), the hinges can be at most $2/\sqrt{5} \approx 0.8944$ times as long as the other four edges in each tetrahedron. In the case of the maximum hinge length, all four faces of each tetrahedron contact with faces of their corresponding neighboring tetrahedra at certain times during the motion, see Figure 1 (left column, colors red and blue are used to distinguish the six tetrahedra more easily). This contact might give the impression that the volume of $\mathcal{S}$ is already maximized. But on the contrary, the volume can be increased considerably, as can be seen from a two-step construction. First, each tetrahedron is cut into two identical smaller tetrahedra, one of them remains in its position while the other one is rejoined at another face, resulting in hexahedral shapes, see Figure 1 (middle column). Second, each hexahedron is extended by two additional polyhedra (colored yellow for the red hexahedra and cyan for the blue hexahedra), leading to decahedral shapes, see Figure 1 (right column). This increases the volume of the shapes by approximately 60% compared to the initial tetrahedra (left column), while still allowing the object to evert without collisions. This can be considered as the first iteration of a volume maximization procedure for $\mathcal{S}$. The shape with maximized volume $\mathcal{S}_{\mathsf{max}}$ will be highly nontrivial.

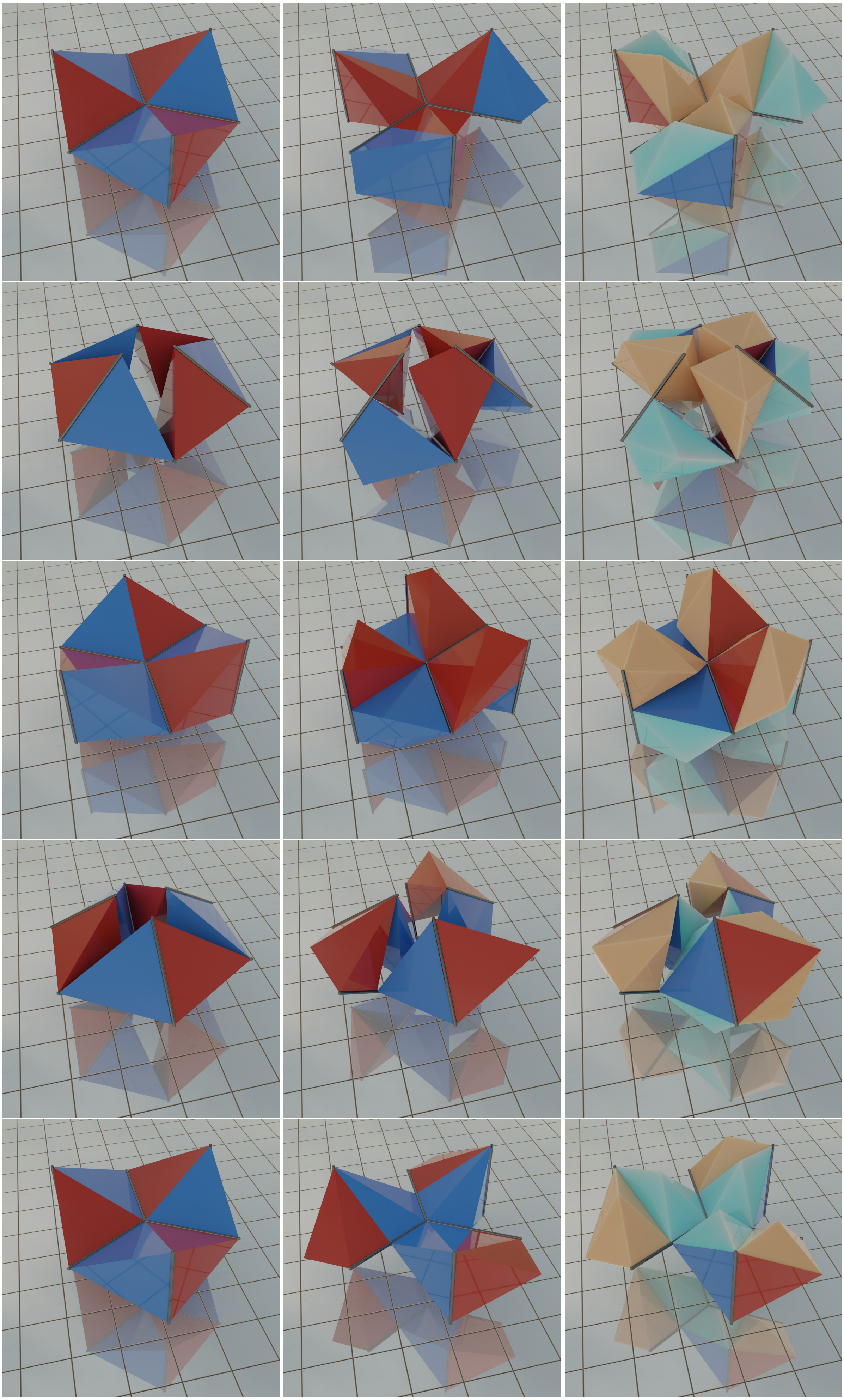

Figure 1: An everting motion sequence (from top to bottom) of a 6-hinged kaleidocycle. Left: classical tetrahedral shapes. Middle: hexahedral shapes obtained through cutting and rejoining of tetrahedra. Right: decahedral shapes obtained through yellow and cyan extensions of the red and blue base hexahedra. Please see the final section of the main text for details.

Largest convex hulls for constant size, convex-hull disjoint clusters

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Abstract

A *cluster* is a set of points, with a predefined similarity measure. In this paper, we study the problem of computing the largest possible convex hulls, measured by length and by area, of the points that are selected from a set of *convex-hull disjoint* clusters, one per cluster. We show that the largest convex hulls for convex-hull disjoint cluster of *constant size*, measured by length or area, can be computed in $O(n^4)$ time, where n is the sum of cardinalities of all clusters. Our solution of either considered problem is *doubly* founded on a structure of clusters, whose all points are in convex position. The restricted problem for the set of clusters, whose points are in convex position, can be reduced to a sequence of subproblems of computing the single-source shortest-paths in a weighted graph. Not only our results significantly improve upon the known time bound $O(n^9)$, but also the obtained solutions are unified and simple. Moreover, our algorithms can be used to improve the known results on several other variants of the considered problem.

Keywords: Computational geometry; Convex hull; Clusters; Imprecise points single-source shortest-paths problem

All Paths Lead to Rome

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The field of computational complexity of games and computer games is a broad vivid field as it also allows a playful entry to the field of computational complexity theory, see for instance the surveys by Demaine et al. [2] and Kendall *et al.* [7]. The rich history and impact of research on puzzle games is also reflected in a huge number of publications at different international conferences over decades, such as in the conferences JCDCG^3 [1] and FUN [4].

All roads lead to Rome is the core idea of the Japanese puzzle game *Roma* (ろーま), see [6] for a playable version of the game. It is played on an $n \times n$ grid consisting of quadratic cells. Those cells are grouped into boxes of at most four neighboring cells and are either filled, or to be filled, with arrows pointing in cardinal directions. The goal of the game is to fill the empty cells with arrows such that each box contains at most one arrow of each direction and regardless where we start, if we follow the arrows in the cells, we will always end up in the special Roma-cell. We study the computational complexity of the puzzle game Roma and show that completing a Roma board according to the rules is an NP-complete task, counting the number of valid completions is #P-complete, and determining the number of preset arrows needed to make the instance *uniquely* solvable is Σ_2^P-complete. We further show that the problem of completing a given Roma instance on an $n \times n$ board cannot be solved in time $\mathcal{O}\left(2^{o(n)}\right)$ under ETH and give a matching algorithm.

A full version of this article can be found in [5].

The Rules of Roma Roma is a one-person puzzle game. A Roma board consists of a quadratic game board, which in turn consists of $n \times n$ quadratic individual cells. One of these cells is a previously determined *Roma-cell* which serves as a target cell. Cells which directly border on each other are called *true neighbors*. Cells which are true neighbors can be gathered in a collection called a *box*. These boxes can consist of 1 to 4 cells. The boxes are preset at the beginning of a game and every cell is contained in exactly one box. The boxes can take any form, as long as every cell within a box can be reached from any other cell within that box by only traversing other cells from the same box, where traversal refers to single-cell steps from one cell to one of its true neighbors. The Roma-cell is always contained in its own 1-box. Each empty cell must be filled by the player with an arrow, pointing in one of the four cardinal directions. Cells can contain preset arrows before the game starts. Each box can contain only one arrow pointing in a given cardinal direction. The goal of the game is to fill each cell such that, beginning in any cell, following the arrows will always lead to the Roma-cell. An example game board may look as follows.

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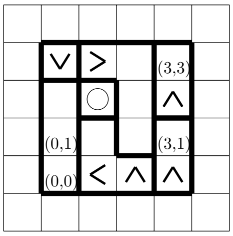

Example of a 4×4 Roma game board where ○ marks the target Roma-cell. The indices explain how cells are addressed on the board. Boxes are defined by drawing thicker boundaries. A typical reasoning is: Consider cell $(3, 1)$. We cannot leave the board, which excludes $\boxed{>}$. The preset 2-box excludes $\boxed{\vee}$, $\boxed{\wedge}$. Hence, we fill cell $(3, 1)$ with $\boxed{<}$. Similarly, $(3, 3)$ gets $\boxed{<}$, which implies $(2, 3)$ getting $\boxed{\vee}$, etc.

Our contribution. We show that the question whether a partially filled instance of Roma can be completed according to the rules of Roma is an NP-complete problem by a reduction from Planar 3SAT. As this reduction is parsimonious, we directly get that the counting variant of Roma, counting the number of solutions, is #P-complete. The parsimonious reduction further implies that the question of how many hints must be added to a Roma instance in order to make it *uniquely* solvable is Σ_2^P-complete. Questions of these types have been introduced under the name *fewest clue problem* in [3] and are of importance when it comes to design *good* and enjoyable puzzles for humans.

We continue our study of the complexity of Roma by giving matching upper and lower bounds. We show that the reduction by Lichtenstein from 3SAT to Planar 3SAT can be translated into our Roma setting with only a constant factor increase in space. Especially, we have that the number of variables and the number of clauses each correspond to the dimension n of an $n \times n$ Roma board and hence, assuming ETH, Roma cannot be solved in time $\mathcal{O}\left(2^{o(n)}\right)$. As our second main result, we match this lower bound by a dynamic programming algorithm, using the idea of Catalan structures.

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Find Routes on a Doughnut

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Crossing Minimization Given a graph and a condition we wish to draw the graph with the minimum number of edge crossings satisfying the given condition. Such problems are called *crossing minimization problems*.

In general, to draw a given graph with the minimum number of edge crossings is $\mathcal{NP}$-hard [3].

Given a bipartite graph $(U \cup V, E)$ and two parallel straight lines it is $\mathcal{NP}$-hard [3] to draw the bipartite graph with the minimum number of edge crossing so that the vertices in U are on a straight line and the vertices in V are on the other straight line. The problem is $\mathcal{NP}$-hard [2] even if the ordering of the vertices in U is given.

Given a bipartite graph $(U \cup V, E)$ and two concentric circles and the cyclic ordering of the vertices in U it is $\mathcal{NP}$-hard [1] to draw the bipartite graph with the minimum number of edge crossing so that the vertices in U are on a circle with the given cyclic ordering and the vertices in V are on the other circle with any ordering.

Our Results In this paper we consider a very restricted version of the problem. Given a matching $M = (U \cup V, E)$ as a bipartite graph, two concentric circles, the cyclic ordering of the vertices in U and the cyclic ordering of the vertices in V, we wish to draw M with the minimum number of edge crossings so that the vertices in U are on the smaller circle with the given cyclic ordering and the vertices in V are on the larger circle with the given cyclic ordering. We call the problem *the doughnut routing problem*. We design an $\mathrm{O}(n^3)$ time algorithm to solve the problem. The main idea of the algorithm is a reduction to a set of the minimum generator sequence problem.

Preliminaries For the doughnut routing problem let n be $|E|$ and we can assume that the cyclic ordering for U is $(1, 2, \cdots, n)$. (Otherwise we can rename them.) So the input has one permutation (for V).

Given a set of generators $\pi_1, \pi_2, \cdots$ of a permutation group G, a target permutation $P = (p(1), p(2), \cdots)$ in G, the *minimum generator sequence problem* ask to find a sequence of the generators with the minimum length so that the composition of them to the identity permutation $(1, 2, \cdots)$ results in P.

If the set of the generators consists of all cyclically adjacent transpositions, that is $(1, 2), (2, 3), \cdots, (n-1), (n, 1)$, then one can solve the problem in $\mathrm{O}(n^2)$ time [4]. In this paper we only consider this set of generators.

We have the following observations.

Observation 1 If the minimum generator sequence problem with the permutaion P has a solution with the length of the sequence k then there is a drawing with k edge crossings so that the vertices in U are on the smaller circle with the cyclic ordering $(1, 2, 3, \cdots, n)$ and the vertices in V are on the larger circle with the given cyclic ordering P.

Observation 2 If the doughnut routing problem with the cyclic permutation P has a solution with k edge crossings then there is a minimum generator sequence problem with a permutation P' (which can be derived from the cyclic permutation P) having a solution with the length of the sequence k.

Also we can compute P' as follows. We regard the direction of each edge in the drawing from the vertex on the smaller circle to the vertex on the larger cycle. For the edge of 1, let L be the number of edges crossing with the edge of 1 from left to right, and R be the number of edges crossing with the edge of 1 from right to left. P' is the permutation derived from the cyclic permutation P so that 1 appears at the $(R - L)$-th element.

Algorithmn Our algorithm to solve the doughnut routing problem is as follows. The input is a cyclic permutation P. We first choose an arbitrary permutation P' derived from a cyclic permutation P.

Then compute the minimum generator sequence problem with the permutation P_i, for $i = 0, 1, 2, \cdots, n-1$ (so in total n problems), where P_i is the permutation derived from the permutation P' by i cyclic shifts (so it corresponds to the cyclic permutation P). Choose the one with the solution of the minimum length of the sequence. Compute the corresponding (doughnut routing type) drawing to the solution of the (selected) minimum generator sequence problem. The derived drawing is the solution of the doughnut routing problem.

Since we solve the minimum generator sequence problem n times and we can solve each problem in $O(n^2)$ time [4], the running time of above algorithm is $O(n^3)$.

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New Formulation for Coloring Circle Graphs

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A circle graph is a graph in which the adjacency of vertices can be represented as the intersection of chords of a circle. The problem of calculating the chromatic number is known to be NP-complete, even on circle graphs. In this paper, we propose a new integer linear programming formulation for a coloring problem on circle graphs.

It is well known that circle graphs and overlap graphs are of the same class. A graph is an *overlap graph* if its vertices are intervals on a line such that two vertices are adjacent if and only if the corresponding intervals partially overlap (that is, they have non-empty intersection), but neither contains the other. Hereinafter, we assume that an input of a given circle graph $G = (V, E)$ is a corresponding *interval representation* $\mathcal{I}(G) = \{I(j) \subseteq \mathbb{R} \mid j \in V\}$, where $I : j \mapsto [l_j, r_j]$. We also assume that all the terminal points of intervals in $\mathcal{I}(G)$ are mutually distinct.

Given an interval representation $\mathcal{I}(G)$ of a circle graph $G = (V, E)$, we introduce a partial order $\preceq$ defined on the vertex set V. For any pair of vertices $i, j \in V$, we define $i \preceq j$ if and only if either $i = j$ or $r_i \leq l_j$ holds, where $I(i) = [l_i, r_i]$ and $I(j) = [l_j, r_j]$. Obviously, $(V, \preceq)$ is a partially ordered set. Although every chain of $(V, \preceq)$ is an independent set of G, the converse implication does not hold.

Given a circle graph $G = (V, E)$ and corresponding interval representation $\mathcal{I}(G)$, we introduce a directed graph Γ as follows. The vertex-set of Γ is defined by $V \cup \{0\}$, where 0 is an artificial vertex called a *root.* The arc-set of Γ, denoted by A, is defined by $A = \{(0, i) \mid i \in V\} \cup \{(i, j) \mid I(i) \supsetneq I(j)\}$. The above definition implies that Γ is acyclic.

An arc subset $T \subseteq A$ is called an *arborescence* if and only if $|T| = |V|$ and each vertex $i \in V$ has a unique incoming-arc in T. When a given arborescence T has an arc (i, j), we say that j is a *child* of i and i is a (unique) *parent* of j with respect to T. For any arborescence T and a vertex $i \in V \cup \{0\}$, $\mathrm{Ch}(T, i)$ denotes the set of children of i with respect to T.

In the following, we associate each coloring with an arborescence on Γ. Let $\phi : V \to \{1, 2, \ldots, c\}$ be a c-coloring of G. For each vertex $j \in V$, we define a parent of j with respect to ϕ, denoted by $\mathrm{Prt}(\phi, j)$, as follows: if $V' = \{i \in V \mid \phi(i) = \phi(j), I(i) \supsetneq I(j)\}$ is empty, then we define $\mathrm{Prt}(\phi, j) = 0$ (root); else, $\mathrm{Prt}(\phi, j)$ denotes a vertex in V corresponding to a unique (inclusion-wise) minimum interval in V'. Given a coloring ϕ of G, $T(\phi)$ denotes an arborescence $\{(\mathrm{Prt}(\phi, j), j) \in A \mid j \in V\}$.

Lemma 1. Let T be an arborescence of Γ. Then, there exists a c-coloring ϕ of a given circle graph G satisfying $T = T(\phi)$ if and only if

C1: for each $i \in V$, $\mathrm{Ch}(T, i)$ is a chain of $(V, \preceq)$ or the empty set and

C2: the size of every antichain of $(V, \preceq)$ contained in $\mathrm{Ch}(T, 0)$ is less than or equal to c.

Let $P \subseteq \mathbb{R}$ be a set of (positions of) terminal points of intervals in $\mathcal{I}(G)$. Recall that terminal points of intervals in $\mathcal{I}(G)$ are mutually distinct and thus $|P| = 2|V|$. Let $M = (m_{pi})$ be a 0-1 matrix whose entries are indexed by $P \times V$, satisfying

$$m_{pi} = \begin{cases} 1 & (\text{if } p \in I(i)), \\ 0 & (\text{otherwise}). \end{cases}$$

Now, we give our formulation for a circle graph coloring problem. For any vertex $i \in V \cup \{0\}$, we define a vertex subset $V^{[i]} = \{j \in V \mid (i,j) \in A\}$. We define $V^{\bullet} = \{i \in V \mid V^{[i]} \neq \emptyset\}$. Here, we note that $V^{[0]} = V$ and $0 \notin V^{\bullet}$ hold. For each arc $(i,j) \in A$, we introduce a 0-1 variable x^i_j. The vector of all 0-1 variables is denoted by $\boldsymbol{x} \in \{0,1\}^A$. For any vertex $i \in V^{\bullet} \cup \{0\}$, $\boldsymbol{x}^{[i]}$ denotes a subvector of $\boldsymbol{x}$ indexed by arcs emanating from i, and $M^{[i]}$ denotes a submatrix of M consisting of column vectors of M indexed by $V^{[i]}$. Then, we have the following formulation for a circle graph coloring problem:

$$\begin{aligned} \text{CG} : \text{min. } & c \\ \text{s.t. } & M\boldsymbol{x}^{[0]} \leq\ c\mathbf{1}, \\ & M^{[i]}\boldsymbol{x}^{[i]} \leq \mathbf{1} \quad (\forall i \in V^{\bullet}), \\ & \sum_{i:(i,j)\in A} x^i_j = 1 \ (\forall j \in V), \\ & x^i_j \in \{0,1\} \qquad (\forall (i,j) \in A), \\ & c \in \mathbb{Z}_+. \end{aligned}$$

Theorem 1. A pair $(\widehat{\boldsymbol{x}}, \widehat{c}) \in \{0,1\}^A \times \mathbb{Z}_+$ is optimal to CG if and only if $\widehat{c} = \chi(G)$ and there exists a $\widehat{c}$-coloring ϕ satisfying $T(\phi) = \{(i,j) \in A \mid \widehat{x}^i_j = 1\}$.

The above formulation satisfies the following property.

Theorem 2. Let LR be a linear relaxation problem of CG obtained by substituting $x^i_j \geq 0$ and $c \geq 0$ for $x^i_j \in \{0,1\}$ and $c \in \mathbb{Z}_+$, respectively. Then, the optimal value of LR is equal to the fractional chromatic number of a given circle graph.

A Sufficient condition for an n-vertex plane graph to have a dominating set of size at most $n/4$

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A *dominating set* $D \subset V(G)$ of a graph G is a set such that each vertex $v \in V(G)$ is either in the set or a adjacent to a vertex in the set. The *domination number* of G is the minimum cardinality over all dominating sets of G and denoted by $\gamma(G)$. In 1996, Matheson and Tarjan[1] proved that $\gamma(G) \leq n/3$ for any n-vertex triangulated disc G, and conjectured that $\gamma(G) \leq n/4$ for any n-vertex triangulation G with n sufficiently large.

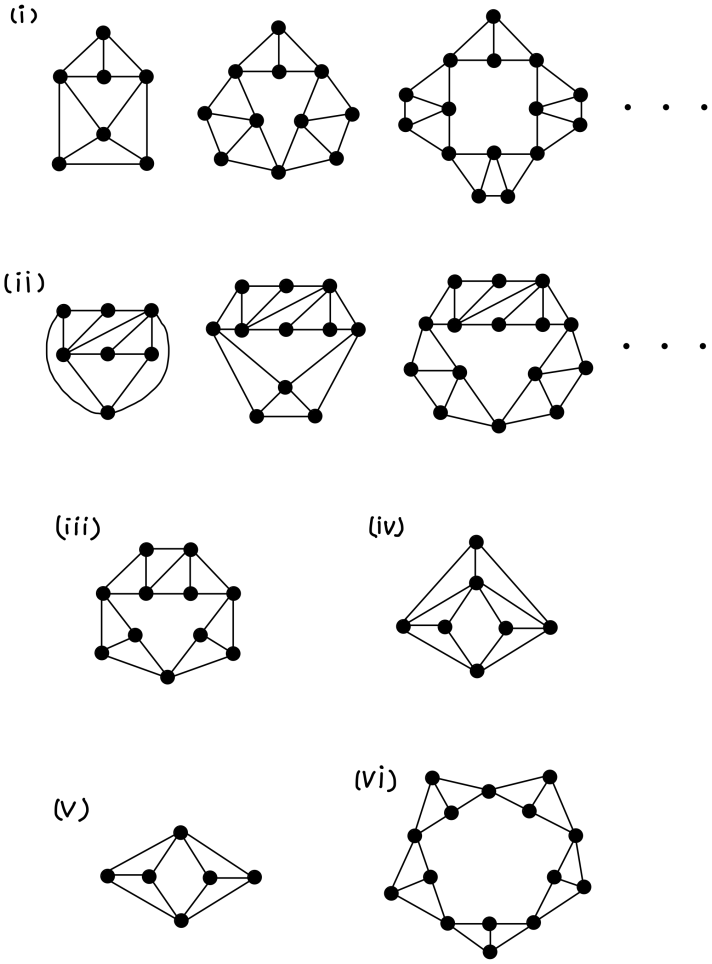
In 2013, Tokunaga[2] (and independently Campos and Wakabayashi[3]) proved that if G is an n-vertex maximal outerplanar graph with $n \geq 3$ having k vertices of degree 2, then $\gamma(G) \leq \lfloor \frac{n+k}{4} \rfloor$ by a simple coloring method.

An *annulus triangulation* is a 2-connected plane graph with two disjoint special faces f_1 and f_2 such that every face of G except for f_1 and f_2 are triangular, and that every vertex of G is contained in the boundary cycle of f_1 or f_2. Using a similar method as in [2], Abe, Higa and Tokunaga[4] proved the following theorem

Theorem A (Abe, Higa and Tokunaga, 2020) *Let G be an annulus triangulation with n vertices and k vertices of degree 2. If $n \geq 7$, then $\gamma(G) \leq \lfloor \frac{n+k+1}{4} \rfloor$, where this estimation is sharp.*

Note that an annulus triangulation might not have vertices of degree 2, so we could consider the case $k = 0$. Further, the case when an n-vertex annulus triangulation with no vertex of degree 2 does not have a dominating set of size at most $n/4$ seems to be restricted. In this talk, we will show a sufficient condition for an n-vertex annulus triangulation with no vertex of degree 2 to have a dominating set of size at most $n/4$ as follows.

Theorem 1 *Let G be an annulus triangulation with n vertices and no vertex of degree 2. If G does not contain any of graphs which belongs to any of the six types illustrated by the following figures as a spanning subgraph, then $\gamma(G) \leq \lfloor \frac{n}{4} \rfloor$.*



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Counting 2-factors of 4-regular bipartite graphs is #P-complete

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Extended Abstract. We prove that counting 2-factors (i.e., spanning 2-regular subgraphs or vertex disjoint cycle covers) of 4-regular bipartite graphs is $\#P$-complete under many-one counting ("weakly parsimonious") reductions. This resolves a missing case in the proof by Felsner & Zickfeld [3] that counting 2-factors of k-regular bipartite graphs is $\#P$-complete $\forall\,(k \geq 3 \mid k \neq 4)$. Furthermore, observing the bijection between the number of Eulerian orientations (i.e., orientations where every vertex has equal in- and out-degree) and number of 2-factors in 4-regular bipartite graphs [3], this concomitantly extends a $\#P$-completeness (under Turing reductions) proof for counting Eulerian orientations of 4-regular graphs [4]. We remark that, due to additional bijective correspondences (see, e.g., ref. [2, 5, 8, 10]) between objects known as Fully Packed Loop (FPL) configurations, where FPLs are modeled in many settings as 2-factors of toroidal or higher genus 4-regular bipartite graphs, Alternating Sign Matrices (ASMs), and microstates of the 6-vertex model, our results have implications for the hardness of calculating partition functions of "ice-type" systems (e.g., Pauling's model [6, 7] for the residual entropy of water ice).

Briefly, recall that to establish a many-one counting reduction from a $\#P$ problem $f : \Sigma^* \longrightarrow N$, with input string x, to a $\#P$ problem $h : \Sigma^* \longrightarrow \mathbb{N}$, it suffices to show the existence of a pair of $\mathcal{O}\left(poly\left(|x|\right)\right)$ functions, $R_1 : \Sigma^* \longrightarrow \Sigma^*$ and $R_2 : \mathbb{N} \longrightarrow \mathbb{N}$, where we have that $f(x) = R_2(h(R_1(x)))$. A sketch for the proof of the main theorem is now as follows:

Theorem 1. Counting 2-factors of 4-regular bipartite graphs is $\#P$-complete under many-one counting reductions.

Proof sketch. We will proceed via reduction from the $\#P$-complete problem of counting 2-factors of 3-regular bipartite graphs [3]. To begin, let G be an arbitrary 3-regular bipartite graph with vertex partite sets V_X and V_Y, where we necessarily have that $|V_X| = |V_Y|$, vertex set $V_G = V_X \cup V_Y$, and a set of 2-factors $\mathcal{F}_G$. As a consequence of G being a cubic graph, each perfect matching of G will induce a unique 2-factor. Here, we construct a graph G' from G by sequentially replacing each vertex $v_i \in V_G$, adjacent to vertices $v_a, v_b, v_c \in V_G$, with a 6-cycle given by the edge set $\{v_{(i,1)} \leftrightarrow v_{(i,6)}, v_{(i,1)} \leftrightarrow v_{(i,2)}, v_{(i,2)} \leftrightarrow v_{(i,3)}, v_{(i,3)} \leftrightarrow v_{(i,4)}, v_{(i,4)} \leftrightarrow v_{(i,5)}, v_{(i,5)} \leftrightarrow v_{(i,6)}\}$, where $v_{(i,1)}$, $v_{(i,3)}$, and $v_{(i,5)}$ are made adjacent to vertices v_a, v_b, and v_c, respectively.

We next construct a graph H from G' by replacing all $(3 \cdot |V_G|/2)$ edges of the form $v_{(i,\ldots)} \leftrightarrow v_{(j,\ldots)}$ in G' with linear assemblies, denoted $\mathcal{B}_\omega$ gadgets, consisting of $\omega = 2 \cdot |V_G| + 1$ copies of the *boost gadget* shown in Fig. 1(a–c). More specifically, to be build $\mathcal{B}_\omega$, for each $i \in [1, \omega - 1]$ we assemble the ω copies of the boost gadget together by identifying the vertex v_t on the ith gadget with the vertex v_s on the $(i+1)$th gadget (e.g., as shown in Fig. 1(d)), then identify the remaining degree 2 vertices v_s and v_t at either ends of the assembly with the vertices $v_{(i,\ldots)}$ and $v_{(j,\ldots)}$, respectively (deleting the original $v_{(i,\ldots)} \leftrightarrow v_{(j,\ldots)}$ edge). Subsequently, we construct a graph H' from H by first creating $(3 \cdot |V_G|/2)$ linear assemblies of $\Psi = |V_G|^3$ copies of the boost gadget in the same manner as before, though here denoting these assemblies $\mathcal{B}_\Psi$ gadgets. Then for an arbitrary pairing of the vertices in opposite partite sets having labels of the form $v_{(i,2)}$, $v_{(i,4)}$, or $v_{(i,6)}$ – where we note that such a pairing must exist as a consequence of the fact that the vertex partite sets of G' have equal cardinality – for every vertex pair $\{v_i, v_j\}$, we identify the degree 2 vertices v_s and v_t on a distinct $\mathcal{B}_\Psi$ gadget with v_i and v_j, respectively. We can observe that H' will be a 4-regular bipartite graph if G is cubic bipartite.

We remark that the boost gadget shown in Fig. 1(a–c) can be covered by a 2-factor in three distinct manners: (type α; example shown in Fig. 1(a); 92 instances) a cycle ingresses and egresses via the labeled vertices v_s and v_t; (type β; example shown in Fig. 1(b); 25 instances) two cycles ingress and egress while covering only the vertices v_s and v_t; (type γ; example shown in Fig. 1(c); 25 instances) no cycles ingress or egress. Observe that, for any instance of the $\mathcal{B}_\omega$ or $\mathcal{B}_\Psi$ gadgets, a 2-factor must cover all boost gadget subgraphs in the type α manner, or in the type β and γ manners in an alternating fashion. Now let $\mathcal{F}_{H'}$

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be the set of 2-factors for H', and let $\mathcal{F}^*_{H'} \subset \mathcal{F}_{H'}$ be a subset of 2-factors satisfying the constraint that exactly $|V_G|$ of the $(3 \cdot |V_G|/2)$ instances of $\mathcal{B}_\omega$ gadgets have boost gadget subgraphs uniformly covered in the type α manner (this being the maximum possible number), and that all of the boost gadget subgraphs of $\mathcal{B}_\Psi$ gadgets are covered in either the β or γ manner.

Here, an analysis (omitted due to space constraints) yields that: (1) every 2-factor in $\mathcal{F}_G$ corresponds to $\Upsilon = 2^{|V_G|/2} \cdot (92^\omega)^{|V_G|} \cdot (25^\omega)^{|V_G|/2} \cdot \left(25^\Psi\right)^{3 \cdot |V_G|/2}$ 2-factors in $\mathcal{F}^*_{H'}$, where we additionally have that $|\mathcal{F}^*_H| = \Upsilon \cdot |\mathcal{F}_G|$; (2) assuming $|V_G| \geq 12$, for each $k \in [1, 3 \cdot |V_G|/2]$, the number n_k of 2-factors in $\mathcal{F}'_H$ with at least k instances of $\mathcal{B}_\Psi$ gadgets having boost gadget subgraphs uniformly covered in the type α manner, is exponentially larger by a known value ($> 2^{|V_G|}$) than the number of 2-factors where at most $k-1$ instances of $\mathcal{B}_\Psi$ gadgets are covered in the same manner, and one can recover each n_k via an integer division operation by another known value; (3) that with the value of n_k for each $k \in [1, 3 \cdot |V_G|/2]$, $|\mathcal{F}^*_{H'}|$ and subsequently $|\mathcal{F}_G|$ can be recovered via a similar integer division operation (again assuming $|V_G| \geq 12$). Putting everything together, we have that (1) through (3) implies the existence of a simple recursive algorithm to determine the cardinality of $\mathcal{F}_G$ provided the cardinality of $\mathcal{F}_{H'}$.

Finally, we can observe that the $\#P$-completeness proof of Felsner & Zickfeld [3] for counting 2-factors of 4-regular bipartite graphs, and hence the current result, transitively depends on a proof by Dagum & Luby [1] that counting perfect matchings in k-regular bipartite graphs is $\#P$-complete $\forall k \geq 3$. It now suffices to note that many-one counting reductions are transitive, that Dagum & Luby's proof [1] corresponds to a many-one counting reduction from computing the 0-1 permanent, and that this latter permanent computation is known to be $\#P$-complete under many-one counting reductions [9]. □

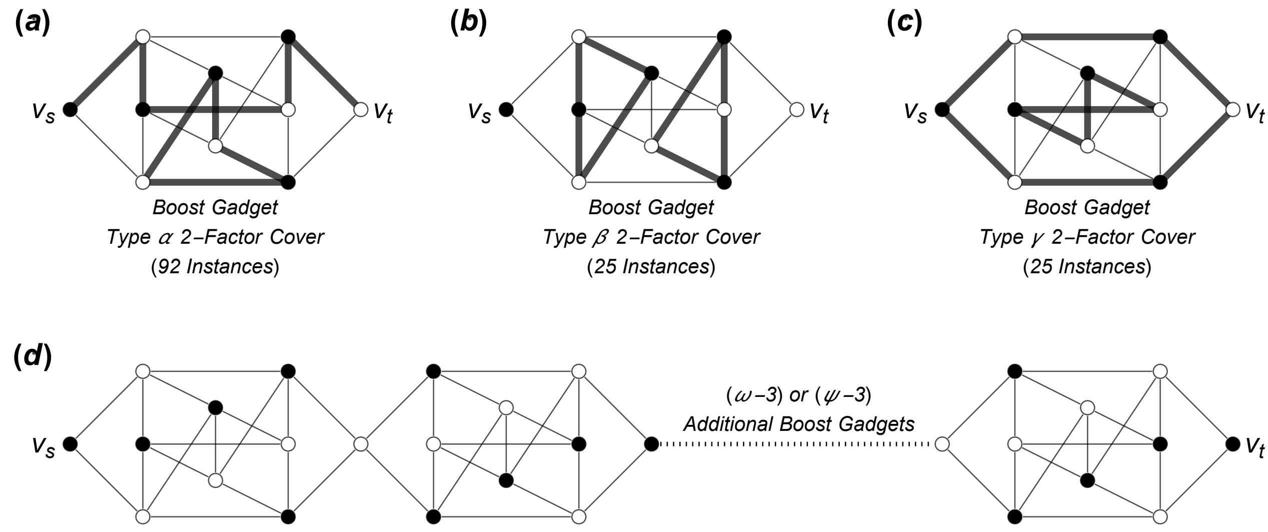

Fig. 1. Illustrations for the Theorem 1 reduction from counting 2-factors of 3-regular bipartite graphs to counting 2-factors of 4-regular bipartite graphs.

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Extended Abstract
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An Improved Algorithm for the Weighted k-Center Problem on Cactus Graphs

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Abstract

Facility location problems are a class of problems which have been well studied in operations research. The *weighted k-center problem* in graphs is a classical example of a facility location problem. Kariv and Hakimi [4] introduced this problem in 1979 and showed it to be NP-hard for general graphs. However, the problem has efficient polynomial runtime for restricted graph classes. The algorithm of Wang and Zhang [5] for the weighted k-center problem on trees uses Frederickson's parametric search technique [3] to bring down the runtime to $O(n \log n)$. We generalize Frederickson's technique to present an $O(kn \log n + n \log^2 n)$ time algorithm for the problem on cactus graphs. This is the first time that such a generalization could be achieved for a graph class other than trees and is an improvement on the previous state of the art [1] for $k = o(n/ \log n)$. We also propose an optimal $O(n)$ time algorithm for the decision version of the problem.

Keywords: k-center problem, facility location problem, parametric search, cactus, cactus stem.

2010 MSC: Primary 68W01; Secondary 05C85, 68R05.

1 Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ with $|V(G)| = n$. We say G is a *cactus* if no two cycles of G share a common edge. Each vertex $v \in V(G)$ is associated with a non-negative weight $w(v)$ and each edge $e \in E(G)$ is associated with a positive length $l(e)$. We interpret e as a line segment of length $l(e)$ so that any point x on e can be referred to. The set of all points on all edges of G is denoted by $A(G)$. For any two points $x, y \in A(G)$, the distance between x and y, denoted by $d(x, y)$, is the length of the shortest path between them in G. For a point $x \in A(G)$ and a vertex $v \in V(G)$, the *weighted distance* between the point x and the vertex v is equal to $w(v) \cdot d(x, v)$. Given a set of points X, the *cost* of covering the vertices of G by X is given by the expression

$$\max_{v \in V(G)} \left\{ \min_{x \in X} \{w(v) \cdot d(x, v)\} \right\}$$

The objective of the *weighted k-center* problem in G is to find a set of k points X such that the cost of covering $V(G)$ is minimized. This cost is also called the *optimal cost* of the weighted k-center problem in G. We say a point (or a center) $x \in A(G)$ *covers* a vertex $v \in V(G)$ with cost λ, if $w(v) \cdot d(x, v) \leq \lambda$.

2 Feasibility Test

The decision version of the problem is commonly referred to as the *feasibility test* in literature. Let $\lambda^* \in \mathbb{R}_{\geq 0}$ be the optimal cost of the weighted k-center problem in G. The feasibility test takes as input a value $\lambda \in \mathbb{R}_{\geq 0}$, and returns *feasible* if $\lambda \geq \lambda^*$, and *infeasible* otherwise. Note that λ^* is an unknown quantity. For a given λ, our algorithm works by placing a minimum number of centers X_λ, such that every vertex in G is covered by some center in X_λ with cost λ. If $|X_\lambda| \leq k$, then λ is feasible; otherwise λ is infeasible. We have the following result.

Theorem 2.1. *The feasibility test for the weighted k-center problem on cactus takes $O(n)$ time, where n is the number of vertices in the cactus.*

3 k-Center on Cactus Stems

A *cactus stem*, is a generalization of the (path) stem defined by Frederickson [3] for trees. Let S' be a graph whose structure is an alternating sequence of paths and cycles. We call a vertex a *pendant vertex* if its degree is 1. Let S be the graph generated from S' by attaching any number of pendant vertices to the vertices in $V(S')$. We call S a *cactus stem*. For two vertices u and v and a path π_{uv} connecting them in S, let $\mathrm{cost}(u, v, \pi_{uv})$ represent the cost of covering u and v by a single center restricted to lie on π_{uv}. We have the following observation.

Observation 1. The optimal cost λ^* of the weighted k-center problem in S equals $\mathrm{cost}(u, v, \pi_{uv})$, for some vertices u and v and a path π_{uv} of S.

We can map the vertices of S into a collection of set of lines $\mathcal{L} = \{L_1, L_2, \ldots, L_{O(k)}\}$ in $\mathbb{R}^2$ such that for every $u, v \in V(S)$ and π_{uv}, the value $\mathrm{cost}(u, v, \pi_{uv})$, and hence the optimal cost λ^*, is represented as the y-coordinate value of the intersection point of a pair of lines which belong to some set $L_i \in \mathcal{L}$.

With respect to the feasibility test as defined in Section 2, let p_1 (resp. p_2) be the highest (resp. lowest) point in the set $L_i \in \mathcal{L}$ whose y-coordinate value is infeasible (resp. feasible). We have the following result by Chen and Wang [2].

Lemma 3.1 (Chen and Wang [2])**.** *Given L_i, both p_1 and p_2 can be found in $O((m + \tau) \log m)$ where $m = |L|$ and τ is the running time of the feasibility test.*

Using Observation 1 and Lemma 3.1 we can show the following result.

Theorem 3.2. *We can find $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ in $O(km \log m)$ time such that $\lambda^* \in (\lambda_1, \lambda_2]$ and no value of type $\mathrm{cost}(u, v, \pi_{uv})$ lies in the range (λ_1, λ_2).*

4 The Main Algorithm

For a cactus G, the subgraph S_i is a *leaf cactus stem* of G if S_i is a maximal cactus stem of G and the rest of the graph $G \setminus S_i$ is connected to S_i at one of its two endpoints. Our algorithm for the weight k-center problem on the cactus G is as follows.

Initialize λ_1 to 0 and λ_2 to ∞. Let $\mathcal{S} = \{S_1, S_2, \ldots\}$ be the collection of leaf cactus stems of G. We run the cactus stem algorithm of Section 3 on all leaf stems in $\mathcal{S}$ simultaneously and update λ_1 and λ_2 accordingly such that $\lambda^* \in (\lambda_1, \lambda_2]$ and no value of type $\mathrm{cost}(u, v, \pi_{uv})$, where $u, v \in V(S_i)$ for some $S_i \in \mathcal{S}$, lies inside (λ_1, λ_2). If we perform a feasibility test on G with input $\lambda \in (\lambda_1, \lambda_2)$, then the centers placed by the feasibility test on the leaf stems are optimal. Let k' be the number of centers placed on those leaf stems. We replace these leaf cactus stems with pendant vertices to get a new graph G' such that the $(k - k')$-center solution in G' along with the k' centers in the leaf stems is the k-center solution of G. We repeat this process $O(\log n)$ time until a single cactus stem remains for which we can find the solution optimally. We have the following result.

Theorem 4.1. *The weighted k-center problem on a cactus G can be solved in $O(kn \log n + n \log^2 n)$ time, where n is the number of vertices of G.*

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PSPACE-completeness of generalized Quarto!

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Abstract. Quarto! is a board game for two players, in which one player chooses a piece to place, and the other player places it in a vacant cell on the board. In this paper, we define a generalized Quarto! and analyze its computational complexity using a reduction from generalized geography. In our reduction, we construct some gadgets on the board for generalized Quarto! and fill almost all cells on the board due to its unique rule. As a result, we prove that the generalized Quarto! is PSPACE-complete.

Keywords: Combinatorial game theory · Generalized geography · Board game · PSPACE-complete · Quarto!

1 Introduction

Quarto! is a board game for two players invented by the Swiss mathematician Brades Mueller [1]. The game has a board with 4×4 cells, and 16 pieces that have four types. The first player (after we denote N) chooses a piece, and then the second player (after we denote P) places that chosen piece in any vacant cell. After that placement, player P chooses a new piece, and then player N places that piece and chooses the next piece after placement. These steps are repeated alternately until a player makes a run of row or column, or diagonal of 4 pieces having the same type. When the run is completed, the player who places the last piece wins the game. It would be said that Quarto! is a unique game because of the rule that the opponent chooses pieces to be placed.

In this paper, we analyze the computational complexity of the decision problem to determine the winner on an arbitrary board of generalized Quarto!. To simplify, we restrict the number of types of piece into two; shape ($\circ/\times$) and color (black/red). We also discuss on a board with $m \times m$ cells. In this setting, the player who makes a run of 4 pieces with the same type in vertically, horizontally, or diagonally wins the game. Then we define the decision problem of the generalized Quarto! as follows.

Generalized Quarto!
Input : A $m \times m$ board that may contain some pieces.
Output : If there is a winning strategy for the player who chooses the next piece to place on the input board *yes*, otherwise *no*.

It is easy to show that the generalized Quarto! is in PSPACE since players can place a piece at most once on finite cells. Therefore, using a DFS, we can solve the problem using $O(n^2)$ space. Then we focus on its hardness and prove that generalized Quarto! is PSPACE-hard.

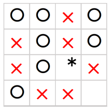

Fig. 1. board example

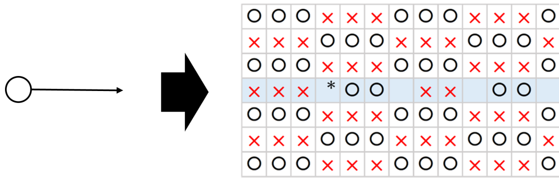

Fig. 2. A gadget for the starting vertex

2 Our contribution

Our proof technique is based on a reduction from generalized geography [3]. It is known that generalized geography is PSPACE-complete even if its input graph satisfies the following constraints [2] :

1. Plane bipartite graph.
2. All vertices are on grid points and edges are horizontal or vertical.
3. All edges lengths are 1 or 2.
4. The angle between two edges starting at the same vertex is 180 degrees.
5. All vertices except the starting vertex must have at least 1 in-degree and at least 1 out-degree.

Therefore, it is sufficient for the reduction to construct gadgets for subgraphs under the above constraints. In our reduction, players are forced to choose a patient piece and place it in a patient cell. Fig. 1 shows a case where players are forced. In this case, player N must choose a ($\circ$,black) piece. If not, player P wins to place a $\times$ or red piece in the cell labeled *. After player N chooses a ($\circ$,black) piece, player P must place it in the cell labeled *. If not, whatever piece player P chooses after this placement, player N wins. Using this technique, we make gadgets on generalized Quarto! which leave players only a few way for preventing the opponent's win.

Fig. 2 shows a gadget for reduction. This gadget forces player N to choose a ($\circ$,black) piece, and forces player P to place it in the cell labeled *. Then a token's move on generalized geography is reduced to a consequential move on the shaded cells.

To complete the proof, the input must not have free cells except for patient moves. Therefore, our reduction fills almost all cells. Furthermore, each gadget needs only a constant number of cells. Then we show the following theorem.

Theorem 1. *Generalized Quarto! is PSPACE-complete.*

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On the Computational Complexity of Pushing Machine (extended abstract)

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1 Introduction

Many combinatorial puzzles are hard, and pinpointing the exact computational complexity classes for the puzzles are also difficult. Sokoban is one of the most famous puzzles, and its computional complexity was shown to be NP-hard [4]. Culberson proved Sokoban is PSPACE-complete by constructing several sophisticated Sokoban gadgets to emulate Turing machine [3]. A new proof of the PSPACE-completeness of Sokoban was obtained by Hearn and Demaine in 2005 [6]. The new proof is based on a new framework called non-deterministic constraint logic (NCL). The NCL framework is quite powerful in determining the computational complexity of other puzzles with a motion planing flavor [5].

Another puzzle that has been extensively studied is Rush Hour. With the NCL framework, Hearn and Demaine were able to prove that 1×2 Rush Hour is PSPACE-complete [6]. But the complexity of 1×1 Rush Hour remains open for almost twenty years before finally being solved in [2]. Yet another new framework was invented to tackle the 1×1 case. This new framework is named oriented Subway Shuffle, which was inspired by [1].

In this paper, we study the complexity of a puzzle called *Pushing Machine* which first appeared as an Android game[1] in 2012. The game mechanism of Pushing Machine looks like Sokoban or sliding block puzzles at first glance, but is siginificantly different. In Pushing Machine, there are two kinds of objects in the 2-dimensional maze with walls (i.e. fixed blocks). One is the pushers, the other is the boxes. Unlike Sokoban, there is no agent in Pushing Machine, and the pushers cannot move by their own. Instead, each pusher has an arm that can be extended or retracted. When extending, the arm can push other pushers or boxes. Like Sokoban, the extending arm can only push one box or one other pusher at a time. The goal is to push all the boxes to marked locations (see Figure 1).

There is another game named *Push It Sokoban Edition*[2] which generalizes the Pushing Machine in two ways. First, each pusher may have up to four pushing arms (in four directions) instead of just one. Second, boxes have different colors, and they should be pushed to marked locations with the same colors (see Figure 1).

2 Main results

We first give the following definition of the decision problem related to the puzzle of Pushing Machine.

Definition 1 (k-PushingMachine Problem). *Given a 2-dimensional maze with walls, several pushers with up to k ($1 \leq k \leq 4$) arms each, several boxes (with no color distinction), decide whether the boxes can be all pushed to marked locations.*

[1] https://play.google.com/store/apps/details?id=com.reactor.pushingmachine
[2] https://en.gruv-apps.com/push-it-sokoban-edition/

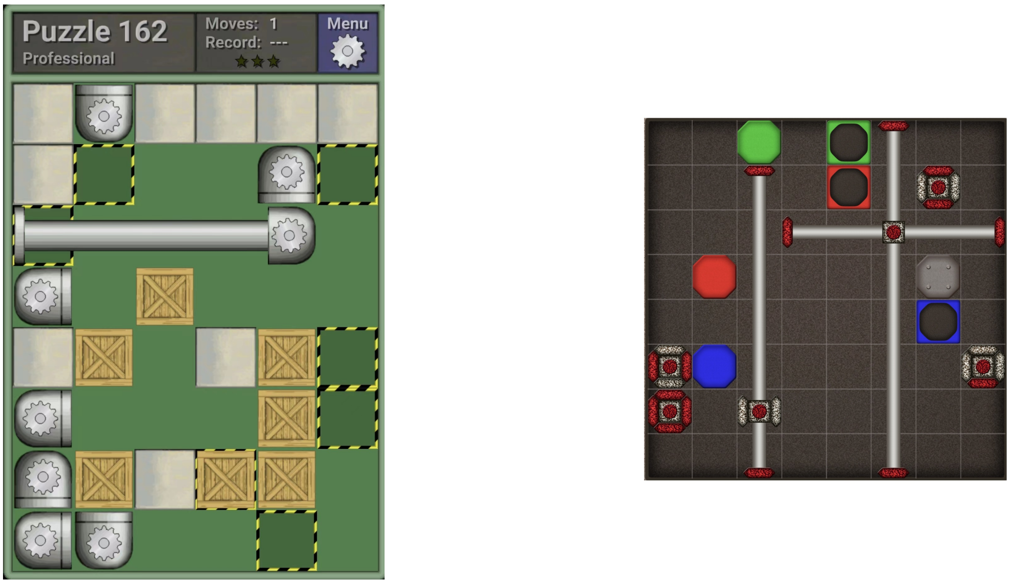

Figure 1: Screenshots of the games Pushing Machine (left) and Push It Sokoban Edition (right)

Despite the similarity with Sokoban or sliding block puzzles, it is not easy to determine the computational complexity of k-PushingMachine. We have work on this problem for several years. At first, we try using the NCL framework, but with no success. Then we switch to the framework of oriented Subway Shuffle which is recently developed in [2]. With this new tool at hand, we are able to prove the following main result for $k = 2$.

Theorem 1. *The 2-PushingMachine problem is PSPACE-complete.*

It remains open to decide the computational complexity of the 1-PushingMachine problem.

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Extended Abstract
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Constructive comparison in bidding combinatorial games

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Abstract

A class of discrete Bidding Combinatorial Games that generalize alternating normal play was introduced by Kant, Larsson, Rai, and Upasany (2022). The major questions concerning optimal outcomes were resolved. By generalizing standard game comparison techniques from alternating normal play, we propose an algorithmic play-solution to the problem of game comparison for a class of bidding games that include game forms that are defined numbers. We demonstrate a number of consequences of this result that, in some cases, generalize classical results in alternating play (from the famous books Winning Ways and On Numbers and Games). We state a couple of conjectures and open problems for readers to dive into this promising path of bidding combinatorial games.

Keywords: Bidding Combinatorial Game, Constructive Comparison, 0-bid strategy, Number game, Dyadic rational game, 0-Game, Group structure.

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1 Introduction

Normal play is the Combinatorial Game convention where a player who cannot move loses [1, 2, 7]. Recently [4], we generalize classical 2-player alternating play to infinitely many game families, by means of a Discrete Richman Auction [3, 5, 6]. The two players have split a total budget, TB, between them, and winning bids are transferred to the loser of the auction. In [4] we establish that games have pure subgame perfect equilibria, and we resolve all essential structure problems regarding their perfect play outcomes.

In this paper our focus is on generalizing the results available for alternating play (*i.e.*, $\mathrm{TB} = 0$) to bidding combinatorial games with $\mathrm{TB} > 0$. We generalize standard game comparison techniques from alternating play. Our main result proposes a constructive i.e., algorithmic solution to the problem of game comparison. Further we establish results on the structural and arithmetical aspects of games, both in general as well as for some particular types of games like Numbers, Integers and Dyadic Rationals.

2 Preliminaries

For a given TB, a game is a triple $(\mathrm{TB}, G, \widetilde{p})$, where Left's part of the budget is $\widetilde{p} \in \{0, \ldots, \mathrm{TB}, \widehat{0}, \ldots, \widehat{\mathrm{TB}}\}$, and where $\widehat{\cdot}$ indicates that Left holds the tie-breaking marker. Here the bidding convention is that the marker may be included in a bid. Right's monetary budget is $\mathrm{TB} - p$. If TB is given, we generally write $(G, \widetilde{p})$. The *game form* is G. Games are recursively defined, with $G = \left\{G^{\mathcal{L}} \,|\, G^{\mathcal{R}}\right\}$, where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are the current Left and Right options from the game G. If $G^{\mathcal{L}} = \varnothing$ or $G^{\mathcal{R}} = \varnothing$ then G is terminal, i.e., the current player, Left or Right, cannot move. In case $G^{\mathcal{L}} = G^{\mathcal{R}} = \varnothing$, then $G := 0$ is terminal irrespective of move order. Games are finite and contain no cycles, i.e., each game has finitely many options and the rank of the game tree is finite. We define $o(G, \widetilde{p})$ as the perfect play outcome of the game G if the game is played with Left's budget equal to $\widetilde{p}$.

If the current player cannot move, they lose. Hence a player does not want to win an auction at a terminal position.

The disjunctive sum of the game forms G and H is defined recursively as:

$$G + H = \left\{G + H^{\mathcal{L}}, G^{\mathcal{L}} + H \,|\, G + H^{\mathcal{R}}, G^{\mathcal{R}} + H\right\},$$

Where $G + H^{\mathcal{L}} = \{G + H^{L} : H^{L} \in H^{\mathcal{L}}\}$, in case $H^{\mathcal{L}} \neq \varnothing$, and otherwise the set is not defined and omitted. The *conjugate* of a game form G is the game form where players have swapped positions, and is recursively defined as $\overline{G} = \left\{\overline{G^{\mathcal{R}}} \mid \overline{G^{\mathcal{L}}}\right\}$.

The *partial order* of games is defined as usual. Consider games G, H. Then $G \geqslant H$ if, for all games X, $o(G + X) \geqslant o(H + X)$. Game equality satisfies $G = H$ if $G \geqslant H$ and $H \geqslant G$. The games $G > H$ if $G \geqslant H$ but $H \not\geqslant G$. The games G and H are confused if $G \not\geqslant H$ and $H \not\geqslant G$; this is denoted $G \parallel H$.

A game form G is a *number* if for all $G^{L} \in G^{\mathcal{L}}$ and for all $G^{R} \in G^{\mathcal{R}}$, $G^{L} < G < G^{R}$, and all options are numbers.

Some of the standard games are: I) $0 = \{\varnothing|\varnothing\}$, II) $* = \{0|0\}$, III) For $k \in \mathbb{N}$: Integer games are i) $k = \{k-1|\varnothing\}$, ii) $-k = \{\varnothing| -(k-1)\}$, and Dyadic rational games are i) $1/2^{k} = \{0|1/2^{k-1}\}$, ii) $-1/2^{k} = \{-1/2^{k-1}|0\}$. These notations are motivated by Theorem 3.6.

3 Main Results

Theorem 3.1 (Main Theorem, Case Study Numbers). *Consider any total budget and a number game G. Then $G \geqslant 0$ if and only if $o(G, 0) = \mathrm{L}$.*

Theorem 3.2 (Constructive Main Theorem). *Consider a game form G and any total budget. Suppose that Left has an optimal 0-bid strategy in $(G, 0)$. Then $G \geqslant 0$ if and only if $o(G, 0) = \mathrm{L}$.*

Observation 3.3 (No Generic 0-optimal Strategy). We demonstrate that, for any non-zero total budget, there is a game form G such that $o(G, 0) = \mathrm{L}$, but Left does not optimally bid 0 at each follower.

Theorem 3.4 (0-Games). *For all positive integers k, $\{-k\,|\,k\} = 0$ and $\left\{-1/2^{k}\,|\,1/2^{k}\right\} = 0$.*

Theorem 3.5 (No Group Structure in general). *Consider* $\mathrm{TB} > 0$. *There is no game G such that $* + G = 0$.*

Theorem 3.6. *Consider game forms that are integers and dyadic rationals, respectively. Both are numbers and subgroups of the full game monoid.*

Theorem 3.7. *Standard arithmetic rules and total order continue to hold for bidding games that are dyadic rationals.*

Conjecture 3.8. If a bidding game G has an inverse H, then $H = \overline{G}$.

We believe that several classical results on alternating play numbers, such as number avoidance, continue to hold in general bidding.

Conjecture 3.9. All numbers are dyadic rationals.

However, we conjecture that the Archimedean property distinguishes non-trivial bidding convention from positive total budgets.

Conjecture 3.10. Consider $\mathrm{TB} > 0$. All positive games are dyadic rationals.

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The Constructor-Blocker Game

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Abstract

We study the following game version of generalized graph Turán problems. For two fixed graphs F and H, two players, Constructor and Blocker, alternately claim unclaimed edges of the complete graph K_n. Constructor can only claim edges so that he never claims all edges of any copy of F, i.e. his graph must remain F-free, while Blocker can claim unclaimed edges without restrictions. The game ends when Constructor cannot claim further edges or when all edges have been claimed. The score of the game is the number of copies of H with all edges claimed by Constructor. Constructor's aim is to maximize the score, while Blocker tries to keep the score as low as possible. By $g(n, H, F)$ we denote the score of the game when both players play optimally and Constructor starts the game.

We obtain the exact value of $g(n, H, F)$ when both F and H are stars and when $F = P_4$, $H = P_3$. We determine the asymptotics of $g(n, H, F)$ when F is a star and H is a tree and when $F = P_5$, $H = K_3$, and we derive upper and lower bounds on $g(n, P_4, P_5)$.

1 Introduction

The Turán problem for a set F of graphs asks the following: What is the maximum number $ex(n, F)$ of edges that a graph on n vertices can have without containing F as a subgraph? This function has been intensively studied, starting with Mantel and Turán who determined $ex(n, K_r)$ where K_r denotes the complete graph on r vertices with $r \geq 3$.

There is a generalization of the Turán problem, when we count the maximum number of copies of a certain graph H in a graph G on n vertices, provided that G does not contain a graph F as a subgraph. To be more precise, let us introduce some notation: for two graphs H and G, let $\mathcal{N}(H, G)$ denote the number of copies of H in G. Given graphs H and F, let

$$ex(n, H, F) = \max_G \{\mathcal{N}(H, G) : G \text{ is an } F\text{-free graph on } n \text{ vertices}\}.$$

This problem was initiated by Zykov in 1949, who determined $ex(n, K_s, K_t)$ exactly, however the systematic study of the function $ex(n, H, F)$ started just recently in [1] by Alon and Shikhelman, followed by a number of other results in the recent years.

Our goal is to introduce a game analogue of the parameter $ex(n, H, F)$ and provide some results. For two fixed graphs F and H, two players, Constructor and Blocker alternately claim unclaimed edges of the complete graph K_n. Constructor can only claim edges so that he never claims all edges of any copy of F, i.e. his graph must remain F-free, while Blocker can claim unclaimed edges without restrictions. The game ends when Constructor cannot claim further edges or when all edges have been claimed. The score of the game is the number of copies of H with all edges claimed by Constructor. Constructor's aim is to maximize the score, while Blocker tries to keep the score as low as possible. We denote by $g(n, H, F)$ the score of the game when both players play optimally and Constructor starts the game.

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Our Constructor-Blocker games borrow some aspects of two well-studied classes of combinatorial games on graphs – the Maker-Breaker positional games (see, e.g., [3]), and the saturation games (see, e.g., [2]). In our Constructor-Blocker games each player builds his own graph, as it is the case in Maker-Breaker games, and in addition Blocker, like Breaker, has no restrictions on his moves. As for Constructor, his graph must remain F-free resembling the settings in saturation games. Note that for every two graphs H, F and $n \geq 1$ we trivially have $g(n, H, F) \leq ex(n, H, F)$.

2 Our results

The first case we consider is when both H and F are stars, determining the game score exactly.

Theorem 2.1. *For $2 \leq \ell \leq k$ there exists $n_0(k, \ell)$ such that if $n \geq n_0(k, \ell)$ we have*

$$g(n, S_\ell, S_{k+1}) = \begin{cases} \binom{k}{\ell} \cdot (n-2) + 2 \cdot \binom{k-1}{\ell} & \text{if } nk \text{ is even,} \\ \\ \binom{k}{\ell}(n-1) + \binom{k-1}{\ell} & \text{if } nk \text{ is odd.} \end{cases}$$

For $1 = \ell \leq k$, we have $g(n, S_1, S_{k+1}) = \lfloor \frac{nk-1}{2} \rfloor$.

Next we look at the case when H is a tree and F is a star.

Theorem 2.2. *For any $k \geq 2$ and tree T with maximum degree at most k, we have*

$$g(n, T, S_{k+1}) = ex(n, T, S_{k+1}) - O_{k,T}(1).$$

We consider two games with both H and F being paths. Let $B(n) := \binom{\lfloor \frac{n-2}{2} \rfloor}{2} + \binom{\lceil \frac{n-2}{2} \rceil}{2}$. In the case $H = P_3, F = P_4$ we can determine the exact game score.

Theorem 2.3. *There exists an integer n_0 such that for every $n \geq n_0$, we have*

$$g(n, P_3, P_4) = B(n).$$

When $H = P_4$ and $F = P_5$ we provide the following bounds.

Theorem 2.4.

$$\frac{8}{49}n^2 - o(n^2) \leq g(n, P_4, P_5) \leq \frac{4}{23}n^2 + o(n^2).$$

Finally, we study the case $H = K_3, F = P_5$.

Theorem 2.5.

$$g(n, K_3, P_5) = \frac{n}{4} - o(n).$$

More details and all of the proofs can be found in the full version of our paper [4].

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Geodesic paths passing through all faces on a polyhedron

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Abstract

A shortest path passing on the surface of a polyhedron is called a geodesic path. A geodesic path of a polyhedron has the property that it becomes a single line segment on a development. A geodesic path is a shortest path and it mostly passes a small number of face. We, however, consider a problem "is there a case that a geodesic path passes all faces of a polyhedron?" For this problem the answer is "yes": we found that a regular tetrahedron has such a geodesic path. The next question is "what polyhedra have such geodesic paths?" We define a face-guard geodesic path (FGG path) as a geodesic path connecting two points on a polyhedron and passing through all its faces, and propose an FGG path problem which asks whether a given polyhedron has an FGG path. We considers that this is a new idea. In this talk, we will present some results on this problem.

Keywords: Geodesic path, convex polyhedron, security problem

1 Introduction

It is known that a geodesic path connecting two points on opposite sides of some rectangle passes through five faces (Fig.1). However, no two points have been found where the geodesic path passes through all the faces of a cube/cuboid. This naturally raises the question of whether there is a case where the geodesic passes through all the faces. We call a geodesic path connecting two points on a polyhedron and passing through all its faces an *face-guard geodesic path (FGG path for short)*. We propose a *face-guard geodesic path problem (FGG path problem)*, which asks whether a given polyhedron has an FGG path or not. The FGG path problem also has an aspect of a security problem, i.e., whether a geodesic path guards all faces. This idea was first discussed at the 32nd Bellairs Winter Workshop on Computational Geometry held in Barbados in 2017.

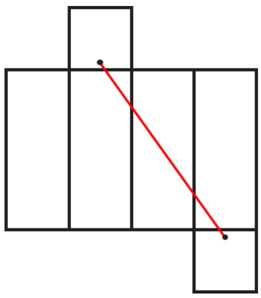

Figure 1: The geodetic path through five faces

2 Preliminaries

All polyhedra in this research are assumed to be convex polyhedra.

2.1 Definitions

Definition 2.1 (geodesic path)**.** For a given polyhedron and two points s and t on its face, the shortest paths passing on the face of the polyhedron between s and t is called a *geodesic path*. They are sometimes called an *s-t geodesic paths* for explicitly indicating the endpoints. A path through the faces of a polyhedron with s and t as its two ends, such that it is locally shortest, is called an *s-t local geodesic path*. From the definition, the shortest one among s-t local geodesic paths is the s-t geodesic path.

Definition 2.2 (FGG path)**.** A geodesic path passes through all the faces of the polyhedron is called a *face-guard geodesic path* or an *FGG path* for short. A polyhedron that has an FGG path is called an *FGG polyhedron*. For a polygon, an *FGG path* is the shortest path connecting two points on the perimeter and passing on the perimeter of the polygon.

Definition 2.3 (FGG number and face-pair FGG number)**.** On a given polyhedron, the maximum number of faces guarded (passed) by a geodesic path is called the *FGG number*. Given a polyhedron and its two faces, the maximum number of faces guarded by a geodesic path whose endpoints are selected from each face is called the *face-pair FGG number*.

2.2 Problems

Following problems are proposed.

Problem 1. FGG path problem
Input polyhedron Π
Output Does Π have a FGG path?

Problem 2. FGG number problem
Input polyhedron Π
Output FGG number of Π

Problem 3. Face pair FGG number problem
Input Polyhedron Π and its faces S_1, S_2
Output face pair FGG number of Π between S_1 and S_2.

3 Results

Theorem 3.1. *Tetrahedra (Fig.2), and triangular prisms are FGG polyhedra, conversely, regular hexahedra (cubes), regular octahedra, regular dodecahedra, regular icosahedra, and rectangular prisms are not FGG polyhedra. Furthermore, the FGG number of a cube is 5.*

Theorem 3.2. *For any positive integer $n \geq 4$, there exists an FGG n-faced polyhedron.*

Lemma 3.3. *A polygon has an FGG path if and only if there is a pair of adjacent edges AB and BC such that the sum of the lengths of AB and BC is larger than the sum of the lengths of the other edges.*

Theorem 3.4. *If a polygon has an FGG path, then a prism whose base is the polygon has a FGG path if the height is large enough.*

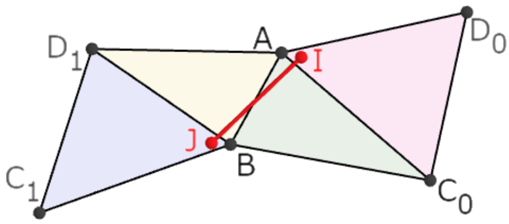

Figure 2: An example of FGG paths of a tetrahedron

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Nonrealizable Planar and Spherical Occlusion Diagrams

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The *art gallery problem* in $\mathbb{R}^3$ asks for the minimum number of guards that can jointly see all points in a nonconvex polyhedron P, where points s and t *see* each other if $st \subset P$. It is well known that guards stationed at vertices do not always suffice, as some points in P may not see any of the vertices [1, Sec. 10.2]. Viglietta [4] recently introduced *spherical occlusion diagrams* (SOD) to analyze the visibility map of such points. Formally, a SOD is a finite nonempty collection of arcs of great circle on the unit sphere satisfying the following axioms: (A1) if two arcs intersect, then an endpoint of one is in the interior of the other, (A2) each endpoint of an arc lies in the interior of another arc, and (A3) all arcs that end in the interior of an arc a lie on the same side of a.

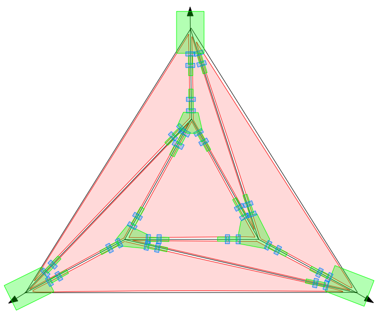
It is not difficult to verify that if a point s in the interior of a polyhedron P (in general position) and s does not see any vertices of P, then its *visibility map* is a SOD. Alternatively, for a *polygonal scene* P (i.e., a finite collection of interior-disjoint polygonal faces in $\mathbb{R}^3$), if a point $s \in \mathbb{R}^3$ does not see any vertices of P, then its visibility map is a SOD. Viglietta [4, version 1] conjectured that every SOD is the visibility map of a point s in some polyhedron P in $\mathbb{R}^3$, that is, every SOD is *realizable* as a visibility map. The main result of this paper is to disprove this conjecture: We construct a SOD that is not realizable.

Our result raises several open problems: Can one recognize efficiently whether a given SOD is a visibility map? If so, can one find a realization efficiently? What is the (combinatorial or topological) complexity of the realization space? Many further open problems are related to SODs, and in general to visibility problems in 3-space, see [3].

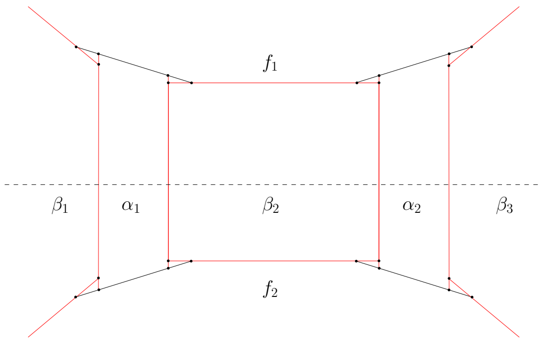
1 Construction

Our construction of a nonrealizable SOD is reduced to *planar occlusion diagrams* (POD), which are defined analogously to a SOD, but with line segments in the plane. Formally, a POD is a finite nonempty collection of line segments and rays (for short, *segments*) in $\mathbb{R}^2$ satisfying the following axioms: (A1') if two segments intersect, then an endpoint of one is in the interior of the other, (A2') each endpoint of a segment lies in the interior of another segment or the boundary of B, and (A3') all segments that end in the interior of a segment ℓ lie on the same side of ℓ. Again, it is not difficult to verify that if P is a polygonal scene in $\mathbb{R}^3$, and its lower envelope does not contain any vertices of P, then the orthogonal projection of the lower envelope to the xy-plane is a POD. We construct a POD that is not realizable in this way.

The key component of the construction is a nonregular triangulation. A triangulation in $\mathbb{R}^2$ is *regular* if it is the orthogonal projection of the lower envelope of a convex polytope [2]. Let $\varepsilon > 0$. We start with the triangulation T in Fig. 1a, which is known to be nonregular, and then modify it in several steps as follows. Each triangle t in T is replaced by a convex polygon $c(t) \subset \mathrm{int}(t)$ that ε-closely follows the boundary of t; these are called *sub-faces*.



(a) Sub-faces in the triangles of a nonregular triangulation.



(b) A gadget between sub-faces f_1 and f_2.

Figure 1: Modification of a nonregular triangulation to construct a planar occlusion diagram.

To complete the construction, we replace every edge between a pair of triangles in T by a sequence of small faces, $\beta_1, \alpha_1, \beta_2, \ldots, \beta_k$, shown in Fig. 1a, where α_i and β_i are trapezoids: Two opposite sides of α_i (resp., β_i) are contained in (resp., contain) some sides of the two sub-faces in the two triangles. A subsequence $(\beta_i, \alpha_i, \beta_{i+1})$ of faces is called a *gadget* (Fig. 1b). These replacement operations yield a POD, satisfying axioms (A1')—(A3').

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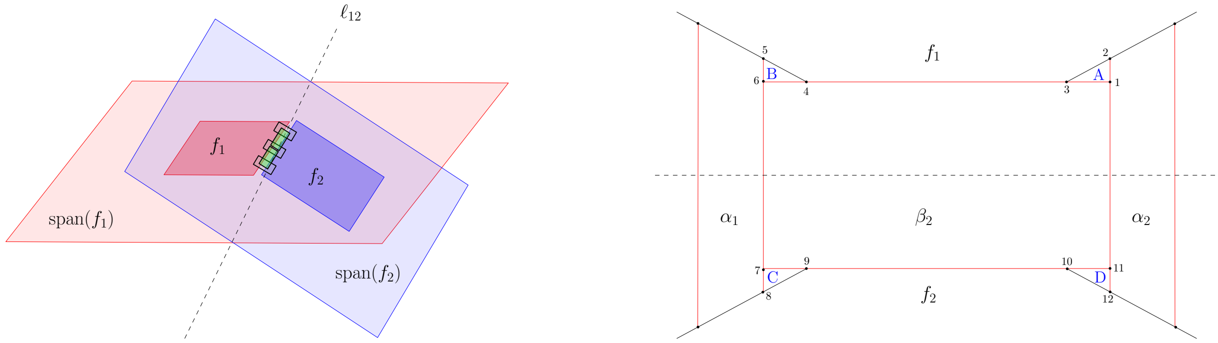
2 Non-Realizability

Suppose, for the sake of contradiction, that the POD constructed above is realizable, that is, it is the orthogonal projection of the lower envelope of some polygonal scene P_ε in $\mathbb{R}^3$. Note that the lower envelope and the orthogonal projection are invariant under orientation-preserving linear transformations that modify only the z-coordinates. In particular, we may assume that the minimum and maximum z-coordinates, resp., of the polygons in P_ε are 0 and 1.

The gadgets ensure that for any pair of adjacent sub-faces, f_1 and f_2, and in any realization, f_1 lies below the plane $\mathrm{span}(f_2)$ and f_2 lies below the plane $\mathrm{span}(f_1)$, with the exception of at most one or two vertices (Lemma 2.1). In particular, the dihedral angle between $\mathrm{span}(f_1)$ and $\mathrm{span}(f_2)$ is convex from below. Furthermore, the orthogonal projection of the line $\ell_{12} = \mathrm{span}(f_1) \cap \mathrm{span}(f_2)$ separates the sub-faces f_1 and f_2 in the POD, with the exception of one or two vertices (Lemma 2.3); see Fig. 2a. Denote by $\mathrm{span}(f_i)^-$ the lower half-space bounded by the plane $\mathrm{span}(f_i)$ for a sub-faces f_i. Then $Q_\varepsilon = \bigcap_{i=1}^{7} \mathrm{span}(f_i)^-$ is a convex polytope. As ε goes to zero, Q_ε converges to a polytope Q whose upper envelope is the triangulation T. However, T is a nonregular triangulation, and it can be neither the lower nor the upper envelope of a convex polytope: a contradiction.



(a) Convex dihedral angle betwee sub-faces f_1 and f_2.

(b) Labelling of vertices in a gadget.

Figure 2

In the remainder of this section, we sketch how the gadgets between sub-faces f_1 and f_2 are used in the proof. Consider a single gadget in Fig. 1b. Note that β_1, β_2, and β_3 occlude the sub-faces f_1 and f_2, and α_1 and α_2 occlude β_1, β_2, and β_3, but are occluded by f_1 and f_2.

Let t be the an endpoint of a segment ℓ_1 in the POD. By axiom (A2'), t lies in the interior of another segment ℓ_2. In the realization P, ℓ_1 and ℓ_2 are edges of some polygons p_1 and p_2 in $\mathbb{R}^3$; and q is the orthogonal projection of two points: a point $p_1(t) \in p_1$ and a point $p_2(t) \in p_2$. Since t is the endpoint of segment $\ell_1 \subset p_1$, then polygon p_2 occludes p_1 at t. Thus the z-coordinate of $p_2(t)$ is less than the z-coordinate of $p_1(t)$; denoted as $p_2(t) \preceq p_1(t)$.

We use above-below relationships to show that in any gadget between sub-faces f_1 and f_2, at least one vertex of f_2 lies below $\mathrm{span}(f_1)$ (and similarly, at least one vertex of f_1 lies below $\mathrm{span}(f_2)$).

Lemma 2.1 *In any realization, $\alpha_1(6) \preceq f_1(6)$ or $\alpha_2(1) \preceq f_1(1)$, and $f_2(8) \preceq f_1(8)$ or $f_2(12) \preceq f_1(12)$; that is, the point $f_2(8)$ or $f_2(12)$ is below the plane* $\mathrm{span}(f_1)$.

A *swirl* in a POD is a cyclic sequence of line segments in which one endpoint of each segment is in the interior of the next segment. Each gadget contains four swirls, denoted by A, B, C, and D in Fig. 2b.

Lemma 2.2 *Assume that the ratio between the diameter of a swirl and the distance between two swirls in a gadget is less than some fixed $\delta > 0$. Then the altitudes of the points in the swirls are less than 10δ.*

It follows that each swirl approximates a single point in a realization. Since the four corners of the polygon β_2 are co-planar, the four swirls around these corners are almost co-planar (and, in particular, cannot be on a saddle surface). This, in turn, determines the position of the line $\ell_{12} = \mathrm{span}(f_1) \cap \mathrm{span}(f_2)$ approximately.

Lemma 2.3 *The vertices of f_1 and f_2 are on opposite sides of ℓ_{12} with an exception of at most two vertices of each.*

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The Two-Squirrel Problem and Its Relatives

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Imagine that two squirrels try to fetch and divide $2n$ nuts to their nests. Since each time a squirrel can only carry a nut back, this naturally gives the following problem: they should travel along the edges of an n-star, centered at the corresponding nest, such that each leaf (e.g., nut) is visited exactly once (in and out) and the maximum distance they visit should be minimized (assuming that they travel at the same speed, there is no better way to enforce the fair division under such a circumstance). See Figure 1 for an illustration.

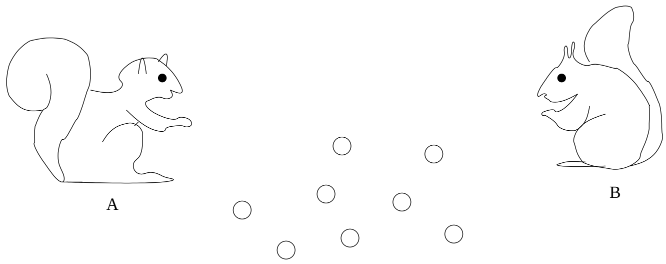

Fig. 1: Two squirrels A and B try to fetch and divide $2n$ nuts.

A star S is a tree where all vertices are leaves except one (which is called the *center* of the star). An n-star is a star with n leaf nodes. When the edges in S carry weights, the weight of S is the sum of weights of all the edges in S. Given two points p, q in the plane, with $p = (x_p, y_p)$ and $q = (x_q, y_q)$, we define the Euclidean distance between p, q as $d(p, q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$. (Note that the coordinates of the input points change the complexities of the problems in some way — see below.)

Formally, the *Two-Squirrel* problem can be defined as: Given a set P of $2n$ points in the plane and two extra point sites c_1 and c_2, compute two n-stars S_1 and S_2 centered at c_1 and c_2 respectively such that each point $p_j \in P$ is a leaf in exactly one of S_1 and S_2; moreover, the maximum weight of S_1 and S_2 is minimized. Here the weight of an edge (c_i, p_j) in S_i is $w(c_i, p_j) = d(c_i, p_j)$ for $i = 1, 2$.

This problem can be thought of as an example for destination and capacity constrained commodity transportation, or as a constrained version of the twomedian problem. A more general version of the problem is when the two squirrels only need to split the $2n$ nuts and each could travel along a Minimum Spanning Tree (MST) of the n points representing the locations of the corresponding nuts, which we call the *Two-MST* problem: Compute a partition of P into n points each, P_1 and P_2, such that the maximum weight of the MST of $P_1 \cup \{c_1\}$ (say T_1) and $P_2 \cup \{c_2\}$ (say T_2), i.e., $\max\{w(T_1\}), w(T_2)\}$, is minimized. Likewise, we can replace 'MST' with TSP (Traveling Salesman Problem) to have the Two-TSP problem. These kind of 'equal-size' constraint is interesting by nature, and has not been considered in geometric optimization (to the best of our knowledge). On the other hand, there have been results on covering a (weighted) graph with stars and trees $[1, 6, 2]$.

It turns out that Two-Squirrel, with rational inputs, is strongly NP-hard under both the Euclidean and L_1 metric. (With integral inputs, it is obviously weakly NP-hard.) The proofs can be directly from a variation of the famous Set-Partition problem $[3, 4]$, namely, (Equal-Size) Set-Partition for Rationals, which is strongly NP-hard with the recent result of Wojtczak [5]. For the approximation algorithms, Two-Squirrel (with rational inputs) admits an FPTAS (note that this does not contradict the known result that a strongly NP-hard problem with an integral objective function cannot be approximated with an FPTAS unless P=NP, simply because our objective functions are not integral). This can be done by first designing a polynomial-time dynamic programming algorithm through scaling and rounding the distances to integers, obtaining the corresponding optimal solutions, and then tracing back to obtain the approximate solutions. For Two-MST and Two-TSP such a dynamic programming based technique does not work, due to the fact that the structure of MST's and TSP's are much more complex than stars with fixed centers c_1 and c_2. Also, for Two-MST and Two-TSP we mainly focus on rational/real coordinates.

We state our main results as follows:

Theorem 1. *Two-MST is strongly NP-hard and can be approximated with a factor of 3.6402.*

Theorem 2. *Two-TSP can be approximated with a factor of* $4 + \varepsilon$.

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A note on odd-graceful trees

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Extended Absract

A *graceful labeling* of a tree T with m edges is a labeling f of the verticies of T with the integers $0, 1, 2, \ldots, m$ such that $\{|f(u) - f(v)| : uv \in E(T)\} = \{1, 2, \ldots, m\}$. A tree is *graceful* if it admits a graceful labeling. A well-known conjecture due to Ringel, Kotzig and Rosa [3, 2, 5] asserts that all trees are graceful.

Later, Gnana Jothi [1] conjectured that all trees admit an odd-graceful labeling, where an *odd-graceful labeling* of a tree T with m edges is a labeling g of $V(T)$ with integers in the set $\{0, 1, 2, \ldots, 2m - 1\}$ such that $\{|g(u) - g(v)| : uv \in E(T)\} = \{1, 3, \ldots, 2m - 1\}$.

As for graceful labelings, many classes of trees (like caterpillars and binary trees, among others) are known to admit odd-graceful labelings.

A graceful labeling f of a tree T is called an *α-labeling* if there is an integer λ_g such that either $f(u) \leq \lambda_g < f(v)$ or $f(v) \leq \lambda_g < f(u)$ for every edge uv of T. It is an easy exercise to show that if f is an α-labeling of a tree T, then an odd-graceful labeling g of T is obtained as follows:

$$g(v) = \begin{cases} 2f(v) & \text{if } f(v) \leq \lambda_g \\ 2f(v) - 1 & \text{if } f(v) > \lambda_g \end{cases}$$

Therefore, if a tree admits an α-labeling, then it also admits an odd-graceful labeling. Unfortunately, not all graceful trees admit an α-labeling (see for instance Fig. 1).

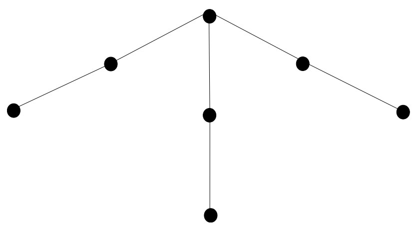

Figure 1: Tree R admits no α-labelings.

A tree T with root r belongs to class Γ if: i) There is a plane representation of T which is symmetric with respect to a vertical line through r, and ii) For each vertex v of T the number of vertices in all branches of T at v is the same. Tree R, above, is in class Γ as well as tree T in Fig. 2. Rivera-Campo [4] proved that all trees in class Γ are graceful.

In this paper we show every tree in Γ also admits an odd-graceful labeling in spite of the fact that many trees in Γ do not admit α-labelings. We also present a class Δ of *edge-centered* symmetric

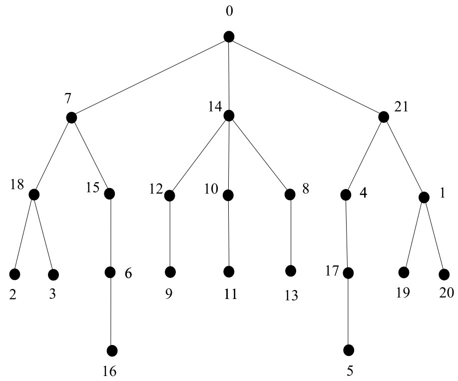

Figure 2: A tree T in class Γ with a graceful labeling.

trees (see Fig. 3 for an example) and prove that all trees in Δ admit an α-labeling and therefore also an odd-graceful labeling.

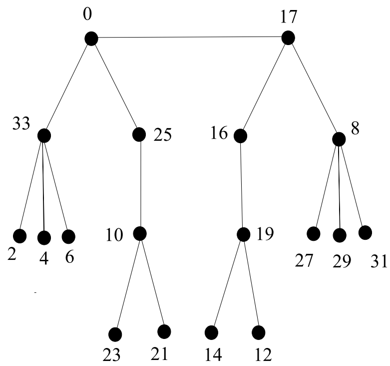

Figure 3: An odd-graceful labeling of a tree in class Δ.

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Pair crossing number, crossing number, and cutwidth

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Abstract

Deciding whether there exists a graph such that its crossing number and pair crossing number are distinct is an important open problem in geometric graph theory. We show that $cr(G) = O(\mathrm{pcr}(G)^{3/2})$ for every graph G; this improves the previous bound by a logarithmic factor. Answering a question of Pach and Tóth, we prove that the bisection width (and, in fact, the cutwidth as well) of a graph G with degree sequence $d_1, d_2, \ldots, d_n$ satisfies $\mathrm{bw}(G) = O(\sqrt{\mathrm{pcr}(G) + \sum_{k=1}^{n} d_k^2})$. The proofs of these results rely on a separator theorem for string graphs, proven by Lee. We also obtain a bound on the pair crossing number of toroidal graphs and graphs that can be made planar by removing few edges.

1 Introduction

We consider only simple undirected graphs.

Given a graph $G = (V, E)$, a *drawing* of G is a representation such that the vertices correspond to distinct points in the plane and the edges are represented by simple continuous curves which go from one endpoint to the other but do not pass through any point representing a vertex. We further assume that no two curves are tangent or share an infinite number of points, and no three curves have a point in common. The *crossing number* of G, $\mathrm{cr}(G)$, is the least number of crossing points amongst all drawings of G. The crossing number is one of the most important and studied measures of non-planarity, and it is also relevant from a practical point of view, particularly in graph visualization and VLSI. Similarly, the *pair crossing number*, $\mathrm{pcr}(G)$, corresponds to the minimum number of pairs of crossing edges in all drawings of G. The systematic study of the pair crossing number was started by Pach and Tóth [10]. Deciding whether $\mathrm{cr}(G) = \mathrm{pcr}(G)$ for all graphs is one of the most tantalizing open problem in geometric graph theory, and a considerable amount of effort has been put into bounding $\mathrm{cr}(G)$ in terms of $\mathrm{pcr}(G)$.

2 Previous work and our main results

Tóth [12] showed that $cr(G) = O(\mathrm{pcr}(G)^{7/4} \log^{3/2} \mathrm{pcr}(G))$ and, using the same technique in conjunction with a result by Lee [6], the bound was later improved to $cr(G) = O(\mathrm{pcr}(G)^{3/2} \log \mathrm{pcr}(G))$ [7], [4]. We strengthen this result by getting rid of the logarithmic factor.

Theorem 1. *For every graph G we have that* $\mathrm{cr}(G) = O(\mathrm{pcr}(G)^{3/2})$.

The *bisection width* of a graph G, denoted by $\mathrm{bw}(G)$, is the least number of edges whose removal disconnects the graph into two (not necessarily connected) subgraphs having no more than $\frac{2}{3}|V|$ vertices each. The bisection width and the crossing number are related by the following inequality (see [11], [8]): If G is a graph with degree sequence $d_1, d_2, \ldots, d_n$, then

$$\mathrm{bw}(G) \leq 1.58\sqrt{16\,\mathrm{cr}(G) + \sum_{k=1}^{n} d_k^2}.$$

This has become one of the most useful tools for studying the crossing number. From now on, we write $\mathrm{ssqd}(G) = \sum_{k=1}^{n} d_k^2$. Pach and Tóth [9] asked whether an analogous result holds for the pair crossing number, namely, is there a constant C such that $\mathrm{bw}(G) \le C(\sqrt{\mathrm{pcr}(G)} + \sqrt{\mathrm{ssqd}(G)})$ for every graph? Almost providing a positive answer to this question, Kolman and Matoušek [5] showed that $\mathrm{bw}(G) = O(\log n \sqrt{\mathrm{pcr}(G) + \mathrm{ssqd}(G)})$.

The *cutwidth* of G, $\mathrm{cr}(G)$, is the least integer ℓ for which there is an ordering $v_1, v_2, \ldots, v_n$ of its vertices such that the number of edges which have one endpoint in $\{v_1, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$ is at most ℓ for every i with $1 \le i \le n-1$. Djidjev and Vrt'o [1] showed that $\mathrm{cr}(G) = O(\sqrt{\mathrm{cr}(G) + \mathrm{ssqd}(G)})$. We prove that this holds for the pair crossing number as well, which settles the question of Pach and Toth mentioned in the previous paragraph.

Theorem 2. *For every graph G, $\mathrm{cr}(G) = O(\sqrt{\mathrm{pcr}(G) + \mathrm{ssqd}(G)})$. Since $\mathrm{cw}(G) \ge \mathrm{bw}(G)$, this implies that $\mathrm{bw}(G) = O(\sqrt{\mathrm{pcr}(G) + \mathrm{ssqd}(G)})$.*

A *string graph* is an intersection graph of a collection of curves in the plane. The aforementioned results rely on a modified version of a separator theorem for string graphs by Lee [6].

Finally, we provide a bound on the crossing number of toroidal graphs[1] and graphs that can be made planar by removing k edges.

Theorem 3. *Let G be a graph of maximum degree Δ. If G is toroidal, then $\mathrm{cr}(G) = O(\Delta^2 \,\mathrm{pcr}(G))$. If G can be made planar by deleting k edges, then $\mathrm{cr}(G) \le k\Delta \,\mathrm{pcr}(G) + k^2$.*

This result performs better than Theorem 1 for a wide variety of graphs, and its proof relies on the techniques developed by Hliněný and Salazar in [3] and [2].

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[1] A graph is *toroidal* if it can be drawn on a torus without crossings.

Edge-intersection graph of induced paths in a graph

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Extended Abstract

This work is parallel to the work of Egawa, Kano, and Tan [8] *On cycle graphs*. For any graph G, the *cycle graph* [4] of G, denoted by $\mathscr{C}(G)$ is the graph whose vertices are the induced cycles of G and where two vertices are adjacent if and only if the associated cycles share at least one edge in common. For any integer $n > 1$, we define $\mathscr{C}^n(G) = \mathscr{C}^{n-1}(\mathscr{C}(G))$. If G has no cycles, we put $\mathscr{C}(G) = \emptyset$ and also we define $\mathscr{C}^0(G) = G$ for convenience. Egawa, *et al.* showed that for any graph G exactly one of the following holds:

(1) There exists an integer $n \geq 1$ such that $\mathscr{C}^n(G) = \emptyset$.

(2) $\lim_{n\to\infty} |V(\mathscr{C}^n(G))| = \infty$.

(3) There exist integers $m \geq 0$, $p > 0$, such that $\mathscr{C}^{n+p}(G) = \mathscr{C}^n(G)$ for all $n \geq m$.

The graphs satisfying (1), (2), (3) are called *cycle-vanishing graphs*, *cycle-expanding graphs*, and *cycle-periodic graphs* respectively. Characterizations of these three classes of graphs were established in their paper. Further results on cycle graphs can be found in the paper of Seema, *et al.* [9]

We introduce here a graph operator $\mathscr{P}$ similar to the graph operator $\mathscr{C}$.

Defnition 1. For a given graph G, the *path graph of* G, denoted by $\mathscr{P}(G)$, is the graph whose vertices are the nontrivial induced paths in G, and where two vertices are adjacent if the corresponding induced paths share at least one common edge. If G has no nontrivial paths, we put $\mathscr{P}(G) = \emptyset$ and refer to this as the *null graph*.

For any integer $n > 1$, and any graph G, we define $\mathscr{P}^n(G) = \mathscr{P}^{n-1}(\mathscr{P}(G))$. For convenience, we set $\mathscr{P}^0(G) = G$.

We show here that for any graph G, exactly one of the following holds:

1. There exists an integer $n \geq 1$ such that $\mathscr{P}^n(G) = \emptyset$.
2. $\lim_{n\to\infty} |V(\mathscr{P}^n(G))| = \infty$.
3. There exist integers $m \geq 0$, $p > 0$, such that $\mathscr{P}^{n+p}(G) = \mathscr{C}^n(G)$ for all $n \geq m$.

Following the terminology of Egawa, *et al.*, we shall call a graph *path-vanishing*, *path-expanding*, or *path-periodic* if G satisfies 1 or 2 or 3, respectively.

The following three theorems characterize these three classes of graphs.

Theorem 1. *A graph G is path-vanishing if and only if every component of G is a complete graph.*

Theorem 2. *The following statements are equivalent.*

(1) G is path-expanding.

(2) G has a component which is neither complete nor isomorphic to $K_n - e$.

(3) G has two induced paths of length 2 sharing a common edge.

(4) G has an induced subgraph isomorphic to one of the graphs $K_{1,3}, K_4 - E(P_3), P_4, C_4$.

(5) G has a component of order n and size less than $\binom{n}{2} - 1$.

Theorem 3. *A graph G is path-periodic if and only if it has at least one component isomorphic to $K_n - e$, and all other components are complete graphs.*

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Fewest Moves Inequality Constraint Puzzles[1]

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Abstract

Inequality constraint games are normal play impartial games where users place available numbers, while respecting existing inequalities on the board. While the 1D and 2D 1-player versions are known to be in P, other versions have yet to be explored. Here, we show that the 2D 1-player fewest moves version is NP-complete, i.e., given an $n \times n$ board (with $\lceil n^2/2 \rceil$ spaces), and an integer $k > 0$, can you place k unique numbers from $\{1, \ldots, \lceil n^2/2 \rceil\}$ such that there are no more available moves? This is true whether the inequalities are strict or not, and with only one direction of inequalities on all left/right comparisons (only $<$'s or $>$'s).

1 Introduction

An inequality constraint puzzle is a game where 1-player plays on an $n \times n$ checkered board, placing $\lceil n^2/2 \rceil$ numbers in spots where inequalities are satisfied. The numbers may only be used once (Fig. 1a). Although it could be more general, for any north/south comparison, we use north on left, south on right.

Problem 1.1 (Inequality Constraint Game)**.** Given an $n \times n$ checkerboard with $\lceil \frac{n^2}{2} \rceil$ spaces and inequalities (inequality constraint board), and a $k \in \mathbb{N}$, can k numbers be placed from universe U?

There are many variations to the 1-player game, such as non-unique numbers, a limited universe (only 0 or 1), pattern objectives (four consecutive numbers or four 0's in a row), specific inequality patterns, more placement constraints (a row must sum to less than a value), etc. There are also the obvious extensions to 2-player normal-play impartial (or partizan) combinatorial games.

The 1D 1-player version is simple (see [4]). The 2D 1-player version was proven to be in P if all numbers must be placed, regardless of the inequalities used [3]. A similar generalized version has been explored where 4-sided tiles are used with a different number on each side [1]. If the tiles have the same number on all sides, the models are similar, but slightly different (in ICG, an inequality used to compare a north/south pair is also used to compare a west/east pair). There is also considerable combinatorial work counting the permutations of numbers given a sequence of inequalities (Up-Down Sequences) [2].

2 Unique Fewest Moves 1-player ICG is NP-complete

We show that, if a number may only be placed once, deciding whether k numbers can be placed is NP-complete. We use a reduction from a restricted version of Set Element-Label Cover, which we show is hard by a reduction from Vertex Cover in cubic graphs (VC3) [5] following a similar reduction as [6].[4] Given space constraints, the ideas are outlined without the formal proof.

Problem 2.1 (Unique Fewest Moves Inequality Constraint (k-UIC))**.** Given an Inequality Constraint board B with strict inequalities, a universe of numbers $U = \{1, \ldots, n\}$, and an integer $k > 0$. Can k unique numbers from U be placed on B such that no more moves are possible?

Problem 2.2 (k-Set Element-Label Cover (SELC))**.** Given sets $S = \{s_1, \ldots, s_m\}$ over some universe $U = \{x_1, \ldots, x_n\}$ and $k \in \mathbb{N}$. A Set Element-Label Cover is an injective mapping $M : S' \to U'$ where $S' \subseteq S$ and $U' \subseteq U$, and $M(s_i) \in s_i$. Does there exist a maximal partial cover such that $|S'| \leq k$?

Theorem 2.3 (SELC is NP-complete)**.** *Set Element-Label Cover is NP-complete even with sets of size 2 and when an element may occur in at most 4 elements.*

Proof. For clarity, we reduce to a graph variant of Set Element-Label Cover where, for an undirected simple graph $G = (V, E)$, let $S = E$ and $U = V$ such that each edge is defined as $\{v_i, v_j\}$ where $v_i, v_j \in V$ and $i \neq j$. This is essentially uniquely labeling edges with an incident vertex, and the labels can not be used more than once. Using only k edges for the bipartite labeling graph (Examples in Fig. 1) is equivalent to finding a minimal maximum matching.

Given a cubic VC3 graph $G = (V, E)$, create a graph $G' = (V', E')$ where every $v_i \in V$ is replaced with the 3-clique $u_i, m_i, p_i \in V'$ and edges $\{u_i, m_i\}, \{m_i, p_i\}, \{p_i, u_i\} \in E'$ as a vertex gadget (Fig. 1b). WLOG, for an edge $\{v_i, v_j\} \in E$, we add one vertex $z_{ij} \in V'$ and edges $\{X_i, Y_j\}, \{X_i, z_{ij}\}, \{z_{ij}, Y_j\} \in E'$ where X, Y are one of the u, m, p vertices from v_i, v_j, respectively (Fig. 1b). G' has max degree 4 and $|V'| = 3|V| + |E|$ and $|E'| = 3|V| + 3|E|$.

We show a bound on the minimum number of labels assigned with no other labeling possible. Let $\beta_{G'}$ be the size of a minimum labeling on G', and C_G be a minimum vertex cover of G. We assert that $\beta_{G'} = 2|V| + 2|E| - 2|C_G|$, and thus, G has a vertex cover of size k if and only if G' has a minimal

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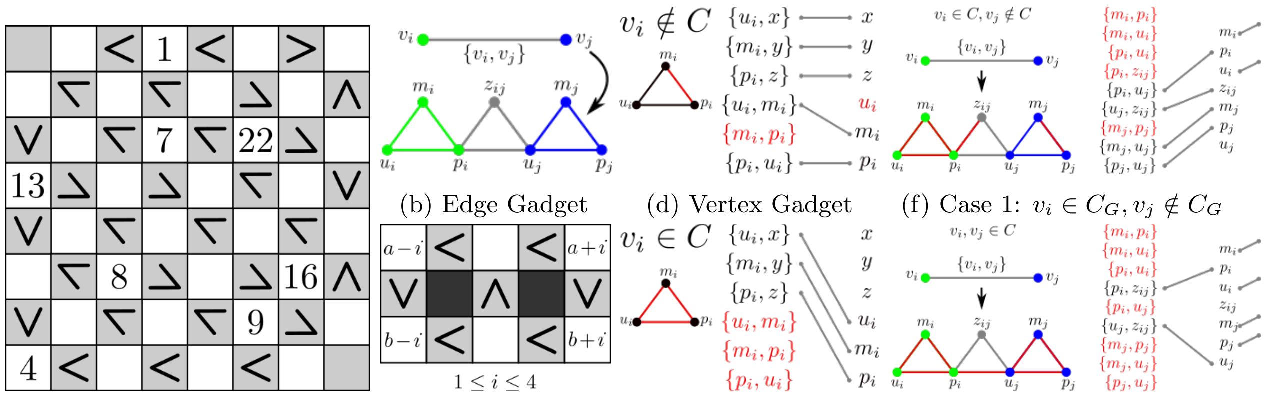
[4]VC3 usually refers to Vertex Cover on graphs of max degree 3, but we use it as cubic graphs for convenience.



(a) Example game (b) Edge Gadget (c) UIC Mapping (d) Vertex Gadget (e) Vertex Gadget (f) Case 1: $v_i \in C_G, v_j \notin C_G$ (g) Case 2: $v_i, v_j \in C_G$

Figure 1: (a) Example game with 32 blanks, thus the pieces are $\{1, \ldots, 32\}$. Given the board with 8 numbers placed, can you fill in 10 numbers s.t. there are no more valid moves? (b) Edge gadget connected to two vertex gadgets. (c) k-UIC gadget for two numbers a, b with $a > b$. Each a, b can be in up to 4 gadgets- each using a different i ($1 \le i \le 4$). (d-e) Vertex gadget and the mapping in SELC based on whether v is part of the vertex cover. x, y, z are vertices of adjacent vertex gadgets. (f-g) Edge gadget labeling based on whether one or both vertices are in the vertex cover. Red indicates an unlabeled edge.

maximal labeling of size $2|V| + 2|E| - 2k$ ($|V| + |E| + 2k$ unlabeled edges). The basic idea is, if a vertex $v_i \notin C_G$, then the edges in the vertex gadget can be labeled by the nodes u_i, m_i, or p_i. If $v_i \in C_G$, then u_i, m_i, and p_i are used to label edges to neighboring nodes. Note that a minimum labeling is not unique.
Obs. 1. A minimum labeling on a vertex gadget (with no other edges) labels 2 edges. (Fig. 1d).
Obs. 2. A minimum labeling on a vertex gadget with vertices of degree 3 can label 0 edges. (Fig. 1e).
Obs. 3. For two adjacent vertices $v_i, v_j \in V$. The edge gadget between the two vertex gadgets in G' can always have one unlabeled edge.
Obs. 4. For the subgraph of two adjacent vertex gadgets connected by an edge gadget, if one is part of a vertex cover, the minimum labeling labels 4 edges (Fig. 1f).
Obs. 5. For the subgraph of two adjacent vertex gadgets connected by an edge gadget, if both are part of a vertex cover, the minimum labeling labels 2 edges (Fig. 1g).

Obs. 4-5 note how to always give a minimum labeling for the 3-clique on the edges with 2 vertex gadgets. The assertion is that any deviation from this labeling will not yield a smaller labeling (since it is not unique). Finally, the problem is in NP since k assignments can be guessed. □

Theorem 2.4. *k-UIC is NP-complete.*

Proof. Given an instance of SELC with (S, U, k), we enumerate U and let $U' = \{1, \ldots, (9|U| + 4)\}$. Further, take S' to be the pairs of enumerated values times 9, i.e., if $s_i = \{x_7, x_{11}\}$, then $s'_i = \{63, 99\}$. Now, since each $s \in S'$ is simply two numbers, say $a, b \in U'$ and WLOG, assume $a > b$. We create $|S'|$ gadgets as shown in Figure 1c. Either a or b may be placed in their space, but not both, nor any numbers besides a or b. Since the game only allows a number to be placed once, this is equivalent to picking one of the numbers as the mapping for the set (gadget). Note that number a may occur in up to 4 pairs, so the multiplication by 9 lets us use unique numbers for each gadget a is in ($a - i << a + i$ for $1 \le i \le 4$). Let $q = \lceil \sqrt{|U'|}\, \rceil$, g_i represent the 5×3 gadget for s_i, and X represent a 3×3 gadget used for spacing that is nonplayable (prefilled with numbers). The gadgets are laid out in a grid with the first row being $g_1, X, g_2, X, \ldots, X, g_q$, then a spacing row of X gadgets. In general, gadget row r is $g_{q(r-1)+1}, X, \ldots, X, g_{q(r-1)+q}$. This grid has size $(5q + 3(q-1)) \times (3q + 3(q-1))$, which we fill out to a square with nonplayable prefilled area. Let $T = 5q + 3(q-1)$, so we have a $T \times T$ board with $U' = \{1, \ldots, \lceil T^2/2 \rceil\}$. Finally, the problem is in NP since k numbers can be guessed with location. □

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Reachability in Population Protocols is PSPACE-Complete[1]

Bin Fu[2], Tim Gomez[3], Elise Grizzel[2], Andrew Rodriguez[2], Robert Schweller[2], Tim Wylie[2]

Abstract

Recently developed motion planning techniques lend powerful tools to formulate insights across numerous research fields. This work investigates the application of motion planning gadgets to reachability problems for the small agent-based Population Protocols model of Distributed Computing and, in turn, the nearly equivalent research fields of Vector Addition Systems [7], Petri-Nets [5], and Chemical Reaction Networks [8]. By two reductions derived from the edge encoding of graphs into the interactions of a Population Protocol, we show that general reachability is PSPACE-complete and is NP-hard in Feed-Forward versions.

1 Introduction

Population Protocols are a model of distributed computation where there is a discrete count of agents that interact pairwise and change their states according to a set of rules. This model has been shown to be able to compute exactly semi-linear sets [1]. Here, we focus on the Reachability problem and show it is PSPACE-complete. For the case of Feed-Forward Population Protocols, where there is an ordering on the rules such that agents transition in a directed way [3], we show it is NP-hard.

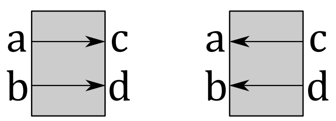
Definition 1.1 (Population Protocol)**.** A Population Protocol $P = (\Lambda, R)$, where Λ is the set of *states* and $R : \Lambda^2 \to \Lambda^2$ is a partial function mapping pairs of states to other pairs, which are called the *rules*. For convenience, the rules of R, $R(x_1, x_2) = (y_1, y_2)$, are usually denoted as $x_1 + x_2 \to y_1 + y_2$. We use $x_1 + x_2 \rightleftharpoons y_1 + y_2$ to denote that a rule is reversible. A configuration c is a multiset over the elements of Λ. Each element of c is called an agent.

Definition 1.2 (Reachability)**.** Given a Population Protocol P and two configurations c and c', does there exist a sequence of rule applications (replacing a pair of agents (a, b) within the domain of R with $R(a, b)$) to reconfigure c to c'?

Proper Chemical Reaction Networks (CRNs) can be seen as a generalization of Population Protocols where there is no bound on the size of the rules, and only that they must keep the same number of agents. Previous work on the Reachability problem in CRNs shows it is PSPACE-complete if using rules up to size 5 (up to 5 agents in a rule) [8]. Our result improves this to rules of only size 2. When considering only rules of size 1, i.e., agents that cannot interact and just change states, the reachability problem is equivalent to a directed path and thus is NL-complete.

2 Reachability is PSPACE-complete

We reduce from the reconfiguration 2-Toggle gadgets presented in [2, 4]. A motion planning system consists of a set of gadgets with labeled ports, a set of wires denoting port connections[4], and an initial signal location. Each gadget (Fig. 1a) has 2 parallel directed tunnels. A gadget's *direction* is the direction of its indicated tunnel arrows. Completing a traversal of a tunnel by a signal toggles the direction of the gadget and associated tunnels. Given an initial gadget system and target configuration, the reconfiguration problem asks if there exists a traversal of the gadgets such that the system reconfiguration meets the target configuration.

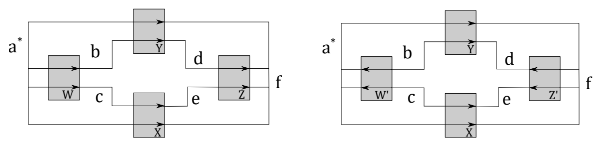


(a) State G (b) State G'

$a + G \rightleftharpoons c + G'$

$b + G \rightleftharpoons d + G'$

(c) Rules for Gadget



(d) Initial $\{a, W, Y, X, Z\}$ (e) Target $\{a, W', Y, X, Z'\}$

Figure 1: (a-b) Two states of a 2-Toggle gadget. (c) Bidirectional rules which implement a single gadget. (d) Example initial system and configuration. (e) Target system and configuration.

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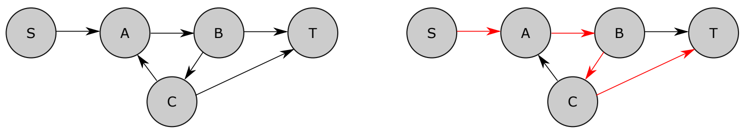
[4]In [2, 4], they define wires using a connection graph, however, the definitions are equivalent.



$$\begin{array}{lllll} \{S_0^* & A & B & C & T\} \\ \{S^v & A_1^* & B & C & T\} \\ \{S^v & A^v & B_2^* & C & T\} \\ \{S^v & A^v & B^v & C_3^* & T\} \\ \{S^v & A^v & B^v & C^v & T^4\} \end{array}$$

(a) Graph (b) Hamilton Path (c) Configurations

Figure 2: Our starting configuration $c = \{S_0^*, A, B, C, T\}$. Our goal configuration is $c' = \{S^v, A^v, B^v, C^v, T^4\}$. In order to reach the target, each vertex must be changed to the visited state and the T must be the last vertex.

We divide our states into two sets. The first set contains the *location* states, denoting each of the wires where the signal might be (states $\{a, b, c, d, f\}$ in the example). Reachable configurations always contain exactly one agent in a single location state that represent the signal's location in the system. The second set contains the *gadget* states that correspond to each gadget of the system and its current direction (states $\{W, W', X, X', Y, Y', Z, Z'\}$ in the example). Each reachable configuration has one agent for each gadget to represent that gadget's current direction.

The rule set contains rules to facilitate traversal of the gadget system. Each rule takes in one agent state and one gadget state if the agent can traverse the gadget in its current state. The location state changes to the location prescribed by the relevant gadget after the traversal, and the gadget state switches to its toggled version. We note these rules are reversible, meaning for every rule the opposite rule also exists. The provided Population Protocol reaches the target agent configuration if and only if the given toggle system is reconfigurable into the destination system, yielding the following theorem:

Theorem 2.1. *The Reachability Problem in Population Protocols is PSPACE-complete.*

3 Reachability in Feed-Forward Systems is NP-hard

Feed-Forward Population Protocols permit an ordering on the rules such that a state occurring in the output of the i^{th} rule cannot be used in the input for the j^{th} rule for $i > j$. We show NP-hardness for such systems by a reduction from the Hamiltonian Path problem with vertices of in and out degree of at most 2 [6]. For each vertex X in the graph $G = (V, E)$, we include $2 + |V|$ states: an initial state X, a visited state X^v, and $|V|$ signal states X_i^*. We encode the edges of the graph in the rules as follows,

$$\text{Rules } R = \left\{ \begin{array}{c|c|c} S_i^* + A \to S^v + A_{i+1}^* & B_i^* + C \to B^v + C_{i+1}^* & B_i^* + T \to B^v + T_{i+1}^* \\ A_i^* + B \to A^v + B_{i+1}^* & C_i^* + A \to C^v + A_{i+1}^* & C_i^* + T \to C^v + T_{i+1}^* \end{array} \right\}$$

An example reduction is shown in Figure 2. Given this reduction, we see that any Hamiltonian path of graph G has a corresponding sequence of rules that end with every vertex, other than T, represented with the *visited* state, and T represented with the signal state matching the count of the vertices. Conversely, the only way to reach such a configuration corresponds directly to a Hamiltonian walk of G from S to T, yielding the following result:

Theorem 3.1. *The Reachability Problem in Feed-Forward Population Protocols is NP-hard, even if each state appears on the left side of at most two rules, and on the right side of at most two rules.*

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Reachability in Pikmin Cave Sublevels is PSPACE-Complete

Andrew Rodriguez[1]

Abstract

Pikmin 2 is a real-time strategy and puzzle video game developed by Nintendo. I prove that Pikmin cave sublevel reachability is PSPACE-complete. The reduction is constructed from a relatively recent motion-planning-through-doors framework shown in [Ani et al. 2020].

1 Introduction

Pikmin 2 is a 3D real-time strategy and puzzle video game developed by Nintendo originally released for the GameCube and later ported to the Wii. You lead plant-like creatures called Pikmin to aid in your quest to pay off a large debt by collecting treasure while surviving the harsh environment filled with hazards and enemies. There are various species of Pikmin each with their own strengths and weaknesses. Pikmin 2 introduced underground caves which are dungeons with various sublevels and a boss. I focus on the decision problem of reachability in a generalization of these cave sublevels.

Definition 1.1 (Pikmin Cave Sublevel Reachability)**.** Given a Pikmin cave sublevel, a starting location and an exit, can the player reach the exit from the starting location?

I show that this problem is PSPACE-complete via a reduction from 1-player motion planning with the symmetric self-closing door shown in [1]. Section 2 gives more detail about the mechanics of the game and highlights the important features used in the reduction. Section 3 shows the gadget construction and proves the problem's PSPACE-completeness.

2 Pikmin 2 Mechanics

The player takes control of one of the protagonists; Captain Olimar, Louie or The President. While you only control one at a time, you can switch who you are controlling to take advantage of multi-tasking, but this feature isn't used in this paper. I will refer to the player as Captain Olimar for the rest of this paper. Olimar can walk around the map but has no ability to jump or climb around the environment. His biggest power is being able to lead Pikmin and direct them to complete certain tasks. Pikmin initially start idle waiting on Olimar's commands and will continue to stay where they are. Olimar can blow his whistle to call nearby Pikmin and have them become part of his party and follow behind him. The whistle has an area of effect which any Pikmin outside of that area won't hear and continue to stay idle.

Under the leadership of Captain Olimar, Pikmin can interact with the world and help Olimar and his party access certain areas not normally accessible. Pikmin can build bridges across bodies of water, destroy hazards, fight enemies and interact with various other obstacles in Olimar's way. Olimar can direct Pikmin at certain tasks by throwing them at the target. Pikmin will begin to interact with their target until the task is complete and sit idle until called again.

An important obstacle used in the reduction is a seasaw block. Seasaw blocks come in pairs and act like a scale that can grant Olimar and his party access to higher elevation. Standing on the low block, Olimar can lift himself and his party by throwing Pikmin on the high block until it is heavier than the low block, swapping the elevations of the blocks. This results in some Pikmin being left behind. In Pikmin 2 specifically, Olimar does not affect the weight of the blocks. This makes it possible to raise a lone Olimar up by throwing a single Pikmin on the high block.

3 Pikmin Cave Sublevel Reachability is PSPACE-Complete

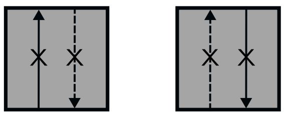

Figure 1: The two states of the symmetric self-closing door from [1]. Each arrow is a tunnel connecting two entrance/exit locations. Dotted line indicates that the tunnel is closed. Solid line indicates that the tunnel is open. The 'X' indicates that the tunnel will close after traversal. The gadget in Figure 2 starts in the left state.

In this section I prove that Pikmin Cave Sublevel Reachability is PSPACE-complete. [1] shows that we need only to construct a symmetric self-closing door gadget and show how to connect these doors to show

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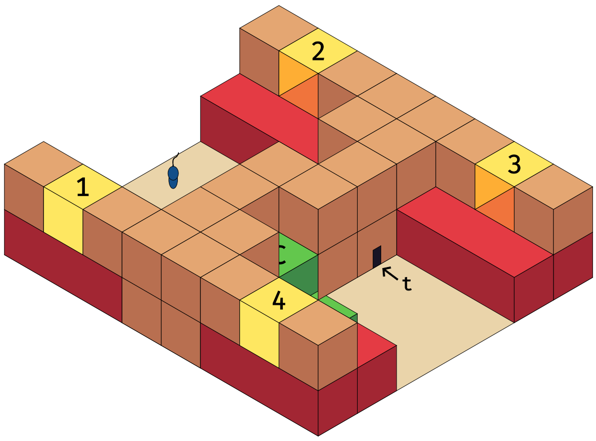
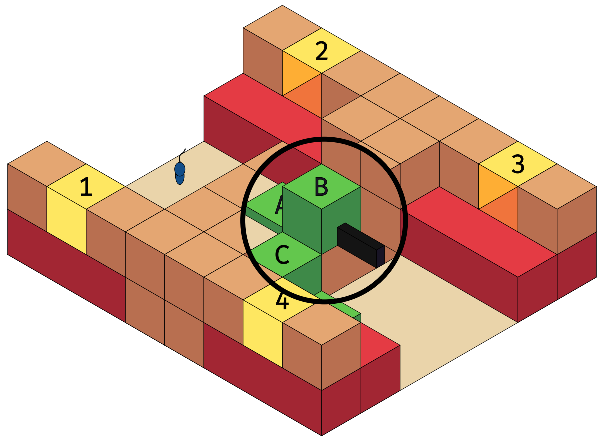
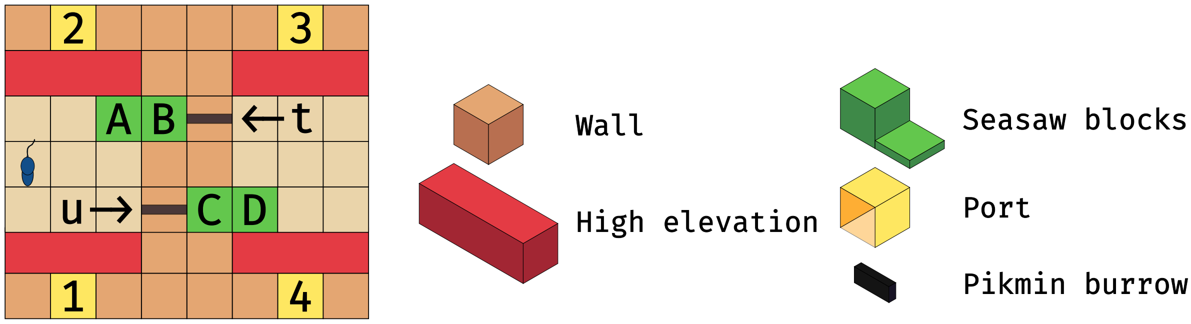


(a) Isometric view.



(b) Isometric view exposing the Pikmin burrow t.



(c) Top-down view.

Figure 2: Various views of the symmetric self-closing door gadget in Pikmin 2.

PSPACE-hardness. The gadget we are going to be constructing in Pikmin 2 can be seen in Figure 1 while the implementation can be seen in Figure 2. It is constructed of two pairs of seasaw blocks, varying elevation, a single Pikmin of no specific species and small burrows big enough for only the Pikmin to walk through. The gadget is split into two tunnels, divided by the center wall, each tunnel named after the ports available on each side; $1 \to 2$ on the left side and $3 \to 4$ on the right. One block from each seasaw pair is designed to be flush with the center wall.

Theorem 3.1. Pikmin Cave Sublevel Reachability *is PSPACE-complete.*

Proof. Containment is given by nondeterministically making moves until we reach the exit, putting the problem in NPSPACE. Using Savitch's theorem, our NPSPACE problem is also in PSPACE. Hardness is shown by reducing from 1-player motion planning with the symmetric self-closing door shown in Figure 2.

Olimar starts in $1 \to 2$ by entering from port 1, and drops down off the high elevation. He then calls the single Pikmin and continues to the seasaw blocks A and B. While standing on A, he throws the Pikmin onto B, lifting him up giving him access to port 2. The Pikmin is now trapped and Olimar can no longer access the Pikmin in $1 \to 2$, switching the state of the gadget. Olimar can repeat the process for $3 \to 4$. Olimar enters from port 3 and drops down from the ledge. When Olimar calls the Pikmin with his whistle, it will traverse through the Pikmin burrow t until he meets with Olimar. Olimar can then use seasaw blocks C and D to reach port 4.

Connecting the door gadgets together is simple as Pikmin 2 is 3D. As long as the terrain isn't too steep, Olimar can simply walk from door to door. ☐

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Weights of Convex Quadrilaterals in Weighted Point Sets

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Extended Abstract

Let P be a set of points in the plane. P is said to be *in general position* if no three of its elements are collinear. All point sets considered in this talk are in general position in the plane. We say P *contains* a convex k-gon if P contains k elements that are vertices of a convex k-gon.

Erdős and Szekeres [5] proved that for any integer $k \geq 3$, there is an integer $N(k)$ such that any set of at least $N(k)$ points contains a convex k-gon. In 1984, Erdős [4] asked the minimum number $\mathrm{conv}_k(n)$ of convex k-gons contained in a point set with n elements. In particular, for $k = 4$, this problem is equivalent to the problem of determining the *rectilinear crossing number* of K_n, and has been studied extensively for a long time [7, 8, 3, 6, 1, 2].

P is called a *weighted point set* if each point is assigned a number called a *weight*. We denote by $\mathcal{P}(n)$ the collection of weighted point sets P with n elements each of which receives a different weight in $\{1, 2, \ldots, n\}$. For a polygon Q with vertices in a weighted point set P, we denote by $w(Q)$ the sum of the weights of its vertices.

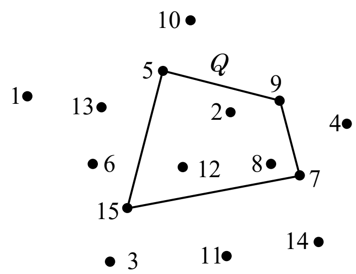

Figure 1: An example of $P \in \mathcal{P}(15)$ and a convex quadrilateral Q with $w(Q) = 5 + 7 + 9 + 15 = 36$.

Obviously, for any $P \in \mathcal{P}(n)$, $n \geq 3$, and for any integer k between $1+2+3 = 6$ and $(n-2)+(n-1)+n = 3n-3$, there exists a triangle T with $w(T) = k$. However, convex quadrilaterals in a weighted point set $P \in \mathcal{P}(n)$, $n \geq 4$, do not necessarily have all integers between $1+2+3+4 = 10$ and $(n-3)+(n-2)+(n-1)+n = 4n-6$ as their weights. For example, the point set $P \in \mathcal{P}(6)$ shown in Figure 2 contains three convex quadrilaterals, but they have the same weight 14. The point set $P \in \mathcal{P}(n)$ shown in Figure 3 does not have any convex quadrilateral with odd weight.

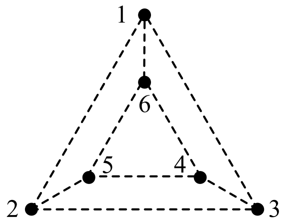

Figure 2.

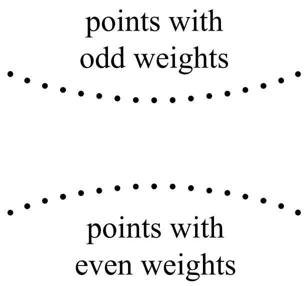

Figure 3.

Let $f(P)$ denote the total number of different weights of convex quadrilaterals contained in $P \in \mathcal{P}(n)$, and let $F(n) = \min_{P \in \mathcal{P}(n)} f(P)$. The point set shown in Figure 3 implies that

$$F(n) \leq 2n - 7 \quad \text{for } n \geq 4.$$

In this talk, we show the following theorem:

Theorem 1 $\quad F(n) \geq n - 5$.

We also show the following result on weights of *empty* triangles, i.e., triangles that have vertices in $P \in \mathcal{P}(n)$ and do not contain points of P in their interiors. Let $g(P)$ denote the total number of different weights of empty triangles contained in P, and let $G(n) = \min_{P \in \mathcal{P}(n)} g(P)$.

Theorem 2 $\quad \left\lceil \frac{n-1}{12} \right\rceil \leq G(n) \leq 2n - 5 \quad$ *for* $n \geq 3$.

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Numerically balanced dice on isohedral polyhedron

Hiro Ito* , Shunsuke Kanaya† , and Risa Tamechika‡

Abstract

Bosh et al., proposed an idear of numerically balanced dice and presented a icosahedral numerical dice in 2018. In this study, we try to extend their idea on other isohedral polyhedra. This is a progress report of our research.

1 Introduction

Even though we commonly use dice forming regular hexahedra, there are dice forming other regular polyhedra and nonregular polyhedra that have high symmetry. It is well known that the sum of the numbers on each pair of the opposite sides of a hexahedral die is fixed to seven. By extending this constraint Bosh et al. [1] proposed new constraints on numerical symmetries, balanced vertices and balanced faces, and presented a icosahedron die satisfying all of the three constraints, including the opposite numbers convention. We are trying to apply these constraints to other polyhedra [3][4]. This is a progress report of our research. We are dealing with all regular polyhedra, all Catalan's polyhedra, and other twelve isohedral polyhedra. First we found that any of these polyhedra never forms a die satisfying all of the three constraints. Next restricting the conditions two or one, we are trying to find dice satisfying the conditions.

2 Isohedral polyhedron

A polyhedron P is isohedral if, given any two faces F_1 and F_2 of P, there exists a symmetry $\sigma \in S(P)$ such that $\sigma(F_1) = F_2$, that is, σ maps one of the given faces onto the other [2]. An isohedral polyhedron is also called a face-transitive polyhedron, or simply an isohedron. Due to its property, it is suitable for designing fair n-sided dice.

3 Numerically balanced dice

Let $P = P_{n,m}$ be an n-hedron whose faces are all congruent m-gons. Throughout this report, $P_{n,m}$ is sometimes expressed as P for notational simplicity. Let V_P and F_P be the set of vertices and faces of P, respectively. We consider a bijection $x : F_P \to \mathbb{N}_n$ from F_P to a set $\mathbb{N}_n = \{1, \ldots, n\}$ of integers from 1 to n. An n-sided die is represented by a pair $(P_{n,m}, x)$. Bosch et al. proposed an idea of "numerically balanced dice" as an ideal arrangement of numbers for dice [1], that is, an n-sided die is called a numerically balanced die when all three constraints, defined as follows, are met.

For showing the definitions of the constraints, we introduce some notations: For a vertex $v \in V_P$, the set of faces meeting at v is denoted by $F(v)$. For a face $f \in F_P$, $o(f)$ denotes the face on the opposite side if exists and $F(f)$ denotes the set of faces adjacent to f. For a set F of faces, the sum of the numbers on the faces in F is denoted by x_F, i.e., $x_F := \sum_{f \in F} x(f)$. In addition, $x(f)$ may be expressed as x_f or sometimes more simply as f unless no misunderstanding arises.

- **Opposite Numbers Convention**

  For any face $f \in F_P$, the following equation holds.

$$x_f + x_{o(f)} = n + 1.$$

- **Balanced Vertices**

  For any vertex $v \in V_P$, the following inequality holds.

$$\left| x_{F(v)} - \frac{|F(v)|(n+1)}{2} \right| < 1.$$

- **Balanced Faces**

  For any surface $f \in F_P$, the following inequality holds.

$$\left| x_{F(f)} - \frac{m(n+1)}{2} \right| < 1.$$

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4 Current Results

The current results are shown in Tables 1 and 2. The abbreviations O, V, and F stand for the constraints of opposite numbers, balanced vertices, and balanced faces, repectively. In addition, ✓ means that there is an arrangement that satisfies the constraint, a number means the number of arrangements that satisfies the constraint, (✓) means that there is an arrangement that satisfies the constraint in the polyhedron that does not have a pair of opposite faces, and — means that there is no pair of opposite faces. (0) indicates that the result is trivially derived from the result on the left (V only, F only, etc.).

Table 1: Current achievements on regular polyhedra and Catalan solids

	O	V	F	O,V	V,F	F,O	O,V,F
Regular tetrahedron	—	0	0	0	0	0	0
Regular hexahedron	✓	0	2	0	0	2	0
Regular octahedron	✓	6	0	0	0	0	0
Regular dodecahedron	✓	0	0	0	0	0	0
Triakis tetrahedron	—	✓	0	(✓)	0	0	0
Rhombic dodecahedron	✓	0	0	0	0	0	0
Regular icosahedron [1]	✓	✓	✓	✓	✓	✓	✓
Triakis octahedron	✓						0
Tetrakis hexahedron	✓		0		0	0	0
Deltoidal icositetrahedron	✓			0	0		0
Pentagonal icositetrahedron	—	0		0	0		0
Rhombic triacontahedron	✓				0		0
Hexakis octahedron	✓		0		0	0	0
Triakis icosahedron	✓				0		0
Pentakis dodecahedron	✓						0
Deltoidal hexecontahedron	✓				0		0
Pentagonal hexecontahedron	—	0		0	0		0
Hexakis icosahedron	✓						0

Table 2: Current achievements on other isohedral polyhedron

	O	V	F	O,V	V,F	F,O	O,V,F
Bipyramid, dipyramid up-down skewed, dipyramid in-out skewed ($n=3$)	n : Even ✓ n : Odd —	2	2	(✓)	0	(✓)	(0)
Same as above ($n=4$)	Same as above	6	0	0	(0)	(0)	(0)
Same as above ($n=5$ or more)	Same as above	✓	0		(0)	(0)	(0)
Trapezohedron, Trapezohedron with asymmetric sides	n : Even — n : Odd ✓	0	0	(0)	(0)	(0)	(0)
Isosceles tetrahedron	—	0	0	(0)	(0)	(0)	(0)
Scalene tetrahedron	—	0	0	(0)	(0)	(0)	(0)
Octahedral pentagonal dodecahedron	✓	0	0	(0)	(0)	(0)	(0)
Tetragonal pentagonal dodecahedron	✓	0	0	(0)	(0)	(0)	(0)
Deltoidal dodecahedron	✓	0	0	(0)	(0)	(0)	(0)
Hexakis tetrahedron	✓		0		(0)	(0)	(0)
Dyakis dodecahedron	✓			0	0		(0)

5 Summary and future work

We still have several unresolved polyhedra regarding balanced vetices and balanced faces. To fix them is a future work. For this purpose, we consider it is necessary to generalize and integrate the proofs. Currently, we are trying to express a part of a geometric development by a graph. It is also an issue to cont the number of arrangements satisfy the constraints for each combination of a polyhedron and constraints.

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Polytopes with low excess degree

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Abstract for JCDCG3
We consider some graph theoretical properties of polytopes, in particular relationships between the number of vertices and the number of edges.

Any vertex of a d-dimensional polytope must have degree at least d. It is natural to define the *excess degree* of a vertex as its degree minus the dimension, and the excess degree *of a polytope* as the sum of the excess degrees of its vertices. The excess degree of a polytope is routinely checked to equal $2f_1 - df_0$, where f_0 and f_1 are the numbers of its vertices and edges.

We show that having low excess degree (less than or not much more than the dimension) imposes some strong restrictions on the structure of a polytope. Some sample results include the following.

1. A d-polytope with excess 0 (i.e., a simple polytope) must satisfy $f_0 \in \{d+1, 2d, 3d-3, 3d-1\}$ or $f_0 \geq 4d-8$.
2. There are no d-polytopes with excess degree in the range $[1, d-3]$.
3. A d-polytope with excess $d-2$ must satisfy $f_0 \in \{d+2, 2d-1, 2d+1, 3d-2\}$ or $f_0 \geq 3d$ (unless $d = 4$). Moreover, there is either a single vertex with excess $d-2$, or there are $d-2$ vertices with excess 1.
4. If a d-polytope has excess degree $d-1$, then all nonsimple vertices have the same degree, and either $d = 3$ or $d = 5$.
5. If a d-polytope has excess d, and $d \neq 3, 4$ or 6, then there are d nonsimple vertices each with excess degree 1, and either $f_0 = d+2$, $f_0 = 2d+1$ or $f_0 \geq 3d$.
6. For $d \geq 8$, there is no d-polytope with excess $d+1$.

Assertion 1 is well known. It can be deduced most easily from the g-theorem, but the Lower Bound Theorem is actually sufficient. Moreover, one can characterise the polytopes with each of these values of f_0.

The following observation is the starting point for this work. Suppose two facets F_1 and F_2 of a d-polytope P intersect in a face K which is not a ridge. If k is the dimension of K, then every vertex in K has at least $d-1-k$ neighbors

in $F_1 \setminus K$, at least $d-1-k$ neighbors in $F_2 \setminus K$ and at least k neighbors in K. Thus each such vertex has excess degree at least $d-k-2$. Since K has at least $k+1$ vertices, the excess degree of P is at least $(k+1)(d-k-2)$, and this expression is at least $d-2$ for k in the range $[0, d-3]$. This almost establishes Assertion 2; the case when every two facets are either disjoint or intersect in a ridge requires separate treatment.

Building on the idea in the previous paragraph and considering the case of equality, we see that k must be 0 or $d-3$. This leads to the second part of Assertion 3. Again, we can characterise the polytopes with each of these values of f_0.

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On the Vertex Identification Spectra of Grids

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Extended Abstract

Let $G = (V(G), E(G))$ be a nontrivial connected graph with order $n \geq 2$. The diameter of G, denoted by $\mathrm{diam}(G)$, is the maximum distance $d(u, v)$ between any two vertices u and v in G. It follows that $1 \leq \mathrm{diam}(G) \leq n - 1$, where n is the order of G.

Consider a coloring c of the vertices of G such that for $v \in V(G)$, $c(v)$ is either red or white and there exists at least one vertex v such that $c(v)$ is red. For a vertex $v \in V(G)$, define the code of v to be the vector $\vec{d}(v) := (a_1, a_2, ...a_{\mathrm{diam}(G)})$, where a_i is the number of red vertices at distance i from v; that is, $a_i = |\{u : d(v, u) = i \text{ and } c(u) \text{ is red}\}|$. The coloring c is called an **identification coloring** or an **ID-coloring** of G if $\vec{d}(v) \neq \vec{d}(u)$ for any two distinct vertices u and v in G. The **identification number** of G is the least number of red vertices among all ID-colorings of G, and is denoted by $\mathrm{ID}(G)$. A graph with an identification coloring is called an **ID-graph**. The concept of ID-coloring of a graph was introduced fairly recently by Chartrand, Yoko, and Zhang in [1].

Given an ID-graph G, a related question that was explored in [1] is that of finding all values of r for which an ID-coloring consisting of r red vertices exists. The set of all positive values of r for which such is the case is called the **identification spectrum** of G. Identification spectra for paths and cycles have been determined in [1].

There have only been a few studies so far on ID-coloring of graphs since its introduction. In [3], ID-caterpillars were characterized, and in [4], ID-numbers of grids and prisms were determined. In this paper, we extend some of the results in [4] by investigating the identification spectra of ladders $P_2 \,\square\, P_n$ and, more generally, of grids $P_m \,\square\, P_n$.

We begin by presenting an alternative formulation of ID-colorings. Given a red-white coloring c of a graph G, for each vertex v, we denote by $M(v)$ the multiset $\{d(v, u) : u \text{ is a red vertex}\}$. Note that when v is colored red, we have $0 \in M(v)$. Then it is evident that c is an ID-coloring if and only if $M(v) \neq M(w)$ for any pair of distinct vertices v, w.

Our first results pertain to ID-colorings of paths and grids. We say that a coloring of the path P_n (say, $1 \to 2 \to \cdots \to n$) is **symmetric** if $c(i) = c(n + 1 - i)$ for each $i \in \{1, 2, ...n\}$. Then we have the following lemma, which provides sufficient conditions for a red-white coloring of a path to be an ID-coloring.

Lemma 1. *Let $n \geq 4$ and let c be a red-white coloring of the path P_n under which the endvertices of P_n are colored red. If c is not symmetric, then c is an ID coloring.*

We then consider two types of red-white colorings of P_n. Let c be a red-white coloring of P_n under which exactly r vertices are red, where $3 \leq r \leq n - 1$. We say that c is of **Type 1** if the r red vertices are $1, 2, ..., r - 1$, and n. On the other hand, we say that c is of **Type 2** if the r red vertices are $1, ..., r - 2$, $n - 2$, and n. By Lemma 1, it is evident that Type 1 and Type 2 colorings of P_n are ID-colorings. In fact, we also establish the following stronger result.

Lemma 2. *Let c be a Type 1 or Type 2 coloring of P_n. Then for any two distinct vertices i and j of P_n, we have*

$$M(i) \neq M(j) + k$$

for any integer k.

Lemma 2 allows us to prove the following theorem, which implies that $\{3, 4, ..., n-1\}$ is a subset of the identification spectrum of any grid $P_m \square P_n$, where m and n are integers with $m \geq 1$ and $n \geq 4$.

Theorem 3. *Let $m \geq 1$ and $n \geq 4$ be integers. For each $r \in \{3, 4, ..., n-1\}$, the grid $P_m \square P_n$ has an ID-coloring in which exactly r vertices are colored red.*

We then proceed to consider the problem of completely determining the identification spectra of ladders $P_2 \square P_n$. Given Theorem 3, it is left to determine if, for each $r \in \{n, n+1, ..., 2n-1\}$, the ladder $P_2 \square P_n$ has an ID-coloring in which r vertices are colored red. The following result, which applies to grids in general, implies that $2n-1$ is not in the ID spectrum of the ladder $P_2 \square P_n$ if n is even.

Lemma 4. *Let m, n be positive integers. If m and n have the same parity, then $P_m \square P_n$ does not have an ID-coloring in which $mn-1$ vertices are colored red.*

By finding ID-colorings with the appropriate number of red vertices, and applying Lemma 4, we determine the spectra of the ladders.

Theorem 5. *Let $n \geq 5$ be odd. Then $P_2 \square P_n$ has identification spectrum $\{3, 4, 5, \ldots, 2n-1\}$.*

Theorem 6. *Let $n \geq 6$ be even. Then $P_2 \square P_n$ has identification spectrum $\{3, 4, 5, \ldots, 2n-2\}$.*

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On the group vertex magic of join of two graphs

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Abstract

All the graphs considered in the paper are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph and let A be an non-trivial additive abelian group with identity 0. A mapping $l : V(G) \to A \setminus \{0\}$ is said to be a A-vertex magic labeling of G if there exist a μ in A such that $w(v) = \sum_{u \in N(v)} l(u) = \mu$ for any vertex v of G. If G admits such a labeling, then it is called an A-vertex magic graph. If G is A-vertex magic for any non-trivial abelian group A, then G is called a group vertex magic graph. The concept of group vertex magic graphs was motivated by the V_4 magic labeling for edges, which was introduced by S. M. Lee[2]. Here V_4 denotes the famous Klein's four group. R. M. Low and S. M. Lee[3] extended the study of V_4-magic labeling for the product graphs by considering an arbitrary abelian group instead of V_4. By getting motivated by this concept, analogously the concept of group vertex magic graphs was introduced and studied in [1]. In this paper we consider A-vertex magic and group vertex magic labeling of join of two graphs. The following theorems, were proved in [1].

Theorem 1. *Let A be any abelian group with $|A| > 2$. Then the complete bipatite graph $k_{m,n}$ where m, n is A-vertex magic.*

Theorem 2. *Let $G = K_{m_1,m_2,\ldots,m_k}$ be complete k-partite graph. Let A be an abelian group having at least one element a such that $o(a) > lcm\{m_1, m_2, \ldots, m_k\}$. Then G is A-vertex magic.*

Theorem 3. *Let T be a tree of order n and diameter 2.*

1. *The tree T is V_4-vertex magic.*
2. *If n is even, then T is group vertex magic.*

In this paper we generalise the above theorems and we prove some results on A-vertex magic, group vertex magic labeling of join operation of graphs. We prove the following results,

Using the Cauchy's theorem, Sylow's first theorem and Fundamental theorem of finite abelian groups, we prove the following results.

Theorem 4. *A graph G is $\mathbb{Z}_p$-vertex magic for all primes p if and only if G is A-vertex magic for all finite abelian groups A.*

Theorem 5. *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph. Then G is A-vertex magic, where $|A| > 2$.*

Corollary 1. *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph with each partite size is of same parity. Then G is group vertex magic.*

Theorem 6. *Let G be an arbitrary r-regular graph with n vertices. Then the graph $G + {K_m}^c$ is A-vertex magic, where $|A| > 2, m > 1$.*

Theorem 7. *The graph $G = C_n + K_m$ is A-vertex magic, where $|A| > 2$ if and only if*

(i) $n = 3$ *or*

(ii) $n = 4$ *or*

(iii) $n = 4m_1 + 2$, *where* $m_1 = 2^k, k \geq 0$.

Theorem 8. *Let G be a graph. If $P_n + G$ is A-vertex magic, where $|A| > 2$ then $n \leq 3$*

Theorem 9. *The graph $G = P_n + C_m$ is A-vertex magic, where $|A| > 2$ if and only if $n \leq 3$ and*

(i) $m = 3$ *or*

(ii) $m = 4$ *or*

(iii) $m = 4m_1 + 2$, *where* $m_1 = 2^k, k \geq 0$.

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On Reconfiguration of Directed Graph Homomorphisms

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Abstract. A reconfiguration problem is a problem which asks the reachability from a solution to another solution in a solution graph defined by feasible solutions and a certain adjacency relation among them. In this paper, we consider a reconfiguration problem of directed graph homomorphisms. As a main result, for a directed acyclic graph G and a reflexive transitive directed complete graph H, we give a sufficient condition for the existence of a symmetric path between homomorphisms from G to H in the solution graph.

1 Introduction and Preliminaries

Reconfiguration is a framework which analyzes the connectivity and reachability of solution graphs for combinatorial problems. We focus on a reconfiguration of directed graph homomorphisms. For directed graphs G and H, the solution graph has homomorphisms from G to H as vertex set, and (f, g) is an arc if $(f(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$. The reconfiguration problem of graph homomorphisms is defined as a problem which asks the reachability of two homomorphisms in the solution graph. Wrochna [5] has shown that for an undirected graph G and a square-free undirected graph H, the reconfiguration problem is solvable in polynomial time. Brewster [1][2] also has shown that for a directed graph G and a reflexive directed cycle H, (i) if G is reflexive or (ii) if H does not contain 4-cycle of algebraic girth 0, the reconfiguration problem is solvable in polynomial time. Furthermore, Dochtermann [3][4] has proved that for any directed graph G and a transitive tournament H, the solution graph is connected.

Let $G = (V(G), E(G))$, $H = (V(H), E(H))$ be directed graphs. A homomorphism from G to H is a mapping $f : V(G) \to V(H)$ such that $(f(u), f(v)) \in E(H)$ if $(u, v) \in E(G)$. Let $\mathrm{Hom}(G, H)$ be the set of all homomorphisms from G to H, and $(H^G)^o$ be the solution graph which has $\mathrm{Hom}(G, H)$ as vertex set and (f, g) is an arc if $(f(u), g(v)) \in E(H)$ for any $(u, v) \in E(G)$.

Let us define the reconfiguration problem of directed graph homomorphisms Recon(G, H) as follows.

Recon(G, H)
Instance : $f, g \in \mathrm{Hom}(G, H)$
Question : Is there a "symmetric" path from f to g in $(H^G)^o$?

In this paper, we consider Recon(G, H) for a directed acyclic graph G and a reflexive transitive directed complete graph H. As a main result, we give a sufficient condition for homomorphisms to be yes-instances of Recol(G, H), and a polynomial time algorithm to find the symmetric path in the solution graph.

2 Main Result

Let G be a directed acyclic graph and H be a reflexive transitive directed complete graph. A *topological ordering* of G is an ordering $X = \{x_i\}_{i=1}^{n=|V(G)|}$ of the vertices of G such that $i \leq j$ if $(x_i, x_j) \in E(G)$. For $f \in \mathrm{Hom}(G, H)$, a topological ordering $X = (x_1, \ldots, x_n)$ of G is said to be *compatible with* f if $(f(x_i), f(x_j)) \in E(H)$ whenever $i \leq j$. For a topological ordering $X = (x_1, \ldots, x_n)$ of G compatible with f, we denote $(f(x_1), f(x_2), \ldots, f(x_n))$ by $f(X)$. By the definition, we can define an equivalence relation $\simeq$ between f and g on $\mathrm{Hom}(G, H)$ such that $f(X) = g(Y)$, where X is a topological ordering of G compatible with f and Y is a topological ordering of G compatible with g.

Remark 1. By using a method to find a topological ordering and priority queue, for $f \in \mathrm{Hom}(G, H)$, we can find a topological ordering of G compatible with f in $O(|E(G)| \log |V(G)|)$.

Theorem 1. *Let G be a directed acyclic graph and H be a reflexive transitive directed complete graph. If $f, g \in \mathrm{Hom}(G, H)$ are $f \simeq g$, then there exists a symmetric path from f to g in $(H^G)^o$. Furthermore, there exists a polynomial time algorithm that finds such a path in $O(|V(G)|^2)$.*

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Ollivier Ricci curvature of Cayley graphs for dihedral, generalized quaternion, and cyclic groups

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We calculate Ricci curvature of Cayley graphs for dihedral groups, general quaternion groups, and cyclic groups. Ricci curvature of graphs are introduced by Ollivier[4]. In [3] [4], some important properties are proposed. The Ricci curvature of an edge of a graph is denoted by κ in this article.

First, we calculate Ricci curvature of Cayley graphs for the dihedral group of the minimum generator set $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$. S makes two types the edges in Cayley graphs. One type of the edge set is $A = \{(g, g\sigma) \mid g \in D_n\}$, and the other is the edge set $B = \{(g, g\tau) \mid g \in D_n\}$. We call the edge in set A and B type A and type B, respectively.

Theorem 1 *Let $\Gamma(D_n, S)$ be a Cayley graph of dihedral group D_n with $S = \{\tau, \tau^{-1}, \sigma, \sigma^{-1}\}$, Ricci curvature of the edges in Cayley graph $\Gamma(D_n, S)$ with $n \geq 6$ is given as follows.*
Ricci curvature of type A is $\kappa = 0$.
Ricci curvature of type B is $\kappa = \frac{2}{3}$.

Second, we calculate the Cayley graph for the generalized quaternion group with the generating set $S = \{a, b, a^{-1}, b^{-1}\}$. We distinguish between the two sets of edges. One is the edge set $A = \{(g, ga) \mid g \in Q_{2m}\}$, and the other is edge set $B = \{(g, gb) \mid g \in Q_{2m}\}$. We call the edge in set A and B type A and type B, respectively.

Ricci curvature of Cayley graph for generalized quaternion group Q_{2m} with minimum generating set $S = \{a, b, a^{-1}, b^{-1}\}$ is obtained as follows.

Theorem 2 *For the Cayley graph $\Gamma(Q_{2m}, S)$ with $m \geq 4$,*
Ricci curvature of type A is $\kappa = 0$, and
Ricci curvature of type B is $\kappa = \frac{1}{2}$.

For D_n and Q_{2m}, the Ricci curvature are obtained for all remaining Cayley graphs for which the generators are the same ones.

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Finally, we calculate the Cayley graph for the cyclic group. It is known that Ricci curvature of infinite or finite paths are known to be zero.[1][2][5][6]

We consider the Cayley graph for the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with a generating set $S_{1,k} = \{+1, +k, -1, -k\}$, where k is a positive integer not equal to 1. We distinguish between the two sets of edges. One is the edges $A = \{(g, g+1) \mid g \in \mathbb{Z}/n\mathbb{Z}\}$, the other is edges $B = \{(g, g+k) \mid g \in \mathbb{Z}/n\mathbb{Z}\}$.

Theorem 3 *Let n and k be positive integers satisfying $n > k$. Consider the Cayley graph for the cyclic group $\Gamma(\mathbb{Z}/n\mathbb{Z}, S_{1,k})$.*
If $k \geq 5$ and $n \geq 2k+4$, then Ricci curvature of the type A is $\kappa = 0$.
If n and k satisfy the following conditions 1, 2, 3, or 4. then the Ricci curvature of type B is $\kappa = 0$:

1. *$k \geq 5$ and $3k+3 \leq n \leq 4k-2$,*

2. *$k \geq 3$ and $4k+2 \leq n \leq 5k-1$,*

3. *$k \geq 3$ and $n \geq 5k+1$,*

4. *$k \geq 6$ and $2k+3 \leq n \leq 3k-3$.*

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Arithmetic Mean and Geometric Mean

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Abstract

Although the arithmetic mean and geometric mean are well-known, very few things are known except the well-known inequality. Comparison between these means is quite difficult even when we use simple numbers such as $1, 2, 3$, and the authors present some methods to compare them.

1 Arithmetic Mean and Geometric Mean

Most people know the following inequality of the arithmetic mean and geometric mean, and nothing more than that.

$$\sqrt{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots a_n}{n} \tag{1.1}$$

These simple means have many kinds of applications. See [2]. But, for some people, comparing the arithmetic mean and geometric mean is an important topic. They have to choose one of them in a certain situation. See [1].

The authors are interested in the situation when we have the following (1.2) or (1.3) for two sequences of real numbers $\{a_i : i = 1, 2, \cdots, n\}$ and $\{b_i : i = 1, 2, \cdots, n\}$.

$$\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i \quad \text{and} \quad \prod_{i=1}^{n} a_i < \prod_{i=1}^{n} b_i. \tag{1.2}$$

$$\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i \quad \text{and} \quad \prod_{i=1}^{n} a_i > \prod_{i=1}^{n} b_i. \tag{1.3}$$

This article aims to answer the following question.

Question. *What is the probability of getting* (1.2) *or* (1.3)*?*
Clearly, the answer to this question depends on the range of the variables a_i and b_i.

1.1 The case when we use only 1,2,3.

Theorem 1.1. *We suppose that $a_i = 1, 2, 3$ and $b_i = 1, 2, 3$ for $i = 1, 2, \cdots, n$. Then,* (1.3) *is satisfied if and only if*

$$\sum_{x=0}^{n} \sum_{p=0}^{n-x} \sum_{q=0}^{n-p-x} \left(\frac{n!}{p!(q+x)!(n-x-p-q)!}\right)\left(\sum_{y=\lceil \frac{x+1}{2} \rceil}^{\lfloor \frac{x \log 2}{\log 3} \rfloor} \frac{n!}{(p+x-y)!q!(n-x-p-q+y)!}\right). \tag{1.4}$$

Theorem 1.2. *The probability of getting* (1.2) *or* (1.3) *is less than $\frac{1}{3}$.*

Example 1.3. By using (1.4), we calculate the probability of getting (1.2) or (1.3). The graph of the probability is Figure 1. It seems that the probability increases and getting nearer and nearer to the limit, but the authors have not managed to prove this fact, and they do not have any prospect of proof at all.

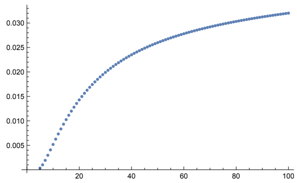

Figure 1: $1, 2, 3$

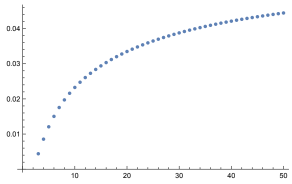

Figure 2: $1, 2, 3, 4$

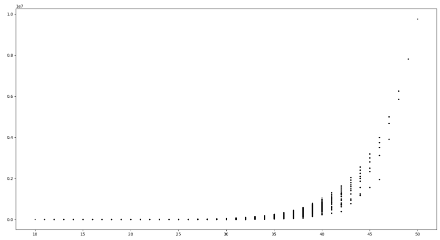

Figure 3: scatter plot

1.2 The case when we use only 1,2,3,4.

When we use $1, 2, 3, 4$ for values of a_i and b_i for $i = 1, 2, \cdots, n$, the problem of finding patters that satisfy (1.3) becomes very difficult. The authors proved Lemma 1.4, but this is not enough to make a mathematical formula as presented in Theorem 1.1. At least this lemma make computer calculation easier. For the graph using this lemma, see Figure 2.

Lemma 1.4. *(a) Suppose that $a_i = 3$ and b_i=$1, 2, 4$ for $i = 1, 2, \cdots, n$. Then we have the following: (i) We never have (1.2).*
(ii) We have (1.3) for some a_i, b_i.
(b) Suppose that $a_i = 3$ for $i = 1, 2, \cdots, n$. Then we have (1.3) for some $b_i = 1, 4$ or some $b_i = 2, 4$ $i = 1, 2, \cdots, n$.
(c) Suppose that $a_i = 2$ for $i = 1, 2, \cdots, n$. Then we have (1.3) for some $b_i = 1, 3, 4$ or some $b_i = 1, 3$ or some $b_i = 1, 4$.
(e) Suppose that $a_i = 2, 3$. Then we have (1.3) for some $b_i = 1, 4$ or some $b_i = 1, 3$.
(f) Suppose that $a_i = 2, 4$. Then We have (1.3) for some $b_i = 1, 3$.

1.3 The case when n is infinitely large.

The situation is different when $n \to \infty$. Then we have the following theorem by prof. Tetsuro Kamae of Osaka Central Advanced Mathematical Institute.

Theorem 1.5. *Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables such that there exists a positive number L such that $0 < X_i < L$. Let $S = \sum_{i=1}^{n} X_i$ and $T = \prod_{i=1}^{n} X_i$. Let S' be a random variable that is independent of S and has the same distribution of S, and let T' be a random variable that is independent of T and has the same distribution of T. Then $P(\{S > S'\} \cap \{T < T'\}) \to \frac{1}{4} - \frac{1}{2\pi} arcsin r$, where r is correlation coefficient of X_i and $log X_i$.*

1.4 The case when we use real numbers.

Suppose that $n = 2$ and a_i, b_i are real numbers.

Theorem 1.6. *The probability of getting (1.3) is $\int_0^{10} \int_0^{b_2} \int_{b_1}^{b_2} (-x + b_1 b_2 - \frac{b_1 b_2}{x}) dx db_1 db_2 + \int_0^{10} \int_{b_2}^{10} \int_{b_2}^{b_1} (-x + b_1 b_2 - \frac{b_1 b_2}{x}) dx db_1 db_2$.*

1.5 The case when we use real numbers.

Let $a_i, b_i = 1, 2, 3, 4, 5$ and $n = 10$. We consider the scatter plot of 5000 randomly generated samples, where the horizontal axis is for large and small relationship based on the arithmetic mean and the vertical axis is for geometric mean. See Figure 3.

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Finding triplets in “Dobble”

Junyi Guo, Hsiang-Chun Hsu

“Dobble” is a very popular card game since it released in 2009. It consists of 55 cards, with 8 different symbols on each card. Any two cards have exactly one symbol in common. The object of dobble is to spot the identical symbol between two cards, name it out loud and then take the card, place it or discard it, depending on the rules of the games you are currently playing.

Fig. 1: What symbol is on both cards?

Suppose there are b different symbols in dobble. According to the condition “Any two cards have exactly one symbol in common”, it’s easy to show that $57 \leq b \leq 386$. In general, suppose there are n cards and $q+1$ symbols on each card. We get $n(q+1) \leq b\left\lfloor\frac{b-1}{q}\right\rfloor$ and $b \leq nq+1$. Indeed, dobble has 57 different symbols that reaches the lower bound.

It is well-known that if q is a prime power and $n = b = q^2+q+1$, then the structure of the symbols arranged on dobble cards is isomorphic to the finite projective plane $PG(2,q)$. Dobble is marketed with 55 cards which is $PG(2,7)$ removing 2 lines.

In the newest official rulebook of dobble, there are 5 mini-games: The Tower, The Well, Hot Potato, The Poisoned Gift, and Triplet. Let’s focus on “Triplet.” The object of “Triplet” is to find any 3 cards which have the same symbol from the 9 face-up cards on the table (as shown in Fig. 2). Does the solution always exist for any 9 cards? The answer is positive. How about any 8 cards? And we proved more general results.

Fig 2: Finding 3 cards which have a common symbol.

Let $e(n)$ be the smallest even integer larger than n. For instance, $e(2)=4$ and $e(7)=8$. We have

Theorem 1. There exist $e(q)$ lines in $PG(2,q)$, no three of which are concurrent.

This theorem extends the dual axiom of the finite projective plane of order q, "There are four points such that no line is incident with more than two of them."

On the other hand, we also prove that the bound $e(q)$ is tight.

> **Lemma 2**. In $PG(2, q)$, for any line L, there exist 2 lines L_1 and L_2 from any other $q + 2$ lines such that L, L_1, and L_2 are concurrent.

This lemma can be concluded by the pigeonhole principle immediately. Additionally, we use Wilson Theorem to prove

> **Theorem 3**. For any $e(q) + 1$ lines in $PG(2, q)$, there must exists three of which are concurrent.

Then we conclude the mini-game "Triplet" of dobble always has a solution. And we designed a new 6th mini-game called "Finding Triplets" according to Lemma 2. In the beginning, deal one card to each player and place 9 cards face-up on the table. All players try to spot the one symbol that appears both on their own card and on 2 cards within 9 cards on the table. If you are the first player to do so, call the symbol out, take these 2 cards, place them on top of your pile, and reload with 2 new cards to the table. When fewer than 9 cards left in the game, the game ends, and the player with the most cards wins.

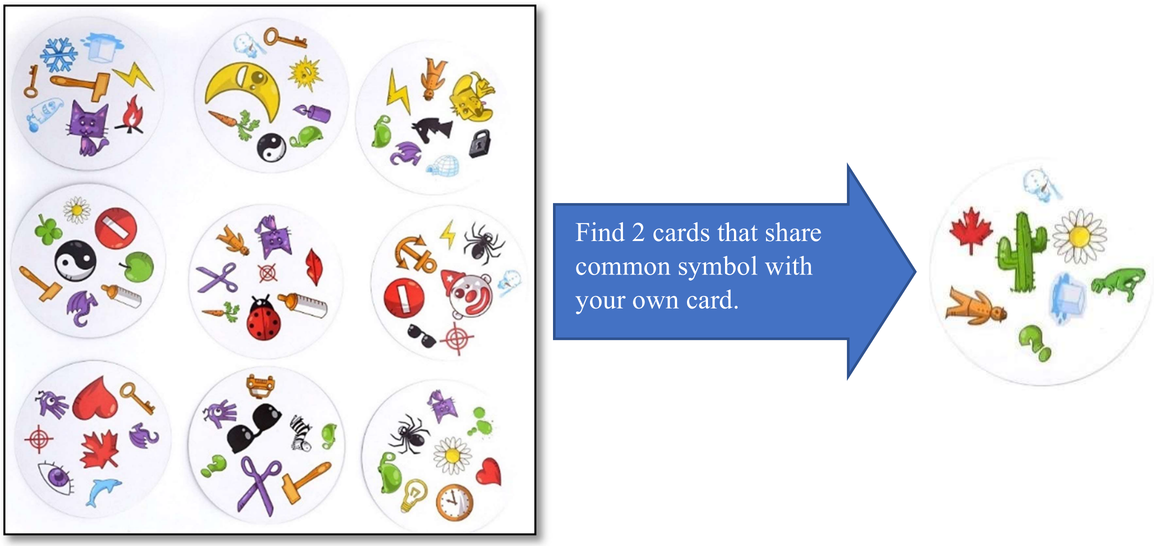

Fig.3: The new game of dobble, "Finding Triplets."

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Some extensions of Delete Nim

Tomoaki Abuku* Ko Sakai† Masato Shinoda‡ Koki Suetsugu§

2022

1 Introduction

Combinatorial games are 2-player games with neither chance elements nor hidden information (for the details on the combinatorial game theory, see, e.g., [2] and [3]).

Delete Nim is a combinatorial game. There are two heaps of tokens. Each of the 2 players on his/her turn selects one heap, deletes it, and then splits the remaining heap into two heaps. The player's turn then ends, and the other player's turn begins. If the player cannot split a heap, then he/she loses. The winning strategy and the closed formula for the Sprague-Grundy value for this game have been fully analyzed by Abuku and Suetsugu [1].

In this study, we propose several rules for the case where the number of heaps is three or more, and we give the winning strategy. Note that in each valuation of Delete Nim in this paper, every heap is always assumed to contain at least one stone after division. The number of heaps after a move is always constant.

2 Main Results

Given a combinatorial game G, a game position is called an $\mathcal{N}$-position (resp., $\mathcal{P}$-position) if the next (resp., previous) player has a winning strategy. Clearly, each game position is either an $\mathcal{N}$-position or a $\mathcal{P}$-position.

Definition 2.1 (Single-delete Nim)**.** There are n heaps of tokens. The player performs the following two operations in succession on his/her turn.

- Selects one heap and deletes it.
- Selects one heap of the remaining $n-1$ heaps and splits it into two heaps.

Denote by $v_p(n)$ the p-adic valuation of n.

Theorem 2.2. If $n = 3$ in the Single-delete Nim, the position (x, y, z) is a $\mathcal{P}$-position if and only if $v_2(x) = v_2(y) = v_2(z)$.

Theorem 2.3. Denote by $I_k(z)$ the k-th digit from the bottom of the binary representation of non-negative integer z. For $n = 4$ in the Single-delete Nim position (w, x, y, z), let $a = v_2(w)$, $b = v_2(x)$, $c = v_2(y)$, $d = v_2(z)$. If $a \le b \le c \le d$, (w, x, y, z) is a $\mathcal{P}$-position if and only if a, b, c, and d satisfy one of the following conditions (1), (2), (3), (4), or (5).

(1) $a = b = c = d$.

(2) $a < b = c = d$ and

- (2A) $I_{d+1}(w) = 0$.

(3) $a < b < c = d$ and the following conditions (3A)-(3C) are satisfied.

- (3A) $I_{d+1}(w) = I_{d+1}(x) = 0$.
- (3B) $I_k(w) + I_k(x) \ge 1$ for $b + 2 \le k \le d$.
- (3C) $I_{b+1}(w) = 1$.

(4) $a < b < c < d$ and the following conditions (4A)-(4E) are satisfied.

- (4A) $I_{d+1}(w) = I_{d+1}(x) = I_{d+1}(y) = 0$.
- (4B) $I_j(w) + I_j(x) + I_j(y) \ge 2$ for $c + 2 \le j \le d$.
- (4C) $I_{c+1}(w) = I_{c+1}(x) = 1$.
- (4D) $I_k(w) + I_k(x) \ge 1$ for $b + 2 \le k \le c$.
- (4E) $I_{b+1}(w) = 1$.

(5) $a < b < c < d$ and the following conditions (5A)-(5F) are satisfied.

- (5A) $I_i(w) + I_i(x) + I_i(y) + I_i(z) \in \{0, 3, 4\}$ for $i \ge d + 2$.
- (5B) $I_{d+1}(w) = I_{d+1}(x) = I_{d+1}(y) = 1$.
- (5C) $I_j(w) + I_j(x) + I_j(y) \ge 2$ for $c + 2 \le j \le d$.
- (5D) $I_{c+1}(w) = I_{c+1}(x) = 1$.
- (5E) $I_k(w) + I_k(x) \ge 1$ for $b + 2 \le k \le c$.
- (5F) $I_{b+1}(w) = 1$.

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Definition 2.4 (All-but-one-delete Nim)**.** There are n heaps of tokens. The player performs the following two operations in succession on his/her turn.

- Selects $n-1$ heaps and deletes them.
- Splits the remaining 1 heap into n heaps.

Theorem 2.5. All-but-one-delete Nim position $(z_1, z_2, \ldots, z_n)$ is a $\mathcal{P}$-position if and only if for every i, the remainder of z_i divided by $n(n-1)$ is between 1 and $n-1$.

Definition 2.6 (Less-than-half-delete Nim)**.** There are n heaps of tokens. The player performs the following two operations in succession on his/her turn.

- Chooses a positive integer k such that $k \leq \frac{n}{2}$, selects k heaps, and deletes them.
- Selects k heaps of the remaining $n-k$ heaps and splits each heap into two heaps.

In particular, if $n = 2$, this game is the same as Delete Nim, and if $n = 3$, this game is the same as Single-delete Nim with $n = 3$.

Theorem 2.7. Less-than-half-delete Nim position $(a_1, a_2, \ldots, a_n)$ is a $\mathcal{P}$-position if and only if $a_1, a_2, \ldots, a_n$ satisfy one of the following conditions:

- if n is odd, $v_2(a_1) = v_2(a_2) = \cdots = v_2(a_n)$,
- if n is even, $v_2(a_1) = v_2(a_2) = \cdots = v_2(a_n) = 0$.

Definition 2.8 (Half-delete Nim)**.** There are $2n$ heaps of tokens. The player performs the following two operations in succession on his/her turn.

- Selects n heaps and deletes them.
- Splits each of the remaining n heaps into two heaps.

In particular, if $n = 2$, this game is the same as Delete Nim.

Hereafter, a heap with an even number of tokens is called an *even heap* and a heap with an odd number of tokens is called an *odd heap*.

Theorem 2.9. Let $(a_1, a_2, \ldots, a_{2n})$ be the Half-delete Nim position, where each a_i is the number of tokens and $a_i < a_{i+1}$. $(a_1, a_2, \ldots, a_{2n})$ is a $\mathcal{P}$-position if and only if $a_1, a_2, \ldots, a_{2n}$ satisfy both of the following two conditions:

(1) all $a_1, a_2, \ldots, a_{n+1}$ are odd,

(2) let 2^m be the smallest power of 2 greater than a_{n+1}. For l $(l > n+1)$, if a_l is even, then $a_l \geq 2^m$.

This theorem is a special case of Theorem 2.13.

Definition 2.10 ($\frac{k-1}{k}n$-delete Nim)**.** There are kn heaps of tokens. The player performs the following two operations in succession on his/her turn.

- Selects $(k-1)n$ heaps and deletes them.
- Splits each of the remaining n heaps into k heaps.

In particular, if $k = 2$, this game is the same as Half-delete Nim, and if $n = 1$, this game is the same as All-but-one-delete Nim.

Definition 2.11. A positive integer whose remainder divided by $k(k-1)$ lies between 1 and $k-1$ is called an *oddoid number*, and any other positive integer is called an *evenoid number*. A heap with an oddoid number of tokens is called an *oddoid heap*, and a heap with an evenoid number of tokens is called an *evenoid heap*.

In particular, if $k = 2$, oddoid and evenoid numbers are consistent with the usual notion of odd and even numbers.

Lemma 2.12.

(1) It is not possible to divide an oddoid number into k oddoid numbers.

(2) All integers x between k and $k(k-1)$ can be divided into k integers that are between 1 and $k-1$.

(3) All evenoid numbers y less than k^m can be divided into k oddoid numbers that are less than k^{m-1}.

Using this lemma, by replacing the odd and even heaps of Theorem 2.9 with oddoid and evenoid heaps, respectively, we can give the following winning strategy for $(k-1/k)n$-delete Nim.

Theorem 2.13. Let $(a_1, a_2, \ldots, a_{kn})$ be the $(k-1/k)n$-delete Nim position, where each a_i is the number of tokens and $a_i < a_{i+1}$. $(a_1, a_2, \ldots, a_{kn})$ is a $\mathcal{P}$-position if and only if $a_1, a_2, \ldots, a_{kn}$ satisfy both of the following two conditions:

(1) all $a_1, a_2, \ldots, a_{(k-1)n+1}$ are oddoid,

(2) let k^m be the smallest power of k greater than $a_{(k-1)n+1}$. For l $(l > (k-1)n+1)$, if a_l is evenoid, then $a_l \geq k^m$.

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Continuous flattening of polyhedra with rigid edges

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We use the terminology *polyhedron* for a polyhedral surface in three-dimensional Euclidean space $\mathbb{R}^3$ that is permitted to touch itself but without self-crossing. The *flat-folding* of a polyhedron refers to its folding by creases, without self-crossing, into a multi-layered flat folded state with a finite number of creases. The *continuous flattening* of a polyhedron is flat-folding through a continuous motion (see [2]). That is, the solution of the continuous flattening of a polyhedron P requires not only the existence of a flat folded state of P but also the existence of a continuous motion from P to the flat folded state. There are many types of continuous flattening of polyhedra described in the literature.

The authors recently proposed the problem of the maximum number of rigid edges (or faces) in flattening motions for a given polyhedron (see [4]). Now, we show two examples of flattening methods of a cube in the literature (Fig. 1). Figs. 1(b) and (c) (respectively, (e) and (f)) are continuous flattening motions with 8 (respectively, 11) rigid edges. Figs. 1(d), (g), and (h) are flat folded states with 8, 11, and 12 rigid edges, respectively. This talk focuses on a continuous flattening motion for a given polyhedron, keeping every edge rigid during the motion, e.g., keeping 12 edges rigid for a cube. We call such flattening *edge-rigid-flattening*, denoted by ER-flattening. Moreover, we also call a folded state of P such that every edge is not folded *edge-rigid-state*, denoted by ER-state.

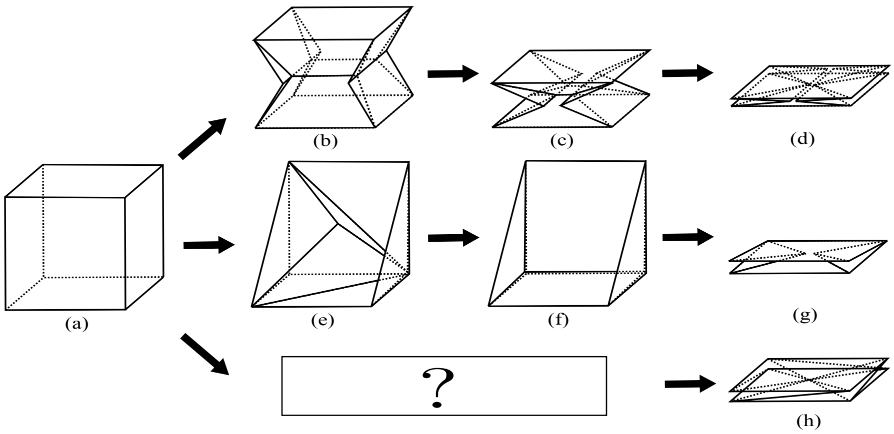

Figure 1: (a) A cube, (b), (c) a flattening motion with eight rigid edges, (d) a flat folded state with eight rigid edges, (e), (f) a flattening motion with 11 rigid edges, (g) a flat folded state with 11 rigid edges, and (h) a flat folded state with 12 rigid edges (ER-state).

To the best of the authors' knowledge, for any convex polyhedron (e.g., a cube in Fig. 1), no continuous ER-flattening motion has been described in the literature, even if a flat folded ER-state exists. However, the non-existence of the motion has not also been proven. On the other hand, it is shown in [4] that some non-convex polyhedra have continuous ER-flattening motions, as shown in Fig. 2. In this talk, we discuss the continuous ER-flattening of several types of polyhedra.

The Bellows theorem in [1] shows that the volume of any polyhedron with every face rigid is invariant even if the polyhedron is flexible. Hence, the following result can be obtained.

Lemma 1. *Let P be a polyhedron with every face triangular. Then there is no continuous ER-flattening motion of P.*

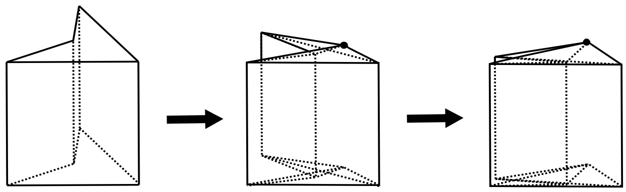

Figure 2: (a) A non-convex quadrangular prism, (b) an ER-flattening motion, and (c) a flat folded ER-state.

Lemma 1 can be applied to three of the regular polyhedra. We can obtain the following table of the continuous ER-flattening of P_k, where P_k is a regular polyhedron with k faces.

	P_4	P_6	P_8	P_{12}	P_{20}
ER-flattening motion	not exist	unknown	not exist	unknown	not exist
flat folded ER-state	not exist	exist	exist	unknown	not exist

Moreover, we can prove the non-existence of the continuous ER-flattening motion for any regular pyramid. On the other hand, we can give the continuous ER-flattening motions for some prisms.

Theorem 2. *There is no continuous ER-flattening motion for any regular n-gonal pyramid with $n \geq 3$.*

Theorem 3. *There exists a continuous ER-flattening motion for any prism whose base is a non-convex polygon $A_1A_2 \cdots A_{2n}$ with edges of equal length, where the interior angles $\angle A_{2i-1}A_{2i}A_{2i+1} = \alpha$ and $\angle A_{2i}A_{2i+1}A_{2i+2} = \beta$ $(i = 1, 2, \ldots, n)$ for some $\alpha < \pi$ and $\beta > \pi$.*

It does not seem easy to show the existence (or non-existence) of the continuous ER-flattening motion of a given polyhedron P in general. However, the assumption of rigidity of one face during the ER-flattening motion yields the following non-existence result.

Theorem 4. *Let P and Q be any regular n-gonal prism and anti-prism with $n \geq 3$, respectively. Then, for each of P and Q, there is no continuous ER-flattening motion such that the base is rigid during the motion.*

Remark. Any regular n-gonal pyramid with even n has a flat folded ER-state, but any regular n-gonal pyramid with odd n does not have a flat folded ER-state. Moreover, a regular n-gonal prism with $n \geq 3$ has a flat folded ER-state with a rigid base if the height of the prism is suitable.

We conjecture that there is no continuous ER-flattening motion for regular prisms and anti-prisms even without the rigidity of the base in Theorem 4. However, the problem is still open, even in the case of a cube. More generally, we also conjecture that there is no continuous ER-flattening motion for any polyhedron with every face convex.

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Continuous Folding of the Surface of a Hypercube onto its Facet

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1 Introduction

Can we flatten the surface of a polyhedron of flexible material such as paper without cutting or stretching? (See [2]). This problem was solved for all convex polyhedra in [1, 4] using moving creases to change the shapes of some faces. In particular, the result in [1] includes 3-dimensional convex polyhedra and some arrangements of hyperplanes in an n-dimensional space, e.g., the surface of an n-dimensional hypercube. However, in [1], if we consider the portions of the surface used for moving the creases of a given convex polyhedron they almost cover the entire surface (except at most two facets). Here, we continuously fold the surface of a four dimensional hypercube using the kite property introduced in [3, 5] such that the volume of the portions used to move creases is less than half of the portions used in [1].

Definition 1. *Let P be a four dimensional regular polytope and $Q = \partial P$ the surface consisting of the facets. We say Q is continuously folded in a hyperplane if there is a set of polyhedral manifolds $\{Q_t : 0 \leq t \leq 1\}$, each of which is intrinsically isometric to Q and possibly overlap each other without self-crossing in some parts, such that $Q_0 = Q$, Q_1 is in a hyperplane, and the mapping from t to P_t is continuous.*

With the moving creases shown by red line segments, Fig. 1 shows two examples of flattening ∂P of a cube P continuously by: (a) the kite property in [3] and (b) the straight skeleton in [1]. We have six square faces in P: the top (T), bottom (BO), right (R), left (L), front (F), and back (BA). In Fig. 1 (a), faces T and BO are rigid (without creases), and the other four faces are folded in half if we ignore self-crossings. To avoid self-crossings, faces R and L are folded in half, but faces F and BA are folded with moving creases, which subjects a motion of R and L, respectively. Here, we extend this result to any higher-dimensional hypercubes.

2 Theorem

Let P be the four dimensional hypercube with 16 vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$ in 4-space with x, y, z, and w axes. C_u^+ and C_u^- denote the cube (facet) in the

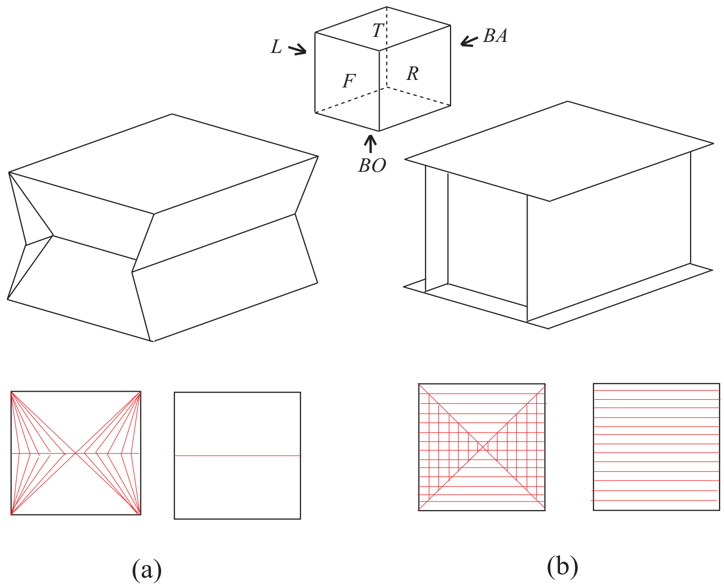

Fig. 1. Continuous flattening of the surface of a cube in two methods with moving creases, which are showed in red for the right and front faces, by: (a) the kite property; (b) the straight skeleton.

hyperspace $\{u = 1\}$ and $\{u = -1\}$, respectively where $u = x, y, z$, or w. We fold the surface ∂P of facets such that C_w^+ and C_w^- are rigid, C_x^+ and C_x^- are folded in half, C_y^+, C_y^-, C_z^+, and C_z^- are folded with moving creases.

Theorem 1. *The surface of a four dimensional hypercube can be continuously folded onto any of its facet such that the total volume used for the moving creases is less than half of the surface volume.*

We believe that we can continuouly fold ∂P of any n-dimensional hypercube P onto any of its facets such that at least four facets have no moving creases.

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Unfoldings of 4D-Hypercube Unfoldings that tile $\mathbb{R}^2$

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Abstract

We prove that every face-unfolding of the 4D hypercube has an edge-unfolding that tiles the plane. Hence the hypercube is a "complete dimension-descending tiler". We also illustrate the algorithm used to manifest this result.

1 Introduction

It is a well-known fact that the cube has 11 distinct (incongruent) edge-unfoldings each of which tiles $\mathbb{R}^2$. So the cube has the delightful property that it itself tiles $\mathbb{R}^3$, and all of its unfoldings tile $\mathbb{R}^2$. The cube is therefore a "complete dimension-descending tiler" (see section 2).

We consider this property in a higher dimensional cube: the 4D cube (which we also refer to as the *hypercube*). The 4D cube tiles $\mathbb{R}^4$ and admits 261 distinct face-unfoldings to order-8 polycubes [1]. A natural question would then be, do all of these unfoldings tile 3D space.

In 2015, G. Diaz and J. O' Rourke [2] explored this very question; and 4 (of the 261) face-unfoldings of the hypercube were shown to tile $\mathbb{R}^3$. Later in 2021, whuts.org [3], M. Firsching [4], and G. Papoutsis [5] independently showed that in fact all 261 unfoldings tile $\mathbb{R}^3$.

Interestingly, G. Diaz and J. O' Rourke [2] also proved that one of the face-unfoldings had an edge-unfolding that then tiled $\mathbb{R}^2$. Likewise, S. Langerman and A. Winslow [6] proved that another face-unfolding, the *dali-cross,* also admits an edge-unfolding that tiles $\mathbb{R}^2$.

As far as we are aware, *only* 2 face-unfoldings of the hypercube were known to have an edge-unfolding that tiles the plane. Therefore, we consider the following question posed by S. Langerman [7]:

Question 1 *Does every face-unfolding of the hypercube have an edge-unfolding that tiles space?*

We answer this positively in Section 3 and illustrate the algorithm adopted in the brute-force search to prove this. We, in a way, complete the journey initiated in 2015.

2 Definitions

Dual Graph. The *dual graph* of a polycube has a vertex for each *square face* and edges between edge-adjacent squares.

Complete Dimension-descending Tiler (c-DDT). A polytope that monohedrally tiles $\mathbb{R}^d$ is a c-DDT if all of its facet-unfoldings tile $\mathbb{R}^{d-1}$, and every $\mathbb{R}^{d-1}$ polytope has a facet-unfolding that tiles $\mathbb{R}^{d-2}$. The $\mathbb{R}^{d-2}$ polytope must then have a facet-unfolding that tiles $\mathbb{R}^{d-3}$ and so on till an edge-unfolding that tiles $\mathbb{R}^2$.

Linear Unfolding [8] Let T be a hamiltonian path on the dual graph of the polycube. The unfolding corresponding to T is linear if direction x is used and direction –x is not.

3 Results

We briefly describe the exhaustive search algorithm used to prove Theorem 3.1 and then demonstrate this in Example 3.2 on the *Dali-cross* unfolding of the hypercube.

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Previous approaches that attempted unfolding the face-unfoldings relied on finding spanning trees, on their *dual graphs,* to generate edge-unfoldings [2]. The algorithm we employ generates hamiltonian paths to enumerate unfoldings and then filters *linear edge-unfoldings*. Any linear unfolding tiles the plane. Jelmer Firet [8] proved that every linear unfolding will tile space (and used a similar algorithm to show that linear *face-unfoldings* of the hypercube tile 3D space).

Theorem 3.1 *Every face-unfolding of the hypercube has an edge-unfolding that tiles* $\mathbb{R}^2$.

Example 3.2 *Consider the Dali-cross face-unfolding of the hypercube (see Figure 1 (a)). A hamiltonian path (blue) on its dual graph (black) is shown in Figure 1 (b). The corresponding unfolding happens to be linear and is therefore a monohedral tile. (see Figure 2).*

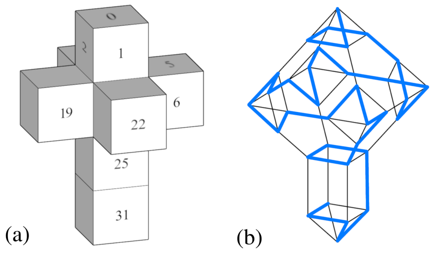

Figure 1: (a) *Dali-cross* unfolding of the hypercube (the numbers correspond to those in Figure 2). (b) A hamiltonian path on the polycube's dual tree *mapping* to the unfolding in Figure 2.

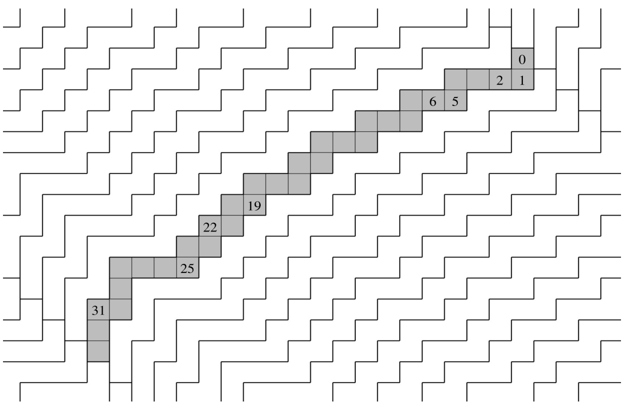

Figure 2: *Linear unfolding* of the *dali-cross* tiling space. Faces are numbered according to position in the unfolding and correspond to Figure 1 (a).

Acknowledgements

We greatly appreciate S. Langerman for motivating this research and commenting on this article.

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The following repository contains edge-unfolding to every face-unfolding of the hypercube:
T. Ramteke. HyperCube Unfoldings Data, 2022. https://github.com/TrunInGitHub/HyperCube_Unfoldings_Data.

On K_2-Hamiltonian Graphs

JAN GOEDGEBEUR, JARNE RENDERS, GÁBOR WIENER, AND CAROL T. ZAMFIRESCU

KU Leuven Kulak

Hamiltonian and hypohamiltonian graphs have been intensely studied since the 1960s [1, 4]. In this abstract we only consider connected simple graphs. A graph is *hamiltonian* if it contains a cycle which visits every vertex. Similarly, a graph is *hypohamiltonian* if it is not hamiltonian, but every vertex-deleted subgraph is. Motivated by a conjecture of Grünbaum [2] and a problem of Katona, Kostochka, Pach and Stechkin [5], we investigate K_2-hamiltonian graphs based on earlier work by Zamfirescu [6].

Grünbaum [2] defined $\Gamma(j, k)$ with $k \geq j$ to be the family of graphs whose order and *circumference*, i.e., the length of a longest cycle, differ by k and in which any j vertices are missed by some longest cycle. $\Gamma(1, 1)$ are then precisely the hypohamiltonian graphs. In 1974, Grünbaum conjectured that $\Gamma(j, j)$ must be empty for $j \geq 2$. So far little is known about the truth of this conjecture. We relax this problem further by investigating the following class of graphs.

A graph is called *K_2-hamiltonian* if the deletion of any two adjacent vertices yields a hamiltonian graph. If this graph is also non-hamiltonian, we call it *K_2-hypohamiltonian*. An example of such a graph is Petersen's graph, which we show to be the smallest K_2-hypohamiltonian graph (Fig. 1). We further study this concept using a combination of theoretical and computational methods.

To this end, we have designed and implemented an algorithm that checks whether a given input graph is K_2-(hypo)hamiltonian and which turns out to be quite efficient in practice [3]. In particular, we use a backtracking algorithm with some heuristics to restrict the search space.

Using this algorithm together with mathematical operations preserving K_2-hypohamiltonicity, we were able to classify for which orders there exist K_2-(hypo)hamiltonian graphs for all orders except two. In the class of cubic graphs,

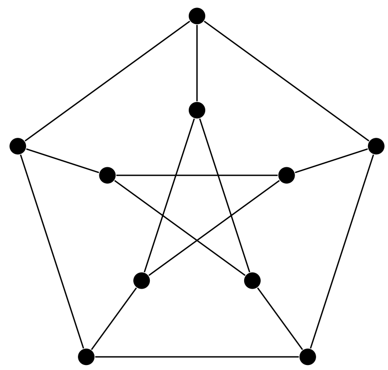

Figure 1: Petersen's graph: the smallest K_2-hypohamiltonian graph.

i.e., graphs where every vertex has three incident edges, and *snarks*, i.e., cubic graphs with a 4-edge coloring, we completely characterize all orders and we have shown that for every order $n \geq 177$ there exists a K_2-hypohamiltonian planar graph. We have also found the smallest planar K_2-hypohamiltonian graph of *girth* 5, i.e., the length of a shortest cycle is 5, which has 48 vertices and the smallest cubic planar K_2-hypohamiltonian graph, which has 68 vertices (Fig. 2).

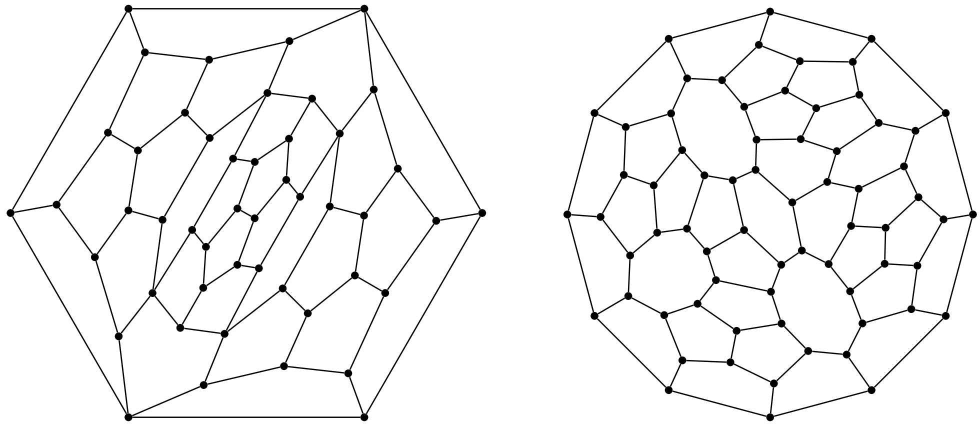

Figure 2: Left: The smallest planar K_2-hypohamiltonian graph of girth 5. Right: The smallest cubic planar K_2-hypohamiltonian graph.

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On new results on extremal and algebraic graphs and their applications.

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Abstract *,New explicit constructions of infinite families of finite small world graphs of large girth with well-defined projective limits which is an infinite tree are described. The applications of these objects to constructions of LDPC codes and cryptographic algorithms are shortly observed. We define families of homogeneous algebraic graphs of large girth over the commutative ring K. For each commutative integrity ring K with |K| > 2, we introduce a family of bipartite homogeneous algebraic graphs of large girth over K formed by graphs with sets of points and lines isomorphic to* K^n, $n > 1$, *and cycle indicator* $\geq 2n + 2$ *such that their projective limit is well defined an*

Keywords*: family of graphs of large girth, small world graphs, cryptographic algorithms, LDPC codes.*

The girth and diameter of a graph are the minimal length of its cycle and the maximal distance of the graph. We can consider the girth indicator *Cind(v)* of a vertex *v* of the graph *Γ* as the minimal length of the cycle through *v* and introduce a cycle indicator *Cind(Γ)* of the graph as the maximal value of Cind(*v*) for its vertices.

The constructions of finite or infinite graphs with prescribed girth and diameter is an important and difficult task of the graph theory. Noteworthy that the incidence of the classical projective geometry over various fields is a graph of girth 6 and diameter 3. J. Tits defined generalized *m*-gons as bipartite graphs of girth 2*m* and diameter *m*. Feit and Higman proved that finite generalized *m*-gons with bi-degrees > 2 exist only in the cases of *m* = 3, 4, 6, 8, and 12. Geometries of finite simple groups of rank 2 are natural examples of generalized *m*-gons for *m* = 3, 4, 6, 8. Classification of flag transitive generalised *m*-gons of the Moufang type were obtained by J. Tits and R. Weiss.

Infinite families of graphs of large girth of bounded degree are important objects of extremal graph theory which were introduced by P. Erdős’. He proved the existence of such families via his well-known probabilistic method. Nowadays,a few explicit constructions of such families are known. The concept of an infinite family of small world graphs of bounded degree turns out to be very important for various applications of graph theory.

Noteworthy that only one family of small world graphs of large girth is known. This is the family *X*(*p, q*) of Ramanujan graphs introduced by G. Margulis [1] and investigated via the computation of their girth, diameter, and the second largest eigenvale by A. Lubotsky, R. Phillips, and P. Sarnak [2].

We have to admit that studies of families of graphs Γ_i with well-defined projective limit Γ, which is isomorphic to an infinite tree, is well motivated.
We refer to such family as a tree approximation. There is only one tree approximation by finite graphs which is a family of large girth. This is the family of *CD*(*n, q*) defined by F. Lazebnik, V. Ustimenko, and A. Woldar [3]. The question whether or not *CD*(*n, q*) form a family of small world graphs has been still open since 1995.

In 2013, the tree approximation by finite graphs *A*(*n, q*) which is a family of small world graphs was presented (see [4]). It was proven that the graph from the family has maximal possible cycle indicator [in fact, Cind(*A*(*n, q*)) = 2*n* + 2].

One of the main statements of this talk is $A(n, q)$, where $n = 2, 3, ...,$ is a family of large girth.

We generalize these results in terms of the theory of algebraic graphs defined over an arbitrary field and consider properties and applications of the above-mentioned graphs.

Case of fini te simple graphs. All graphs we consider are simple, i. e., undirected without loops and multiple edges. Let $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices and the set of edges of Γ, respectively. The parameter $|V(\Gamma)|$ is called the order of Γ, and $|E(G)|$ is called the size of Γ. A path in Γ is called simple, if all its vertices are distinct. When its convenient, we shall identify Γ with the corresponding antireflexive binary relation on $V(\Gamma)$, i. e., $E(\Gamma)$ is a subset of $V(\Gamma) \times V(\Gamma)$. The length of a path is a number of its edges. The girth of a graph Γ, denoted by $g = g(\Gamma)$, is the length of the shortest cycle in Γ. Let $k \geq 3$ and $g \geq 3$ be integers. The distance between vertices v and u of the graph Γ is a minimal length of the path between them. The diameter of the graph is maximal distance between its vertices.

The graph is connected, if its diameter is finite. The graph is k-regular, if each vertex of the graph is incident exactly to k other vertices. The tree is a connected graph which does not contain a cycles.

(1) An infinite family of simple regular graphs Γ_i of constant degree k and order v_i such that $\operatorname{diam}(\Gamma_i) \leq c \log_{k-1}(v_i)$, where c is the constant independent of i and $\operatorname{diam}(\Gamma_i)$ is the diameter of Γ_i, is called a *family of small world graphs*.

(2) Recall that the infinite families of simple regular graphs Γ_i of constant degree k and order v_i such that $g(\Gamma_i) \geq c \log_{k-1}(v_i)$, where $\boldsymbol{c}$ is the constant independent of i and $g(\Gamma_i)$ is a girth of Γ_i are called *families of graphs of large girth.*

One of the main purposes of the paper is to present a special interpretations of q-regular tree (q-regular simple graph without cycles) in terms of the algebraic geometry over a finite field F_q.

Theorem 1. *For each prime power q, $q > 2$ there is a family of q-regular graphs Γ_i satisfying the following properties:*

(i) *Γ_i is a family of small world graphs;*

(ii) *Γ_i is a family of large girth;*

(iii) *Projective limit of graphs Γ_i is well defined and coincides with the q-regulatr tree T_q;*

(iv) *Cind $\Gamma_i = 2 \log_q(v_i / 2) + 2$.*

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Wiener Index of the Ideal-Based Zero-Divisor Graph of Finite Commutative Ring with Unity

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Abstract

All the rings considered in this paper are finite, commutative with unity. The *Wiener index* of a connected graph G is defined as $W(G) = \frac{1}{2}\sum d_G(u,v)$ with the summation running over all distinct pairs of vertices of G where $d_G(u,v)$ is the length of a shortest (u,v)-path in G. The *ideal-based zero-divisor graph* of a ring R with respect to an ideal I, denoted by $\Gamma_I(R)$, is a graph whose vertex set $V(\Gamma_I(R)) = \{x \in R\backslash I : xy \in I, \text{for some } y \in R\backslash I\}$ and two distinct vertices x and y are adjacent if and only if $xy \in I$. This concept was introduced by Redmond in [3]. If $I = (0)$ is the zero ideal of R, then $\Gamma_I(R) = \Gamma(R)$. Therefore, the ideal-based zero-divisor graph is a natural generalisation of zero-divisor graph of a ring. The Wiener index of zero-divisor graph of various rings have been extensively studied in the past. This include [1],[2],[4]. In this paper, we obtain the following results.

Theorem 1. If I is an ideal of R and H is the compressed zero-divisor graph of the quotient ring $\frac{R}{I}$, then

$$W(\Gamma_I(R)) = \sum_{\substack{1\le i\le k\\ a_i\in V(H) \text{ and } a_i^2\in I}} \binom{|A_{\overline{a_i}}||I|}{2} + \sum_{\substack{1\le i\le k\\ a_i\in V(H) \text{ and } a_i^2\notin I}} 2\binom{|A_{\overline{a_i}}||I|}{2} + |I|^2 \sum_{\substack{1\le i<j\le k\\ a_i,a_j\in V(H)}} |A_{\overline{a_i}}||A_{\overline{a_j}}| d_H(a_i,a_j).$$

As a consequence, we deduce the following result in [4].

Corollary 1. (Theorem 1, [4]). For any ring R,

$$W(\Gamma(R)) = \sum_{\substack{1\le i\le k\\ a_i^2=0}} \binom{|A_{a_i}|}{2} + \sum_{\substack{1\le i\le k\\ a_i^2\neq 0}} 2\binom{|A_{a_i}|}{2} + \sum_{1\le i<j\le k} |A_{a_i}||A_{a_j}| d_H(a_i,a_j).$$

Let m and n be the positive integers such that $m \mid n$ and $\mathbb{Z}_n$ be the ring of integers modulo n.

Theorem 2. If I is an ideal of $\mathbb{Z}_n$ generated by m, then

$$W(\Gamma_I(\mathbb{Z}_n)) = \sum_{\substack{m\nmid a_i \\ m|a_i^2}} \binom{\phi\left(\frac{m}{a_i}\right)|I|}{2} + 2\sum_{\substack{m\nmid a_i \\ m\nmid a_i^2}} \binom{\phi\left(\frac{m}{a_i}\right)|I|}{2} + |I|^2\Bigg(\sum_{m|a_ia_j} \phi\Big(\frac{m}{a_i}\Big)\phi\Big(\frac{m}{a_j}\Big)$$
$$+ 2\sum_{\substack{m\nmid a_ia_j \\ gcd(a_i,a_j)\neq 1}} \phi(\frac{m}{a_i})\phi\Big(\frac{m}{a_j}\Big) + 3\sum_{\substack{m\nmid a_ia_j \\ gcd(a_i,a_j)=1}} \phi\Big(\frac{m}{a_i}\Big)\phi\Big(\frac{m}{a_j}\Big)\Bigg).$$

As a consequence, we deduce the following result in [4].
Corollary 2 (Theorem 3, [4]). The Wiener index of $\Gamma(\mathbb{Z}_n)$ is given by

$$W(\Gamma(\mathbb{Z}_n)) = \sum_{a_i^2=0} \binom{\phi(\frac{n}{a_i})}{2} + \sum_{a_i^2\neq 0} 2\binom{\phi(\frac{n}{a_i})}{2} + \sum_{n|a_ia_j} \phi\Big(\frac{n}{a_i}\Big)\phi\Big(\frac{n}{a_j}\Big)$$
$$+ 2\sum_{\substack{n\nmid a_ia_j \\ gcd(a_i,a_j)\neq 1}} \phi\Big(\frac{n}{a_i}\Big)\phi\Big(\frac{n}{a_j}\Big) + 3\sum_{\substack{n\nmid a_ia_j \\ gcd(a_i,a_j)=1}} \phi\Big(\frac{n}{a_i}\Big)\phi\Big(\frac{n}{a_j}\Big).$$

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Lower position numbers of graphs

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1 Introduction

In 1900 the following puzzle was posed by the English mathematician Henry Dudeney:

Problem 1.1. *What is the largest number of pawns that can be placed on an $n \times n$ chessboard such that no three lie on a straight line?*

Despite its simple appearance, this problem remains unsolved in the general case. It has been referred to as 'one of the oldest and most extensively studied geometric questions concerning lattice points' [2]. A trivial upper bound is $2n$, whilst the best known lower bound $\frac{3n}{2} - o(n)$ is due to Hall [5].

This problem was generalised to the setting of graph theory independently by Ullas Chandran & Parthasarathy [3] and Manuel & Klavžar [7].

Definition 1.2. A set $S \subseteq V(G)$ of vertices of a graph G is in *general position* if no shortest path in G contains ≥ 3 vertices of S. The *general position number* $\mathrm{gp}(G)$ is the number of vertices in a largest general position set of G.

An example of a general position set is displayed in Figure 1.

This problem has been generalised to different types of paths, for example induced paths [9] and paths not exceeding a fixed length [6].

2 Lower position numbers

An interesting new direction in the general position problem was opened up by a puzzle of Martin Gardner (the modern Dudeney), drawing on work by Adena, Holton & Kelly [1].

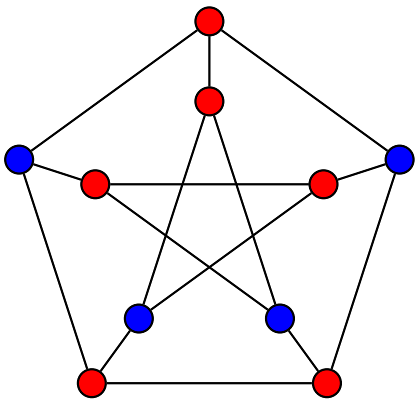

Figure 1: The Petersen graph with a maximum gp-set (red) and a lower gp-set (blue)

Problem 2.1. *What is the smallest number of vertices in a maximal set of vertices in the $n \times n$ grid with the no-three-in-line property that cannot be extended to a larger such set?*

If a greedy algorithm is used to produce a set of vertices of the grid with the no-three-in-line property, then the set asked for in Gardner's problem can be thought of as the worst-case output. We consider the analogous problem for graph theory.

Definition 2.2. The *lower general position number* $\mathrm{gp}^-(G)$ of a graph G is the number of vertices in a smallest maximal general position set in G.

For example, the blue vertices in Figure 1 form a lower gp-set. Another motivation for this problem comes from the theory of graph domination, in which the *upper domination number* of a graph is the largest size of a minimal dominating set. The case $\mathrm{gp}^-(G) = 2$ is also connected to the problem of the existence of a *universal line* in G [8], which originates in a generalisation of the De Bruijn-Erdős Theorem to finite metric spaces [4].

In this talk we will determine the lower general position number for common classes of graphs, discuss the relationship of this parameter with the hull number of a graph, prove the largest size of a graph with order n and given lower general position number, provide realisation results for the lower general position number vs. the general position number and general position sets produced by games, discuss the behaviour of this parameter under Cartesian products, and finally contrast the lower general position number with similar parameters defined using different sets of paths (such as induced paths).

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Combinatorial Games and Genetic Programming

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Abstract

This study proposes a method for solving unsolved combinatorial games using genetic programming. The authors customized a Python genetic programming package "gplearn", and they discovered new facts about combinatorial games using this package.

1 Genetic programming and combinatorial games

The use of mathematical formulas for the representation and description of data lends credence to such data because each representative formula can be mathematically proven. Researchers commonly spend much time discovering formulas by trial and error methods. However, this is not so with artificial intelligence (AI) through which computer systems are able to rapidly simulate human intelligence.

The authors used "gplearn", a Python genetic programming package. See [1], that searches a space of mathematical expressions (or formulas) for the best representation (or description) for a given input dataset. The fitness of a particular formula is determined by the least amount of residual (or error) between the data and formula. The program determines how data is functionally related, and uses a tree structure to represent the formula. See Fig. 1. The authors customized it by implementing new features, such as flexible conditional branching, a new common error metric, and a few discrete functions selected for the study. The research results show the capability of this method by presenting a counterexample to a well-known conjecture on a mathematical game and new facts about combinatorial games.

Only two previous studies have reported the research result on combinatorial games using AI. See [2] and [3], where genetic programming was used to discover formulas for well-known problems. By contrast, this study is the first to discover new facts.

The game chosen by the authors is a chocolate problem, which is a variant of Nim. The authors have published their research on chocolate problems in [4] and [5].

The chocolate game comprises a chocolate bar and rectangular array of squares with a bitter square at the bottom. Two players take turns at breaking the bar into two, either horizontally or vertically, and eat the part without a bitter square. Finally, the player who manages to leave his/her opponent with a single bitter block (black block) is the winner.

The chocolate bars in Fig. 2 is mathematically identical to the classical Nim with piles of 6, 3, and 7 stones. However, a chocolate with the coordinate system shown in Fig. 3 has a different mathematical structure. Combinatorial games have two groups of positions.

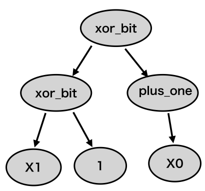

Figure 1: $(X1 \oplus 1) \oplus (X0+1)$.

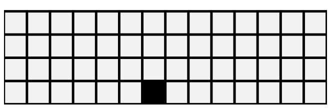

Figure 2: Rectangle chocolate $\{x, y, z\} = \{6, 1, 7\}$.

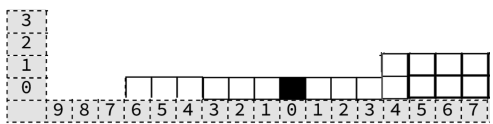

Figure 3: Chocolate $\{x, y, z\} = \{6, 1, 7\}$.

Definition 1.1. (a) A position is called a $\mathcal{P}$-*position* if it is a winning position for the previous player (the player who has just played) provided that he or she plays correctly.
(b) A position is called an $\mathcal{N}$-*position* if it is a winning position for the next player provided that he or she plays correctly.

Definition 1.2. (i) For any position $\mathbf{p}$, there exists a set of positions that can be reached by precisely making one move from $\mathbf{p}$, which we denote by $move(\mathbf{p})$.
(ii) The *minimum excluded value* (*mex*) of a set S of nonnegative integers is the smallest nonnegative integer that is not in S.
(iii) Each position $\mathbf{p}$ of an impartial game has the associated Grundy number, which is denoted by $G(\mathbf{p})$. The Grundy number is found recursively. $G(\mathbf{p}) = mex\{G(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

Theorem 1.3. *For any position* $\mathbf{g}$ *of* $\mathbf{G}$ *we have* $G(\mathbf{g}) = 0$ *if and only if* $\mathbf{g}$ *is a* $\mathcal{P}$*-position.*

For detailed information on combinatorial game theory, see [6]. A major problem in combinatorial games is determining the outcome of a pass move that may be used at most once in the game, and not from a terminal position. A long-standing unsolved problem involves determining a mathematical formula that describes the $\mathcal{P}$-positions of the classical Nim or chocolate game, as shown in Fig. 2 with a pass move. The authors aim to solve this problem through genetic programming.

First, the customized "gplearn" is applied to the chocolate game shown in Fig. 2, which is mathematically identical to the classical Nim. Then the game was analyzed under the condition that a pass-move is allowed. However, the authors did not determine an appropriate formula. They think that they have to revise their algorithm.

Second, the chocolate game, shown in Fig. 3 was analyzed. A pass move was also permitted. Consequently, the authors discovered the following theorem and conjecture: The following theorem was derived from calculation using "gplearn" and proved by the authors.

Theorem 1.4. *Let* x, y, z *be the coordinates of the chocolate bar. (See Fig. 3 as an example.)*
(i) *When there is no pass move,* $\{x, y, z\}$ *is a* $\mathcal{P}$*-position if and only if* $x \oplus y \oplus z = 0$*.*
(ii) *When a pass move is available,* $\{x, y, z\}$ *is a* $\mathcal{P}$*-position if and only if* x *is even and* $(x-1) \oplus y \oplus z = 0$ *or* x *is odd and* $(x+1) \oplus y \oplus z = 0$*.*

According to Theorem 1.4, a simple formula exists for $\mathcal{P}$-positions, regardless of whether a pass-move is permitted. This fact is a counterexample to the conjecture in [7] that combinatorial games with a simple formula for $\mathcal{P}$*-positions* become complicated games without a simple formula for $\mathcal{P}$*-positions* when a pass move is allowed. One of the authors presented another counterexample to the conjecture in [5], but Theorem 1.4 is better in the simplicity of conditions. Then, the following conjecture was derived from calculation using "gplearn".

Conjecture 1.5. $\{\{x, y, z\} : G(\{x, y, z\}) = 3\} = \{\{x, y, z\} : x = 0 \pmod 4$ *and* $(x-3) \oplus y \oplus z \oplus 3 = 0\}$ $\cup \{\{x, y, z\} : x = 1 \pmod 4$ *and* $(1-x) \oplus (-1-y) \oplus z = 0\}$ $\cup \{\{x, y, z\} : x = 2, 3 \pmod 4$ *and* $(x+2) \oplus y \oplus z \oplus 3 = 0\}$

Conjecture 1.5 shows the power of genetic programming. When $x = 1 \pmod 4$, $G(\{x, y, z\}) = 3$ if and only if $(1-x) \oplus (-1-y) \oplus z = 0$. "Bitwise exclusive or" ("BitXor") operations on negative numbers are rare in combinatorial game theory, and this formula is beyond the imagination of many mathematicians. In chess or the game of GO, they say that AI sometimes thinks up a move that is impossible for humans to make. This conjecture shows the remarkable power of AI. This study demonstrated that genetic programming can be used to discover discrete mathematical functions which have numerous applications in mathematics, and this presents a promising field for the application of AI.

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Restricted Nim with a Pass

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Abstract

This paper presents a study of restricted Nim with a pass. Two players take turns and remove stones from the piles. In each turn, each player is allowed to remove at least one stone and at most $\lceil \frac{m}{2} \rceil$ stones from a pile of m stones. It is well-known that in classical Nim, the introduction of the pass alters the underlying structure of the game, significantly increasing its complexity. In the restricted Nim considered in this study, there is a simple relationship between the Grundy numbers of restricted Nim and the Grundy numbers of restricted Nim with a pass.

1 Introduction

An interesting but difficult question in combinatorial game theory has been to determine what happens when standard game rules are modified to allow a one-time pass, that is, a pass move that may be used at most once in the game and not from a terminal position. Once a pass has been used by either player, it is no longer available. The effect of a pass on classical Nim remains an important open question that has defied traditional approaches.

In [1] (p. 370), Friedman and Landsberg conjectured that "solvable combinatorial games are structurally unstable to perturbations, while generic, complex games will be structurally stable." One way to introduce such a perturbation is to allow a pass. One of the authors of the present article reported a counterexample to this conjecture in [2]. The game used in [2] is solvable because there is a simple formula for the Grundy numbers, and even when we introduce a pass move to the game, there is a simple formula for $\mathcal{P}$-positions.

The restricted Nim considered in the present study is of the same type, but there is a simple relationship between the Grundy numbers of the game and the Grundy numbers of the game with a pass move. This result is stated in Theorem 1.8 of the present article. Let $Z_{\geq 0}$ and N be sets of non-negative numbers and natural numbers, respectively. For completeness, we briefly review some of the necessary concepts of combinatorial game theory. Details are presented in [3].

Definition 1.1. (a) A position is referred to as a $\mathcal{P}$-*position* if it is a winning position for the previous player (the player who has just moved), as long as he/she plays correctly at every stage.
(b) A position is referred to as an $\mathcal{N}$-*position* if it is a winning position for the next player as long as he/she plays correctly at every stage.
(c) For any position $\mathbf{p}$ of game $\mathbf{G}$, there is a set of positions that can be reached by precisely one move in $\mathbf{G}$, which we denote as *move*$(\mathbf{p})$.
(d) The *minimum excluded value* (*mex*) of a set S of non-negative integers is the smallest non-negative integer that is not in S.
(e) Let $\mathbf{p}$ be the position of an impartial game. The associated Grundy number is denoted as $G(\mathbf{p})$ and is recursively defined as follows: $\mathcal{G}(\mathbf{p}) = mex(\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\})$.

Theorem 1.2. *For any position of* $\mathbf{G}$, $G(\mathbf{g}) = 0$ *if and only if* $\mathbf{g}$ *is the* $\mathcal{P}$ *position.*

1.1 Maximum Nim

In this section, we study maximum Nim, which is a game of restricted Nim.

Definition 1.3. Suppose that there is a pile of n stones, and two players take turns removing stones from the pile. In each turn, the player is allowed to remove at least one stone and at most $\lceil \frac{x}{2} \rceil$ stones, where m represents the number of stones. The player who removes the last stone is the winner.

Lemma 1.4. *Let $\mathcal{G}$ represent the Grundy number of the maximum Nim with the rule sequence $f(x) = \lceil \frac{x}{2} \rceil$. Then, we have the following properties:*
(i) If t is even and $t \geq 2$, $\mathcal{G}(t) = \mathcal{G}(\frac{t-2}{2})$.
(ii) If t is odd, $\mathcal{G}(t) = \frac{t+1}{2}$.

Definition 1.5. The rule of the game is the same except that there are three piles of stones. The position of the game is represented by three coordinates $\{s, t, u\}$, where s, t, and u represent the numbers of stones in the first, second, and third piles, respectively. Let $\mathcal{G}(s, t, u)$ be the Grundy number of the game.

Theorem 1.6. *The Grundy number $\mathcal{G}(s, t, u)$ of the game of Definition 1.5 satisfies the following equation: $\mathcal{G}(s, t, u) = \mathcal{G}(s) \oplus \mathcal{G}(t) \oplus \mathcal{G}(u)$, where $\oplus$ is the bitxor.*

By Lemma 1.4 and Theorem 1.6 we determine Grundy numbers of Definition 1.3.

1.2 Maximum Nim with a Pass

We modify the standard rules of the games to allow for a one-time pass. The position of this game is represented by $\{t, p\}$, where t is the number of stones in the pile. $p = 1$ if the pass is still available; otherwise, $p = 0$. By Definitions 1.1, we define the Grundy number $\mathcal{G}(t, p)$ of the position $\{t, p\}$.

Theorem 1.7. *For the Grundy number $\mathcal{G}(s, p)$ of position $\{s, p\}$, we obtain the following equations:*
(i) $\mathcal{G}(0, 0) = 0$ and $\mathcal{G}(0, 1) = 0$.
(ii) For $u \in N$, if $\mathcal{G}(u, 0) = 0$, then $\mathcal{G}(u, 1) = 1$.
(iii) For $u \in N$, if $\mathcal{G}(u, 0) = 2$, then $\mathcal{G}(u, 1) = 0$.
(iv) For $u, m \in N$ such that $m > 1$, if $\mathcal{G}(u, 0) = 2m$, then $\mathcal{G}(u, 1) = 2m - 1$.
(v) For $u, m \in N$, if $\mathcal{G}(u, 0) = 2m - 1$, then $\mathcal{G}(u, 1) = 2m$.

Here, we study maximum Nim in Definition 1.5 with a pass. We denote the position of the game with $\{s, t, u, p\}$, where s, t, and u represent the numbers of stones in the first, second, and third piles, respectively. $p = 1$ if the pass is still available, and $p = 0$ otherwise.

Theorem 1.8. *We suppose that $s, t > 0$, $t, u > 0$, or $u, s > 0$. Then, for the Grundy number $\mathcal{G}(s, t, u, p)$ of the position $\{s, t, u, p\}$, the following statements hold:*
(i) $\mathcal{G}(1, 0, 0, 0) = \mathcal{G}(0, 1, 0, 0) = \mathcal{G}(0, 0, 1, 0) = 1$ and $\mathcal{G}(1, 0, 0, 1) = \mathcal{G}(0, 1, 0, 1) = \mathcal{G}(0, 0, 1, 1) = 2$.
(ii) $\mathcal{G}(1, 1, 0, 0) = \mathcal{G}(0, 1, 1, 0) = \mathcal{G}(1, 0, 1, 0) = 0$ and $\mathcal{G}(1, 1, 0, 1) = \mathcal{G}(0, 1, 1, 1) = \mathcal{G}(1, 0, 1, 1) = 1$.
(iii) $\mathcal{G}(1, 1, 1, 0) = 1$ and $\mathcal{G}(1, 1, 1, 1) = 0$.
(iv) $\mathcal{G}(s, 0, 0, 1) = \mathcal{G}(0, s, 0, 1) = \mathcal{G}(0, 0, s, 1) = \mathcal{G}(s, 1)$ for $s \in Z_{\geq 0}$.
(v) For any $m \in Z_{\geq 0}$, if $\mathcal{G}(s, t, u, 0) = 2m$, then $\mathcal{G}(s, t, u, 1) = 2m + 1$.
(vi) For any $m \in Z_{\geq 0}$, if $\mathcal{G}(s, t, u, 0) = 2m + 1$, then $\mathcal{G}(s, t, u, 1) = 2m$.

1.3 Traditional Nim with a pass

Although the traditional Nim with a pass is a very complicated game, it will become very simple if we restrict the use of pass in a certain way.

Theorem 1.9. *Let v be an odd number, and let $\{s, t, u, p\}$ be the position of the three-pile nim with a pass, where s, t, and u represent the numbers of stones in the piles, and p is the pass move. We suppose that the pass move is not available when $s \leq v$, $t \leq v$ and $u \leq v$. Then we have the following:*
(i) $\{s, t, u, 1\}$ is $\mathcal{P}$-position if and only if $s \oplus t \oplus u = 1$.

This theorem is based on the result of [4].

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Previous Player's Positions of Three-Dimensional Chocolate Games with a Restriction on the Size of Chocolate

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Abstract

The authors present a research on three-dimensional chocolate bar games, which are variants of three-pile Nim. They study $\mathcal{P}$-positions from which the next player who will play can force a win, as long as he or she plays correctly at every stage, and present a sufficient condition for the case when the chocolate is a $\mathcal{P}$-position if and only if $p \oplus q \oplus r$, where $p+1, q+1$, and $r+1$ are the chocolate bar's length, height, and width.

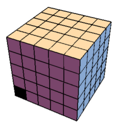
Keywords: Nim, chocolate game, Previous Player's Position.

2010 MSC: Primary 91A46; Secondary 97A20.

1 Introduction

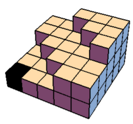
Chocolate bar games are variants of the game of Nim, and they were first presented in [1]. For the details of combinatorial games, we refer to [4]. Let $Z_{\geq 0}$ and N be sets of non-negative integers and natural numbers, respectively.

Definition 1.1. A three-dimensional chocolate bar is a three-dimensional array of cubes in which a bitter cubic box printed in black is present in some part of the bar. Figures 1 and 2 displays examples of three-dimensional chocolate bars. Each player in turn cuts the chocolate horizontally or vertically along the grooves. The player who manages to leave the opponent with the single bitter cubic box is the winner. Examples of how to cut chocolate bars are depicted in Fig. 3, Fig. 4, and Fig. 5.



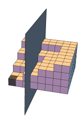
Cubic choco

Figure 1.



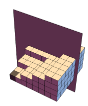
3D choco

Figure 2.



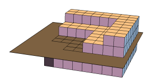
A vertical cut.

Figure 3.



A vertical cut.

Figure 4.



A horizontal cut.

Figure 5.

The following are important concepts in combinatorial game theory.

Definition 1.2. (a) $\mathcal{N}$-positions, from which the next player can force a win, as long as he plays correctly at every stage.
(b) $\mathcal{P}$-positions, from which the previous player (the player who will play after the next player) can force a win, as long as he plays correctly at every stage.
(c) For any position $\mathbf{p}$, there is a set of positions that can be reached by precisely one move , which we denote as $move(\mathbf{p})$.
(d) The *minimum excluded value* (*mex*) of a set S of non-negative integers is the least non-negative integer that is not in S.
(e) Each position $\mathbf{p}$ of an impartial game has an associated Grundy number, and we denote it as $\mathcal{G}(\mathbf{p})$. The Grundy number is found recursively: $\mathcal{G}(\mathbf{p}) = mex(\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\})$.

Theorem 1.3. *For any position* $\mathbf{p}$ *of the game,* $\mathcal{G}(\mathbf{p}) = 0$ *if and only if* $\mathbf{p}$ *is a* $\mathcal{P}$*-position.*

It can be easily determined that the $5 \times 5 \times 5$ cubic chocolate in Fig.1 is mathematically the same as Nim with heaps of 4, 4, and 4 stones, and the Grundy number of this is $4 \oplus 4 \oplus 4$. Hence, it is natural to ask the following question.

Question 1. *What is the necessary and sufficient condition whereby a three dimensional chocolate bar may have a Grundy number $p \oplus q \oplus r$, where $p+1, q+1$, and $z+1$ are the length, height, and width of the bar?*

The answer to this question is presented by one of the authors in [2], but the result of this research is omitted here.

When the Grundy number of a chocolate bar with $p+1, q+1$, and $r+1$ as the length, height, and width, is $p \oplus q \oplus r$, by Theorem 1.3 this chocolate bar is a $\mathcal{P}$-position if and only if $p \oplus q \oplus r = 0$.

Therefore, it is natural to ask the following question.

Question 2. *Under what condition a three dimensional chocolate bar with the length $p+1$, the height $q+1$, and the width $r+1$ is $\mathcal{P}$-position if and only if $p \oplus q \oplus r = 0$?*

Definition 1.4. Let $k \in N$ and $f(x,y) = \lfloor \frac{x+y}{k} \rfloor$. The three-dimensional chocolate bar comprises of a set of $1 \times 1 \times 1$ sized boxes. For $u, w \in Z_{\geq 0}$ such that $u \leq x$ and $w \leq z$, the height of the column of position (u, w) is $\min(f(u,w), y) + 1$. There is a bitter box in position $(0,0)$. We denote this chocolate bar as $CB(f, x, y, z)$. Note that $x+1, y+1$, and $z+1$ are the length, height, and width of the bar.

The authors presented the following sufficient conditions for Question 2 in [3].

Theorem 1.5. *Let $f(x,z) = \lfloor \frac{x+z}{k} \rfloor$ for $k = 4m+3$. Then, the chocolate bar $CB(f,x,y,z)$ is a $\mathcal{P}$-position if and only if*

$$x \oplus y \oplus z = 0. \tag{1.1}$$

After the authors proved Theorem 1.5, they conjectured that Theorem 1.5 is not valid for an even number k, but they discovered the following theorem.

Theorem 1.6. *Let $f(x,z) = \lfloor \frac{x+z}{k} \rfloor$ for $k = 2^{a+2}m + 2^{a+1}$, where $a, m \in Z_{\geq 0}$.*

(i) For

$$x, z \leq (2^{2a+2} - 2^{a+1})m + 2^{2a+1} - 1 \tag{1.2}$$

chocolate bar $CB(f,x,y,z)$ is a $\mathcal{P}$-position if and only if

$$x \oplus y \oplus z = 0. \tag{1.3}$$

(ii) When (1.2) is not satisfied, there exist some $\mathcal{P}$-positions that do not satisfy (1.3).

Remark 1.7. Note that the set $\{2^{a+2}m + 2^{a+1} : a, m \in Z_{\geq 0}\}$ is the set of even numbers.

The following conjecture is based on computer calculation.

Conjecture 1.8. *Let $f(x,z) = \lfloor \frac{x}{k} \rfloor + \lfloor \frac{z}{k} \rfloor$. Chocolate bar $CB(f,x,y,z)$ for $x, y \leq s$ is a $\mathcal{P}$-position if and only if*

$$x \oplus y \oplus z = 0. \tag{1.4}$$

when one of the following conditions is satisfied.
(i) k is an odd number and $s = k$
(ii) $k = 4m+2$, where $m \in Z_{\geq 0}$ and $s = 4m+3$.
(iii) $k = 2^m$, where $m \in N$, $m \geq 2$ and $s = 2^{3m-1}$.

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Unfolding Skeletons

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Unfolding polyhedra has been an active area of mathematical research since the 70s [5]. Generally, an unfolding is a development of the surface of a given polyhedron to a polygon in the plane. Such polygon must be weakly simple (the image of the interior of faces of the polyhedron do not overlap). The *edge unfolding* further requires that the edges of the polygon correspond to the edges of the original polyhedron. The polygon resulted from an edge unfolding together with the image of the polyhedron edges (*crease lines*) is called a *net*. It is a long standing open problem whether every convex polyhedron admits an edge unfolding [5].

The notion of unfolding has typically been generalized for higher-dimensional polytopes as follows. We view a n-polytope as a polyhedral complex, composed by "faces" of dimension $< n$. An $(n-1)$-face of the polytope is called a *facet* and a $(n-2)$-face is called a *ridge*. A *ridge unfolding* of a given n-polytope is then a continuous isometric injective map from its $(n-1)$-dimensional surface (minus a subset of its ridges) into a connected polytope in $\mathbb{R}^{n-1}$. The subset left out of the domain of the map can be seen as the ridges at which we must cut in order to lay out the $(n-1)$-dimensional surface in $\mathbb{R}^{n-1}$.

For small polyhedra such as the cube, we can enumerate all nets. It is folklore that the number of such combinatorially different nets is 11 (up to symmetry). Turney showed in 1984 [6] that there are 261 different ways (up to symmetry) to cut the tesseract in order to get a ridge unfolding, and Desplinter et al. [3] showed that all such ways lead to a net, i.e., they can be unfolded in $\mathbb{R}^3$ without overlap (not only for the tesseract but for an arbitrary n-cube). When trying to reduce the dimension from 4 to 2, Diaz and O'Rourke [4] propose the edge unfolding of the polyhedra obtained by the ridge unfolding of the tesseract. Such "unfoldings of unfoldings" do not include 2-faces that are not cut in the ridge unfolding and include twice each 2-face that is cut.

Inspired by a discussion during the 32nd Canadian Conference on Computational Geometry (CCCG 2020), we introduce a new notion of unfolding between two arbitrary dimensions: Given an n-polytope P, its k-skeleton is the minimal subcomplex containing all k-faces of P. We define the $(k-1)$-*face unfolding* of P as an unfolding of P's k-skeleton in $\mathbb{R}^k$. We give a precise definition in Section 1. The purpose of the -1 is so that the 2-face unfolding and the $(n-1)$-face unfolding correspond respectively to the edge and ridge unfoldings. With this generalization, every k-face of P appears in the net exactly once. We also show that we can easily identify whether a polytope admits a 0-face unfolding. In Sections 2 and 3 we describe the enumeration of a family of 2-face unfoldings of the tesseract.

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1 Definitions

We can describe the edge unfolding of a polyhedron using the set of edges that are cut [1]. It is known that, for convex polyhedra, such edges form a spanning tree of the vertices. Let this tree be the *cut tree*. Alternatively, we can define the unfolding using a spanning tree of the dual graph (adjacency graph of the faces of the polyhedron). Intuitively, an edge in such spanning tree correspond to edges that are not cut and, thus, its incident faces are also adjacent in the unfolding. Let this tree be the *uncut tree*. When generalized to ridge unfoldings of an n-polytope P, the cut tree is defined by the cut ridges and the uncut tree is a spanning tree of the 1-skeleton of the dual polytope [2]. Since the $(n-1)$-dimensional surface of P is a manifold, we can still describe the unfolding as a continuous isometric injective map from the surface minus the cut tree to $\mathbb{R}^{n-1}$. The closure of the image must be a connected weakly simple polytope.

However, the same is not true when generalizing to $(k-1)$-face unfoldings since the k-skeleton P_k of P is not a manifold for $k \in \{1, \ldots, n-1\}$. Since the target is a manifold, it is clear that every $(k-1)$-face of P must be cut in some way. Indeed, there are multiple ways in which one could cut at the $(k-1)$-face. See Figure 1 (left) for an example. We first must perform a "topological surgery" on P_k by *splitting* its $(k-1)$-faces into copies, each of which incident to at most two k-faces that were originally incident to the $(k-1)$-face. The result must be a connected manifold

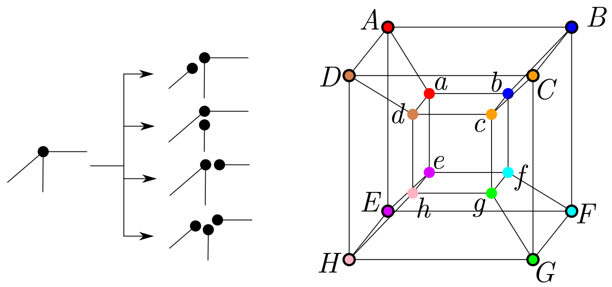

Figure 1: (Left) The four ways we can split a vertex of the cube to get locally a 1-manifold. (Right) Naming convention for the vertices of the tesseract.

P_k^*. The $(k-1)$-face unfolding is then a continuous isometric injective map from P_k^* to $\mathbb{R}^{n-1}$.

Just as before, we can also define the unfolding by an uncut tree. As a consequence of the above definition, the uncut tree of P_k is a spanning tree T_u of the adjacency graph of the k-faces of P_k such that no two k-faces f_1 and f_2 adjacent to a face f_3 in T_u share the same $(k-1)$-face with f_3. The following observation derives directly from the definition.

Observation 1 *An n-polytope P admits a 1-face unfolding if and only if its 1-skeleton admits an Eulerian trail.*

As a corollary, we get that the n-cube admits a 1-face unfolding if and only if n is even.

2 Enumerating the 2-face (edge) unfoldings of the tesseract

Here, we explore edge unfoldings of the tesseract $\mathcal{T}$. The tesseract has 24 2-faces, 32 1-faces, and 16 0-faces (here called *faces*, *edges*, and *vertices* respectively). Enumerating every possible uncut tree of the tesseract may be computationally infeasible. A crude upper bound can be obtained as follows. Each of the 32 edges can be split in 4 different ways so that the result is locally a manifold (pairing two of the incident faces or splitting all 3 incident faces). Then there are 4^{32} possible spanning forests of $\mathcal{T}_1$ (the 1-skeleton of $\mathcal{T}$). Furthermore, even if we get a single component T_u, its developing in the plane might cause two faces to overlap.

Here, we prune the search of 2D nets of $\mathcal{T}$ by gradually growing T_u while checking for overlaps. The idea is to fix a face of T_u and try all possible neighborhoods. To further prune our search, we root T_u at a leaf ℓ that realizes its diameter. This allow us to chose from a limited number of neighborhoods of the root breaking down the search into three cases as follows. We branch based on the degree of the only child f of ℓ. We use the naming convention of vertices shown in Figure 1. Due to symmetry, we can fix $\ell = CcbB$ and $f = CBFG$ without loss of generality. Case 1 occurs when $\deg(f) = 1$. There are two subcases depending on whether f is a turn or not (see Figure 2). Case 2 occurs when $\deg(f) = 2$. On this case, one of the children of f must be a leaf ℓ' or else this would contradict the choice of ℓ. There are again two subcases based on the relative position of ℓ and ℓ'. Furthermore, because each edge is incident to 3 faces, there are two possible choices of ℓ' for each of these cases. Case 2 occurs when $\deg(f) = 3$. For the same reason as above, two children of f must be leaves. In each case we proceed exploring T_u recursively similar to a DFS but trying all available possibilities for children of the current face. At each explored face, we check for overlaps.

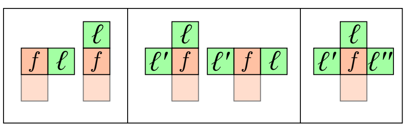

Figure 2: Pinning different neighborhoods of the root ℓ of T_u. From the left Cases 1, 2, and 3.

Once we explore all faces, the obtained T_u corresponds to a 2D net of $\mathcal{T}$. We store all nets found and test if we have already found a congruent net. To do so we store each net T_u now rooted at its center. This is because, unlike a leaf that realizes the diameter, the center of a tree is a unique node or edge. This allows for faster checks of congruency.

3 Results and Conclusion

We have implemented our search in `https://github.com/vasisth/Skeleton-Unfolding`. We could enumerate all 2D nets in Cases 2 and 3 (available at the same link). The numbers of unique nets found are 136,472,616 and 25,338,744 respectively. However our search did not terminate for Case 1, indicating that the majority of nets are of this category. We conjecture that further pruning can make the enumeration of all 2D nets of $\mathcal{T}$ possible.

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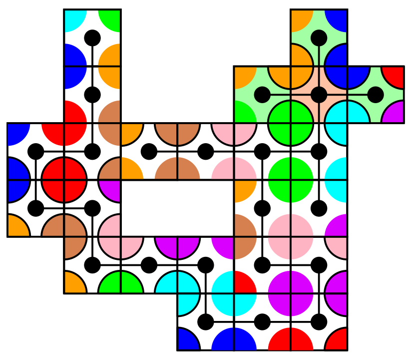


Figure 3: 2D net obtained from Case 3. Vertices of the faces are color coded according to Figure 1. The neighborhood of ℓ is colored as in Figure 2.

A transformation from map folding to Boolean matrix algebra

Yiyang Jia* Thomas C. Hull†

1 Introduction

Map folding and related problems have been studied in terms of computational origami. We introduce a new approach by modeling map folding with Boolean matrices and an associated algebraic structure.

First, we represent each flat-folded state (including partially-folded flat states) of a $1 \times n$ map is by a *state Boolean matrix*. Then, all state matrices form a matrix space, which later serves as the background of our algebraic discussion.

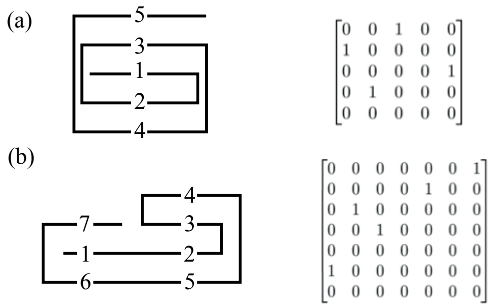
We make associations between flat states by first decomposing them into permutations of layers and foldings that make new face-to-face adjacencies. These are then translated into a sequence of Boolean matrix operations. We employ two different types of operators in our matrix description: *permutation operators* and *addition operators*, which, respectively, use natural addition and multiplication of the two-element Boolean algebra. Permutation operators form an operator group, while addition operators form an operator monoid. These operators allow us to construct a groupoid structure and a Grothendieck topology on the space of state matrices. This study also introduces several solutions to computational origami problems. Proofs of our results will be provided in a forthcoming paper.

2 State Matrix of Map Folding

2.1 Definition of state matrix

For any flat state of a $1 \times n$ map fold, including partly and fully flat-folded states, a certain state matrix can be defined. The definition is as follows.

In a flat state S of the map, for a pair (i, j) of two squares, we determine the (i, j)-entry of the corresponding state matrix by the adjacency between faces i and j. Specifically, if S is a fully folded state of the map and square i lies immediately below square j in S, or if S is some partly folded state and square i is adjacent to (directly touching) square j and must lie below square j in any corresponding fully folded state, then we assign 1 to the (i, j)-entry of the state matrix. Otherwise, we assign 0 to the (i, j)-entry. Figure 1 shows an example.



$$\begin{bmatrix} 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$$

Figure 1: (a) A fully folded flat state of a 1×5 map and its state matrix. (b) A partly folded flat state of a 1×7 map and its state matrix.

2.2 Uses in computational origami

Some related computational problems can be easily solved by matrix computation. We list two below.

(1) **Recover the MV assignment from the state matrix:** We can recover the mountain-valley assignment of the map from the state matrix of a fully folded state. Since the power sum Q of the state matrix corresponds to the transitive closure of the adjacency between layers, the "up-down" relation of all squares is recorded in Q . That is, the fold between i and i+1 is a valley if the (i, i+1)-entry of Q is 1 and i faces its front side up, or if the (i, i+1)-entry of Q is 0 and i faces its front side down. Otherwise, the crease between i and i+1 is a mountain.

(2) **Check for self-intersections:** Self-intersections can be detected by the number of 1s in the 2×2 sub-matrix of the power sum Q of the state matrix. Namely, a self-intersection exists in the corresponding state if and only if there exists at least one 2×2 sub-matrix in Q with an odd number of 1s (or 0s).

3 Matrix Operators

Two types of matrix operators, *permutation operators* and *addition operators*, are defined here. Their combination suffices to represent a folding operation

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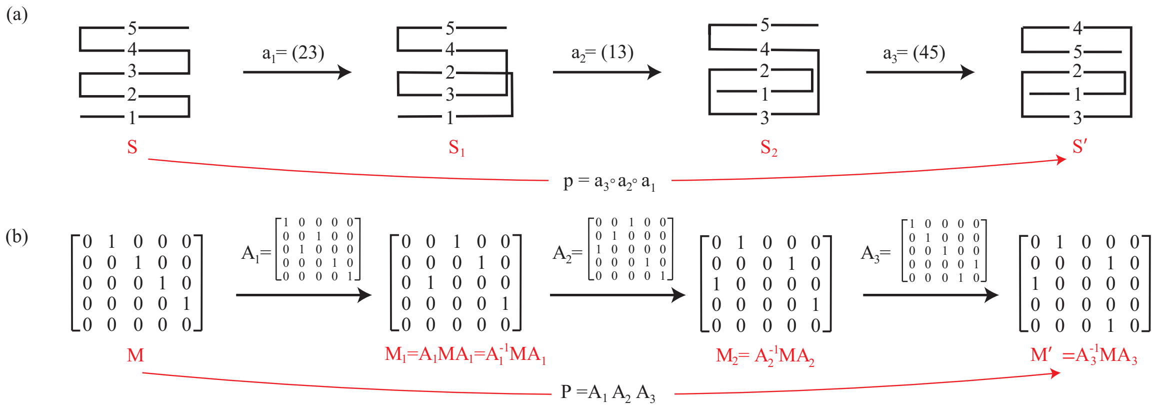

Figure 2: (a) Decomposition of the permutation between two possible states of a 1×5 map, S and S', into three transpositions in order. (b) The corresponding state matrices and matrix operators.

connecting two different flat states.

3.1 Permutation operators

We use permutation operators to represent transitions between flat states that have the same layers but different layers' orderings. The definition is as follows.

Definition 3.1 *An $n \times n$ matrix A is called an transposition operator if (1) there exists a pair of (i, j) ($1 \leq i, j \leq n$) such that the 2×2 sub-matrix formed by the elements $A(i,i)$, $A(i,j)$, $A(j,i)$, $A(j,j)$ is an anti-identity matrix, and (2) the $(n-2) \times (n-2)$ matrix formed by the other rows and columns except the ith and jth ones from (1) is an identity matrix. The pair (i, j) is called a* transposition pair. *A* permutation operator *is the matrix product of a series of transposition operators.*

Our result is that, if the transition between two flat states S_1 and S_2 is achieved only by rearranging the layers, then there exists a unique permutation operator P such that $M_2 = P^{-1}M_1P$ and $M_1 = PM_2P^{-1}$, where M_1 and M_2 denote the state matrices of S_1 and S_2, respectively. Figure 2 shows an example.

Since any permutation operator is a product of some transposition operators, the entire set of permutation operators of $1 \times n$ maps forms a group. Furthermore, it is isomorphic to the symmetric group $Sym(n)$ (i.e., S_n).

It is natural to view the permutation operation as a kind of equivalence relation between the fold states. Based on this equivalence relation, the entire set of flat states of any map, which corresponds to the space of state matrices, forms a groupoid. Furthermore, taking the discrete topology of the set of flat states, the aforementioned groupoid becomes a fundamental groupoid [2]. This structure indicates the possibility of considering the space of flat states in terms of category theory and homotopy theory.

3.2 Addition operators

We further introduce addition operators to describe the creation of new face adjacencies made by folding partially-folded states. The definition is given below.

Definition 3.2 *Any $n \times n$ matrix B where only one non-diagonal matrix element is 1 and the others are 0 is called a **basic addition operator**. Let $\mathbf{B}$ stand for the set of all possible basic addition operators. Any $n \times n$ matrix C is called an **addition operator** if there exists a sequence of basic addition operators B_1, $B_2, \cdots, B_k \in \mathbf{B}$ with $C = B_1 + B_2 + \cdots + B_k$, where $+$ is specified as element-wise Boolean addition.*

Then transitions between arbitrary flat states can be represented by a combination of permutation operators and addition operators.

As for the algebraic structure, the fact that any addition operator is the sum of some basic addition operators makes the entire set of addition operators a monoid. Furthermore, in the space of state matrices, a partial order can be defined by the addition operations. Then, a Grothendieck topology can be built on this poset [1]. This structure shows the possibility of using homological algebra theory to consider the space of flat states.

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Folding every point on a polygon boundary to a point

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Abstract

Given a convex polygon P with n vertices and a point f inside P, we consider a folding problem where every point on the boundary of P is folded to f. This problem, motivated by recent work by Akitaya, Ballinger, Demaine, Hull, and Schmidt [CCCG'21] who considered the problem of folding each corner of P to f, has applications in industrial design. If the vertices of P is given in a circular order, we give a linear-time algorithm for computing the description of the crease lines from the folding. Our algorithm is simple and follows the structure of Graham's convex hull algorithm.

1 Introduction

Paper folding offers a rich computational geometry problems with many real world applications. The topic, typically referred to as *computational origami* or *mathematics of paper folding* [DO08, Hul20], studies both feasibility problems and also structural problems [ABD+21] with the aim to illuminate the connections between physical structures/problems and mathematical geometric objects. As geometric construction using straightedge and compass offers elegant connections between algebra and geometry, paper folding, which can be seen as geometric construction with additional operations, may provide beautiful structural properties worth studying.

This work considers a folding problem where a piece of convex polygonal paper P is given together with a point f inside P. We then fold and unfold every point on P's boundary onto f. We would like to find the inner region with no creases of P containing f. This problem has applications in industrial folding when the target piece has some folding constraint (represented as f in this formulation). See Figure 1 as example.

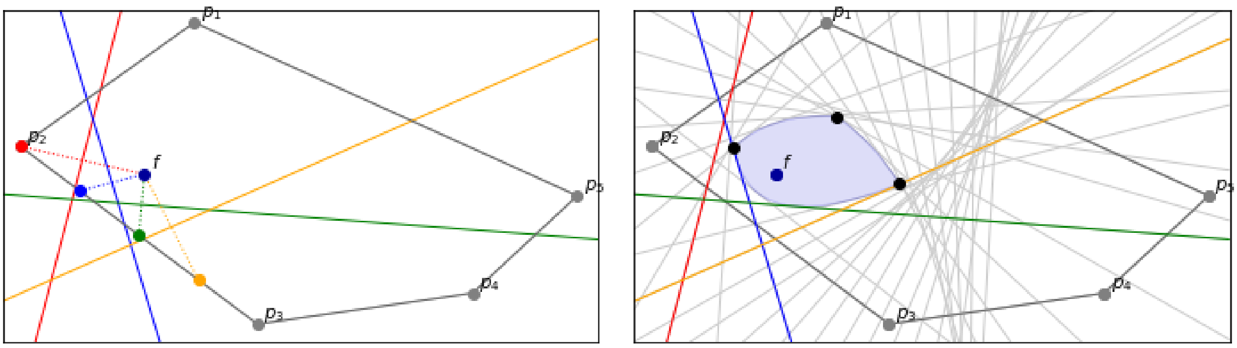

Figure 1: Left, example creases from some points on an edge of the polygon. Right, the region after folding multiple creases close to infinity.

When the points on the boundary considered are only polygon corners, Akitaya, Ballinger, Demaine, Hull, and Schmidt [ABD+21] showed that the crease lines resemble a Voronoi cell where the corners and the marked point acted as seeds and gave a linear-time algorithm for computing it, provided that the polygon corners are given in a circular order. Figure 2 compares the settings of Akitaya *et al.* and ours.

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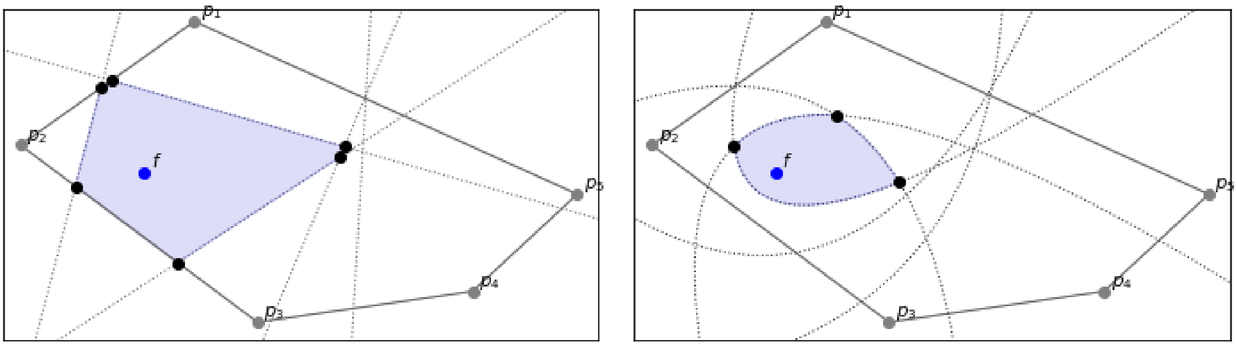

Figure 2: (A) Corner folding in [ABD+21] (B) Boundary folding in our paper

2 Our results

We give a characterization of the resulting region from the crease lines as a region whose boundary is a union of segments from parabolas whose focus is f with P's boundary segments as directrixes. We prove the following theorem.

Theorem 2.1. *Given a polygon P with n vertices provided in a circular order and a point f, there exists a linear-time algorithm that finds the crease-line-free region containing f.*

The algorithm follows the structure of the Graham's scan algorithm [Gra72] for finding convex hulls from a set of points. It iteratively considers each vertex u of P and determines (in constant time) if the parabola segment induced by the focus f and the new polygon segment adjacent to u as directrix should be included in the solution.

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Ising spin model for global flat-foldability of origami

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Origami is a traditional game of folding a sheet of paper to create animals, plants, object, or some geometric patterns. In order to obtain some shape with origami, we have to program the locations of facets and creases on a paper. The design of these locations on an unfolded paper is called an origami diagram here. The problem to determine a given diagram is actually foldable into a plane is called the flat-foldability problem, which is proved to be NP-hard [1].

Flat-foldability problem of an origami is represented with Ising spin model in statistical mechanics [2]. The approaches from statistical mechanics have often considered the problem of assigning mountains and valleys to creases. However, here we see that it is possible to find an intuitive perspective or a more direct discrimination of unfoldability by considering the order for all pairs of facets, instead of focusing only on creases.

If the sum of angles alternation is 0 for all vertices in the origami diagram, then there is a combinatorial problem with layer-ordering of facets. We map this combinatorial problem onto the ground state search problem of spin glass model on random graphs. If the origami diagram is locally flat-foldable around each vertex, a pre-folded diagram, showing the planar-positional relationship of the facet, can be obtained. The Ising-spin variable is assigned for the layer-ordering of each pair of facets which have an overlap in the pre-folded diagram. In addition, the energy function which consists of interactions of spins, namely products of 2- or 4-spin variables, is implemented to prohibit the intrusion of each facet into the other component of the same origami diagram. The 2-spin interactions prohibit the intrusion of a facet into a junction of pair of two adjacent facets. The 4-spin interactions are prepared to attach other certain type of prohibitions. When two edges, which are junctions of pairs of facets, are partially identical as shown in Fig.1(b), no explicit intrusion occurs. However in these cases the layer-orderings are subject to certain constraints so that the stackings shown in Fig. 2(c) are not realizable while those shown in Fig. 2(a) and (b) are accepted. When the facets angle has a continuous random value, for example, when it is generated from a probability distribution, it is rare that the two edges become partially identical. Hence the 4-spin interaction may seem to play no major role. However, in the cases that there is a finite minimum angle and all other angles are given as a multiple of it, as is often seen in some traditional origami, they are thought to become important.

We consider cycles formed by chains of the terms of 2- or 4-spin product. If the "frustrated" ones [3] are included in them, they become sources of unfoldability. In the presentation, we will see the frustration from the interactions. It is considered that this cycle-based discrimination in this model is a refinement description of Justin's ideas [4], which is referred in Hull's literature [5].

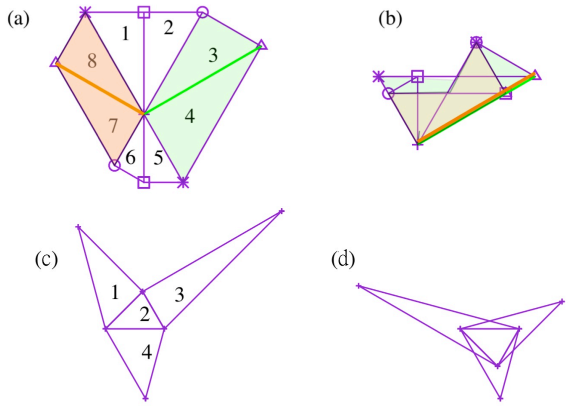

Fig 1. (a)Origami diagram for the case which is flat-foldable. (b)Pre-folded diagram corresponding to the diagram (a). (c)Origami diagram for the case which is not flat-foldable. (d)Pre-folded diagram corresponding to (c).

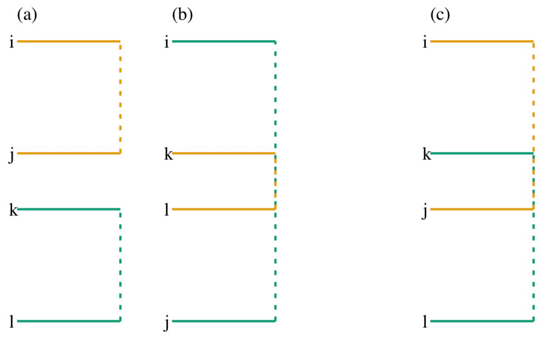

Fig 2. Cross sectional figure of the partially coincident junctions such as shown in Fig. 1(b). Ordering of facets like (c) is prohibited while that like (a) or (b) is accepted.

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Stars in forbidden triples generating a finite set of 4-connected graphs

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Abstract

Let $\mathcal{G}$ and $\mathcal{G}_4(\mathcal{F})$ denote the collection of all connected graphs having at least three vertices and the collection of 4-connected $\mathcal{F}$-free graphs, respectively, where $\mathcal{F} \subseteq \mathcal{G}$. A graph G is said to be $\mathcal{F}$-free if for every $F \in \mathcal{F}$, G does not contain F as an induced subgraph. The members of $\mathcal{F}$ are called forbidden subgraphs and the set $\mathcal{F}$ with order three is called a forbidden triple. In this paper, we determine forbidden triples $\mathcal{F}$ for which $\mathcal{G}_4(\mathcal{F})$ is finite. Specifically, this paper proves that if $\mathcal{G}_4(\{K_3, K_{1,m}, T\})$ is finite, where $m \geq 5$, then T is either a path of order at least four, or a caterpillar having maximum degree at most four such that no degree four vertex is adjacent to a vertex of degree three or higher, and no three vertices of degree three are contiguously adjacent. Moreover, this paper proves that $\mathcal{G}_4(\{K_n, K_{1,m}, T\})$ is finite, where $n \geq 4$ and $m \geq 5$ if and only if T is a path. Also, it will be shown that if $\mathcal{G}_4(\{K_n, K_{1,m}, T\})$ is finite, where $3 \leq m \leq 4$ and $n \geq 4$, then T is a cactus such that all cycles of T are triangles and whose block-cutvertex graph is a path.

Keywords: Forbidden triples, 4-connected family of graphs

In this paper, we will determine forbidden triples of the form $\mathcal{F} = \{K_n, K_{1,m}, T\}$ for which $\mathcal{G}_4(\mathcal{F})$ is finite by describing T as a member of the following sets.

1. $\mathcal{T}_0$ is the set of trees in $\mathcal{G} - \{K_{1,2}, K_{1,3}, K_{1,4}\}$ having maximum degree at most 4.

2. $\mathcal{T}_1$ is the set of caterpillars belonging to $\mathcal{T}_0$ in which no degree four vertex is adjacent to a vertex of degree three or higher, and no three vertices of degree three are contiguously adjacent.

3. $\mathcal{T}_2 = \{P_l, Y_m, Y_n^* : l \geq 4, m \geq 3, n \geq 2\}$.

4. $\mathcal{T}_0^*$ is the set of those cacti T in $\mathcal{G} - \{K_{1,2}, K_3\}$ such that all cycles of T are triangles.

5. $\mathcal{T}_1^*$ is the set of those members of $\mathcal{T}_0^*$ whose block-cutvertex graph is a path.

Figure 1: Graphs Y_3 and Y_2^*

Figure 2: Graphs Q_3 and Q_2^*

6. $\mathcal{T}_2^* = \{P_l, Q_m, Q_{2n}^* : l \geq 4,\ m \geq 2,\ n \geq 1\}$.

The graphs Y_3 and Y_2^* are shown in Figure 1 while Q_3 and Q_2^* are shown in Figure 2.

1 Results

The following are the results proved in this paper.

Theorem 1.1. Let $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}, K_{1,4}\}$. If $\mathcal{F} = \{K_3, K_{1,m}, T\}$ with $m \geq 5$ and $\mathcal{G}_4(\mathcal{F})$ is finite, then $T \in \mathcal{T}_2$.

Theorem 1.2. Let $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}, K_{1,4}\}$ and $\mathcal{F} = \{K_n, K_{1,m}, T\}$ with $n \geq 4, m \geq 5$. $\mathcal{G}_4(\mathcal{F})$ is finite if and only if T is a path.

Theorem 1.3. Let $\mathcal{F} = \{K_n, K_{1,m}, T\}$ where $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}, K_3\}$, $n \geq 4$ and $3 \leq m \leq 4$. If $\mathcal{G}_4(\mathcal{F})$ is finite, then $T \in \mathcal{T}_1^*$ if $n = 4$, and $T \in \mathcal{T}_2^*$ if $n \geq 5$.

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STABLE HOMOLOGY-BASED CENTRALITY MEASURES FOR WEIGHTED GRAPHS

JOHN RICK D. MANZANARES AND PAUL SAMUEL P. IGNACIO AND PAUL SAMUEL P. IGNACIO

1. Introduction

Giscard and Wilson [1] introduced loop-centrality measure that uses the paths intersecting a loop to measure its importance. This centrality measure has the ability to detect essential proteins in protein-protein interaction networks. For the same purpose, Estrada and Ross [2] explored an extension of loop centrality to finite-dimensional substructures where higher-order connectivity networks are represented by triangles and tetrahedra. A widely accepted notion of importance for cycles in a simplicial complex regards long-lived cycles as essential features of data, while short-lived cycles that appear are likely to be due to noise. However, Bubenik et al. [3] demonstrated that short-lived cycles hold important information that can be used to estimate the curvature of surfaces.

In this study, we propose novel centrality measures that leverage the persistence of homology classes and their merge history along the filtration. Integral to this is the development of an algorithm that captures the merge history of homology classes. Accordingly, we establish some properties including stability of these measures under a distance analogous to norms in Lebesgue spaces and persistence landscapes.

2. Methodology

Throughout this paper, we shall use [4] and [5] as standard references for discussions involving simplicial and persistent homology.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Our main objects are (abstract) simplicial complexes which are collections $\mathscr{C}$ of subsets of a finite set V such that $\tau \subseteq \sigma \in \mathscr{C}$ implies $\tau \in \mathscr{C}$. An element $\sigma \in \mathscr{C}$ is called an *(abstract) simplex* with *dimension* $|\sigma| - 1$.

We can build a simplicial complex $\mathscr{C}$ from G by declaring that $\sigma \in \mathscr{C}$ provided that σ is a complete subgraph of G. This construction produces a combinatorial model of G where complete subgraphs represent higher-order interactions between vertices.

The *chain space* C_k is the free group generated by k-simplices with coefficients coming from $\mathbb{Z}/2\mathbb{Z}$, and the *boundary operator* $\partial_k : C_k \to C_{k-1}$ is the linear extension of the map on generators given by

$$\partial_k ([x_0, \ldots, x_j, \ldots, x_k]) = \sum_{i=0}^{k} [x_0, \ldots, \hat{x_i}, \ldots, x_k]$$

where $\hat{x_i}$ indicates that x_i is omitted. Coefficients in the chain space encode the presence or absence of edges.

The *cycle space* Z_k and *boundary space* B_k of C_k are defined by $Z_k = \ker \partial_k$, and $B_k = \text{Im } \partial_{k+1}$. We refer to elements of Z_k as k-cycles and that of B_k as k-boundaries. The *kth homology group* $H_k(\mathscr{C})$ of $\mathscr{C}$ is the quotient Z_k/B_k. Two cycles in $Z_k(\mathscr{C})$ lying in the same homology class in this space are said to be *homologous.*

Now, we endow a graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}_{\geq 0}$ to induce a nested sequence of graphs. This *filters* a graph at a given weight by removing edges in excess of such weight. Hence, we also refer to weights as *thresholds.* The nested sequence of graphs $\{G_{w_i}\}$ generates a sequence of simplicial complexes $\{\mathscr{C}_{w_i}\}$, which induces a sequence of homology groups $\{H_k(\mathscr{C}_{w_i})\}$. In addition, by functoriality, these homology groups are successively related by a sequence of maps induced by the monotonic sequence of weights.

By appealing to persistent homology, we track the generators of each homology class (and their *persistence*) at a given threshold that survive as it is sent by the induced map to a homology group at a larger threshold. This also induces a natural ordering of representative generators for homology classes that merge by choosing the generator formed at the smaller threshold. This is known as the *elder rule.* This situation demonstrates the notion of *transferring* persistence, and is a key observation why we examine the merge dynamics among homology classes.

A standard way to compute persistent homology is via matrix reduction due to Zomorodian and Carlsson [6]. From this method, we can determine the merge instances using the following results. First, we let $c(\sigma)$ be the column associated with the simplex σ in the boundary matrix and $r(\sigma)$ be the representative of $c(\sigma)$ due to the reduction algorithm. Formally, we say that two homology classes $[\sigma]$ and $[\nu]$ *merge* at time $\epsilon' = \min\{d(\sigma), d(\nu)\}$ if they are homologous at ϵ', where $d(\sigma)$ is the death threshold of σ.

Lemma 2.1. *Let $\nu = \sum_p \nu_p$ be a cycle where ν_p gives birth to ν for some p. Suppose that $[\sigma]$ and $[\nu]$ be distinct kth homology classes in a filtered simplicial complex satisfying $b(\nu) \leq b(\sigma)$, where $b(\nu)$ denotes the birth threshold of ν. If σ and ν are adjacent with intersection δ, and $\nu_p \in \delta$, then the cycle representative of $[\sigma]$ produced by the reduction algorithm is given by $\sigma + \nu$.*

Theorem 2.2. *Let $[\sigma]$ and $[\nu]$ be distinct persistent homology classes where $d(\sigma) \neq d(\nu)$. Then $[\sigma]$ and $[\nu]$ merge at $\epsilon = \min\{d(\sigma), d(\nu)\}$ if and only if σ and ν are adjacent.*

Consider the induced homomorphism $H_k(\mathscr{C}_{w_i}) \to H_k(\mathscr{C}_{w_j})$ and let $[\sigma] \in H_k(\mathscr{C}_{w_j})$. We define the *first order merge cluster* $M_1[\sigma, w_j]$ of $[\sigma]$ at w_j as the set of homology classes merging with $[\sigma]$ at threshold w_j. Inductively, for every integer $n \geq 2$, the *nth order merge cluster* of $[\sigma]$ at w_j contains all homology classes in the first order merge cluster of every homology class in $M_{n-1}[\sigma, w_j]$.

We also employ Corollary 2.3 to cut down on the adjacency pairings that need to be checked between cycles

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across homology classes to determine whether or not they merge.

Corollary 2.3. *Suppose that the homology classes $[\sigma]$ and $[\nu]$ merge. If ν is adjacent to a cycle $\delta \notin [\sigma], [\nu]$ where $d(\nu) \leq d(\delta) < d(\sigma)$ then either $[\delta]$ merges with $[\sigma]$ or there exists a homology class that merges with $[\sigma]$ whose nth order merge cluster contains $[\delta]$ for some integer $n \geq 1$.*

3. Homology-based Cycle Centrality

Given a homology class representative σ, we begin by taking the simple aggregate of the persistence of all cycle representatives that directly merge to a single class. Hence, if $P_\epsilon(\sigma)$ is the persistence of σ at ϵ, then

$$J_1(\sigma, \epsilon) = \begin{cases} 0 & \text{for } \epsilon \leq b(\sigma) \\ P_\epsilon(\sigma) + \sum_{[\varsigma] \in M_1[\sigma,\epsilon]} P_\epsilon(\varsigma) & \text{for } \epsilon > b(\sigma) \end{cases}.$$

As each $P_\epsilon(\sigma)$ is monotonic and stabilizes when $\epsilon > d(\sigma)$, this function is piece-wise linear and monotonic. A caveat, however, of the function above is that it treats all merging instances equally. Hence, we also consider a second function that scales each aggregate term by a factor of the merging time between homology classes. We have

$$J_2(\sigma, \epsilon) = P_\epsilon(\sigma) + \sum_{[\varsigma] \in M_1[\sigma,\epsilon]} f_\sigma(\varsigma) P_\epsilon(\varsigma) \text{ for } \epsilon > b(\sigma)$$

where $f_\sigma(\varsigma)$ depends on the merge time of $[\varsigma]$ to $[\sigma]$. Note that we can define the scaling function f to put more value either to merging late by defining $f_\sigma(\varsigma) = d(\varsigma)/d(\sigma)$ or early by defining $f_\sigma(\varsigma) = 1 - d(\varsigma)/d(\sigma)$.

We can also generalize the cascading effect of merging by accounting for higher-order merge clusters yielding

$$J_3(\sigma, \epsilon) = P_\epsilon(\sigma) + \sum_r \sum_{[\varsigma] \in M_r[\sigma,\epsilon]} f_\sigma(\varsigma) P_\epsilon(\varsigma) \text{ for } \epsilon > b(\sigma).$$

In general, our centrality measures are of the form $J_n : \Lambda \times W \to \mathbb{R}_+$ where Λ is the collection of all non-trivial persistent homology classes from a filtration of a weighted graph G induced by the set of edge weights W. Hence, each persistence diagram produces a family of centrality functions.

Now, we will show that the centrality functions defined above are stable with respect to an appropriate metric. For simplicity, we denote $J_n(\sigma, \epsilon)$ by $J_{n,\sigma}(\epsilon)$.

We let $\mathcal{J}_n = \{J_{n,\sigma} | [\sigma] \in \Lambda\}$ denote the collection of centrality functions generated by the set of persistent homology classes Λ, and endow it with the *p-centrality norm* given by

$$\|J_{n,\sigma}\|_p = \begin{cases} \left(\int_0^{d^*} (J_{n,\sigma}(x))^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty \\ J_{n,\sigma}(d(\sigma)) & \text{if } p = \infty \end{cases}$$

where d^* is the minimum between $d(\sigma)$ or the diameter in the largest cycle in $[\sigma]$.

To prove stability, we must define a distance between two collections of centrality functions. Consider the collection $\{\|J_{n,\sigma}\|_p^p : J_{n,\sigma} \in \mathcal{J}_n\}$ from the p-centrality norms in $\mathcal{J}_n$. As no natural order exists between the centrality functions, we appeal to a bottleneck-like distance that considers optimal matchings to capture the cost of transforming one collection to another. For computational efficiency, we match the condition in the bottleneck distance implementation in [7]. Let $\Omega = \{0\} \times \{\|J_{n,\sigma}\|_p^p : J_{n,\sigma} \in \mathcal{J}_n\}$ and $\Omega' = \{0\} \times \{\|J_{n,\sigma'}\|_p^p : J_{n,\sigma'} \in \mathcal{J}_n'\}$ be given and define $\delta_{x_\sigma} = \|J_{n,\sigma}\|_p^p$ for $x_\sigma \in \Omega$. Given $\Delta := \{(x, x) : x \in \mathbb{R}\}$ with infinite multiplicity, for a bijection $\phi : \Omega \cup \Delta \to \Omega' \cup \Delta$,

$$\|x_\sigma - \phi(x_\sigma)\|_\infty = \begin{cases} \frac{1}{2} \max\{\delta_{x_\sigma}, \delta_{\phi(x_\sigma)}\} & \text{if } \phi(x_\sigma) \in \Delta \\ |\delta_x - \delta_{\phi(x)}| & \text{otherwise} \end{cases}.$$

For $1 \leq p < \infty$, the *p-centrality distance* is given by $C_p(\mathcal{J}_n, \mathcal{J}_n') = \inf_\phi \sup_{x \in X} \|x - \phi(x)\|_\infty$ where the infimum is taken over all bijections from $\Omega \cup \Delta$ to $\Omega' \cup \Delta$. For the case where $p = \infty$, we propose a distance akin to p-landscape distance [8, p. 94]. We can order $J_{n,\sigma}(d(\sigma))$ for all $J_{n,\sigma} \in \mathcal{J}_n$ and obtain an increasing sequence $\{J_{n,m}\}_m$. In this case, we define $C_p\left(\mathcal{J}_{n,k}, \mathcal{J}_{n,k}'\right) = \sum_m \left\|J_{n,m} - J_{n,m}'\right\|_\infty$.

Now, we determine a bound for the centrality distance to characterize stability.

Theorem 3.1. *Let D and D' be persistence diagrams corresponding to the collections Λ and Λ' of persistent homology class representative cycles, and let $R'(p) = \sqrt[p]{2}K(1 + q')$ when $1 \leq p < \infty$ and $R'(p) = 2q'(1 + q')$ when $p = \infty$, where $q' := \max\{\sum_r |M_r[\sigma, d(\sigma)]| : \sigma \in \Lambda \cup \Lambda'$ and $P_\epsilon(\sigma) \neq 0\}$ and $K = \max\{P_\epsilon(\nu) : [\nu] \in \Lambda \cup \Lambda'\}$. Then*

$$C_p\left(\mathcal{J}_n(\Lambda), \mathcal{J}_n(\Lambda')\right) \leq \begin{cases} R'(p) \sqrt[p]{d_B(D, D')} & \text{if } 1 \leq p < \infty \\ R'(p) d_B(D, D') & \text{if } p = \infty \end{cases}.$$

Theorem 3.2 (Stability)**.** *Let $w, w' : \mathscr{C} \to \mathbb{R}$ be monotone real-valued functions that filter the simplicial complex $\mathscr{C}$ built over a graph $G = (V, E)$, and let Λ and Λ' denote the respective collections of persistent homology class representative cycles induced by each filtration. If $p = \infty$, or $q' > \left(1/\sqrt[p]{2}K\right) - 1$ when $1 \leq p < \infty$, then*

$$C_p\left(\mathcal{J}_n(\Lambda), \mathcal{J}_n(\Lambda')\right) \leq R'(p) \|w - w'\|_\infty.$$

4. Conclusion

We proposed centrality measures based on persistence and merge dynamics of homology classes that captures attributes in the topology of points clouds missed by existing topological summaries. The computation of persistence is driven by the reduction of the standard boundary matrix. On the other hand, the algorithm for computing the merge dynamics of homology classes is governed by the equivalence of merging and adjacency between two classes. Similar to barcodes, we proved stability by defining a pseudo-metric akin to the bottleneck and landscape distances.

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On the Bounds of the Double Domination Numbers and Game Domination Numbers of Glued Graphs

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Abstract

We propose bounds for two types of generalized domination numbers of glued graphs - namely *double domination numbers* and *game domination numbers*. The obtained bounds are related to the double domination numbers and game domination numbers of the original graphs. The double domination numbers of fan graphs Fn_n and firecrackers $Fc_{n,3}$, as well as the glued graphs with path clones and the original graphs being in these two families were established. As a result, we can construct glued graphs to verify that our bounds for the double domination numbers are tight.

Keywords: double domination number, game domination number, glued graph, bound.

1 Introduction

We consider a binary operation called *glue operator*, introduced by Uiyyasathian [1]. Consider connected graphs G_1 and G_2 where $V(G_1) \cap V(G_2) = \varnothing$. Let H_1 and H_2 be connected subgraphs of G_1 and G_2, respectively, and H_1 is isomorphic to H_2 under an isomorphism ϕ. The identification of H_1 and H_2 using the isomorphism ϕ gives the *glued graph of G_1 and G_2 at H_1 and H_2 with respect to ϕ*, denoted by $G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$. Precisely, $G = G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ is the graph with vertex set

$$V(G) = (V(G_1)\backslash V(H_1)) \cup (V(G_2)\backslash V(H_2)) \cup \{(v, \phi(v)) : v \in V(H_1)\},$$

and edge set

$$\begin{aligned} E(G) = {} & E(G_1\backslash H_1) \cup E(G_2\backslash H_2) \\ & \cup\{u(v,\phi(v)) : uv \in E(G_1)\} \cup \{(v,\phi(v))w : \phi(v)w \in E(G_2)\} \\ & \cup\{(u,\phi(u))(v,\phi(v)) : uv \in E(G_1) \text{ or } \phi(u)\phi(v) \in E(G_2)\}. \end{aligned}$$

Graphs G_1 and G_2 are called the *original graphs*, and subgraphs H_1 and H_2, together with subgraph $H \subseteq G_1 \underset{H_1 \cong_\phi H_2}{\lhd\rhd} G_2$ isomorphic to H_1 and H_2, are said to be the *clones*.

The domination numbers of glued graphs has been considered by various mathematicians. Ruangnai and Panma [2] worked on the domination numbers of glued graphs whose original graphs are the paths P_m and P_n. The domination numbers of the graphs obtained from gluing the cycles C_m and C_n were investigated by Boonmee et al. [3]. Recently, a work on bounds of the domination numbers of glued graph was studied by Sripratak and Panma [4]. As far as we know, there are no publications on the generalized domination number of glued graph.

2 Result

The first type of generalized domination number we consider in this work is the double domination numbers [5]. Let G be a graph. For vertices u and v in $V(G)$, we say that vertex v *dominates* vertex u if $u \in N[v] = \{u \in V(G) : u = v \text{ or } uv \in E(G)\}$. A subset D^2 of $V(G)$ is a *double dominating set* if for any vertex $u \in V(G)$, vertex u is dominated by at least two vertices in D^2. The *double domination number* $dd(G)$ is the minimum cardinality of a double dominating set of graph G.

We obtain a lower bound 2 and an upper bound

$$dd(G) \leq \min\{dd(G_1) + dd(G_2), dd(G_1) + dd(G_2\backslash H_2), dd(G_2) + dd(G_1\backslash H_1)\}$$

for the double domination number of a glued graph G with arbitrary original graphs G_1 and G_2.

The gluing of fan graphs Fn_{3m-4} and Fn_{3m-5} with a clone path P_{3m-4} is an example of a glued graph whose double domination numbers of the original graphs are $m \geq 2$ and that of the glued graph is equal to 2.

As for the upper bound, our example is the glued graph G with path clone P_m, whose original graphs are the glued graphs G_1 and G_2 with firecrackers $Fc_{m,3}$ as original graphs and paths P_m as clones. The double domination number of the original graphs G_1 and G_2 are equal to $4m$, while the double domination number of the glued graph G is equal to $8m = \min\{dd(G_1) + dd(G_2), dd(G_1) + dd(G_2 \backslash H_2), dd(G_2) + dd(G_1 \backslash H_1)\}$.

In conclusion, these examples show that both lower bound and upper bound that we offer in this work are tight.

The other type of generalized domination number presented here is the game domination numbers [6]. Let G be a graph. The *domination game* on G is composed of two players: *Dominator* and *Staller*. In each turn, the players alternately choose a vertex that dominates a vertex that has not been dominated in the previous turns. The game ends when the set of chosen vertices becomes a dominating set. Dominator wants a minimum dominating set, while Staller wants a maximum one. The size of the dominating set when both players play the game optimally is the *game domination number*.

The domination games started by the Dominator and by the Staller result in different game domination numbers. When the Dominator is the first player, the game domination number is denoted by $gd_d(G)$, and when the Staller is the first player, it is denoted by $gd_s(G)$. In case that the Staller is allowed to pass for one turn, we use the notation $gd'_d(G)$ for the game domination number if the Dominator is the first player, and use the notation $gs'_d(G)$ for the game domination number if the Staller is the first player.

Let G be the glued graph of G_1 and G_2 at clones H_1 and H_2 with respect to isomorphism ϕ. Then we obtain these upper bounds:

$$gd_d(G) \leq \min\{gd_d(G_1) + gd'_s(G_2), gd'_d(G_1) + gd_s(G_2), gd_d(G_2) + gd'_s(G_1), gd'_d(G_2) + gd_s(G_1)\}$$

and

$$gd_s(G) \leq \min\{gd_s(G_1) + gd'_s(G_2), gd'_s(G_1) + gd_s(G_2)\}.$$

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Some Graceful C_4-Related Chain Graphs

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Abstract

Let $G(V, E)$ be a finite simple graph. Denote by $|V|$ and $|E|$ the cardinality of the set V and E, respectively. *Graceful labeling* is an injective function f from vertex set V into the set $\{0, 1, 2, ..., |E|\}$ which induces a bijective function f' from edge set E onto the set $\{1, 2, ..., |E|\}$ such that for every edge $uv \in E(G)$ we have $f'(uv) = |f(u) - f(v)|$, with $u, v \in V(G)$. A graph that has graceful labeling is called *graceful*. A *chain graph* is a graph with blocks $B_1, B_2, ..., B_n$ such that for every i, B_i and B_{i+1} have a common vertex, in such a way that the block-cut-vertex graph is a path. A chain graph having n blocks $B_1, B_2, ..., B_n$ is denoted by $[B_1, B_2, ..., B_n]$. Let c_i be the common vertex of B_i and B_{i+1}, in $[B_1, B_2, ..., B_n]$, $1 \leq i \leq n-1$. If a vertex of B_1, $c_0 \neq c_1$, and a vertex of B_n, $c_n \neq c_{n-1}$, are identified, then the chain graph $[B_1, B_2, ..., B_n]$ is called *closed*. On a closed chain graph, any cut vertex of related open chain graph is no longer cut vertex. If $B = B_i$ for every $i \in 1, 2, ..., n$, we denote the open and closed chain graph as $[B^{(n)}]$ and $[B^{(n)}]_c$.
The *join graph* of graphs G and H, denoted by $G + H$, is the graph obtained by joining each vertex of G with every vertex of H. Let $\overline{K_r}$ be the complement of the complete graph on r vertices, K_r, $r \geq 1$. The join graph $\overline{K_r} + \overline{K_2}$ may be considered as a group of 4-vertices cycles, C_4, where two non adjacent vertices of all C_4's are identified. On this notion, we consider this join graph $\overline{K_r} + \overline{K_2}$ as a C_4-realted graph, and is written as rC_4.

In this talk we discuss two C_4-related chain graphs and their graceful labeling: $[rC_4^{(n)}]$ and $[[C_4^{(k)}]_c^{(r)}]$.

Keyword: *graceful labeling, graceful graph, chain graph, four vertices cycle*

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Complexity of Solo Chess with Unlimited Moves

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Abstract

We analyze Solo Chess puzzles, where the input is an $n \times n$ board containing some standard Chess pieces of the same color, and the goal is to make a sequence of capture moves to reduce down to a single piece. Prior work analyzes this puzzle for a single piece type when each piece is limited to make at most two capture moves (as in the Solo Chess puzzles on chess.com). By contrast, we study when each piece can make an unlimited number of capture moves. We show that (among standard Chess piece types), any single piece type can be solved in polynomial time, while any two piece types are NP-complete. We also analyze the restriction (as on chess.com) that there is only one king and it must be the last surviving piece, showing that in this case some pairs of piece types remain hard.

1 Introduction

The classic two-player game of Chess is PSPACE-complete [6] or EXPTIME-complete [4] depending on whether the number of moves is limited to be a polynomial. Recent work analyzes Chess-based puzzles, including Helpmate Chess and Retrograde Chess which are PSPACE-complete [2] and Solo Chess which is NP-complete [1]. In this paper, we extend the analysis of Solo Chess to unbounded moves per piece.

First we review standard Solo Chess as implemented on chess.com [3]. All pieces are of the same color and may capture any piece except a king. Every move must be a capture. The objective is to find a sequence of moves (captures) that results in only one piece remaining on the board. Further, each piece can make a maximum of $k = 2$ moves. Past work [1] generalizes Solo Chess to an arbitrary limit k on the number of moves per piece (and arbitrary board size), denoting this game by (Generalized) Solo-Chess(S, k) where S is the set of allowed piece types. They proved that Solo-Chess({♟}, 2) and Solo-Chess(S, 1) can be solved in linear time, while Solo-Chess({♝}, 2), Solo-Chess({♜}, 2), and Solo-Chess({♛}, 2) are NP-complete.

This paper analyzes the complexity of Solo Chess puzzles without the restriction on the number of moves per piece, or equivalently when the move limit per piece is larger than the number of pieces. We denote this game by Solo-Chess(S), which is equivalent to Solo-Chess(S, ∞). We also consider the game both with and without the restriction of a single uncapturable king, using ♚^{1} to denote the game with this restriction and ♚ to denote the more general case that permits multiple kings.

We prove that any single standard Chess piece type ($|S| = 1$) can be solved in polynomial time, while for any two distinct standard Chess piece types ($|S| = 2$, excluding ♚^{1}), the puzzle is NP-complete. For the single uncapturable king ♚^{1}, we prove that the pairs {♚^{1}, ♟}, {♚^{1}, ♝}, and {♚^{1}, ♞} are NP-complete.

2 One Piece Type is Easy

Theorem 2.1. Solo-Chess(S) *can be solved in polynomial time when S consists of only one standard Chess piece type.*

3 Two Piece Types are NP-Complete

In this section and the next, we prove that Solo-Chess(S) is NP-complete for any set S of two distinct standard Chess pieces. Theorem 3.1 covers two out of three cases when both Chess pieces have constant range; Theorem 3.2 covers when at least one Chess piece has long range; and Theorem 4.1 covers the last constant-range case of king and pawn. All of these reductions are from Hamiltonian path in maximum-degree-3 grid graphs with a specified start vertex and possibly a specified end vertex, or generalizations thereof (e.g., sometimes we do not need the grid-graph property). This problem is NP-hard by a slight modification to [5].

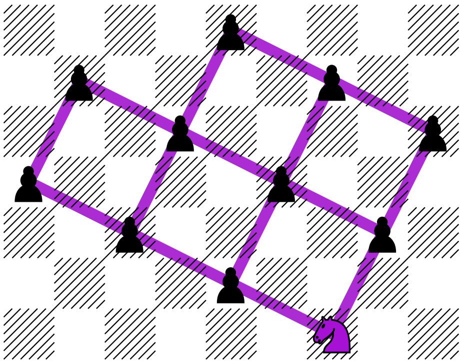

Figure 1: Placing pawns or kings and one knight to form Hamiltonian path in a grid graph with a specified start vertex.

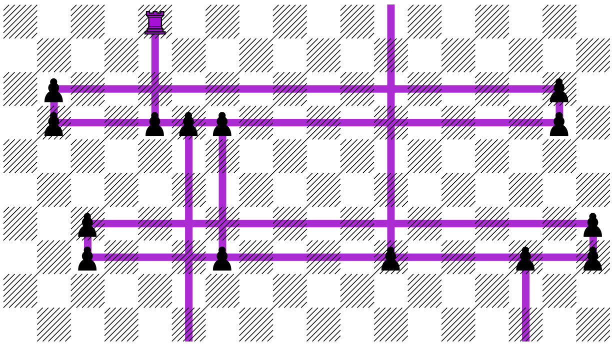

Figure 2: Placing pawns and one rook to form Hamiltonian path in a maximum-degree-3 graph with specified start and end vertices.

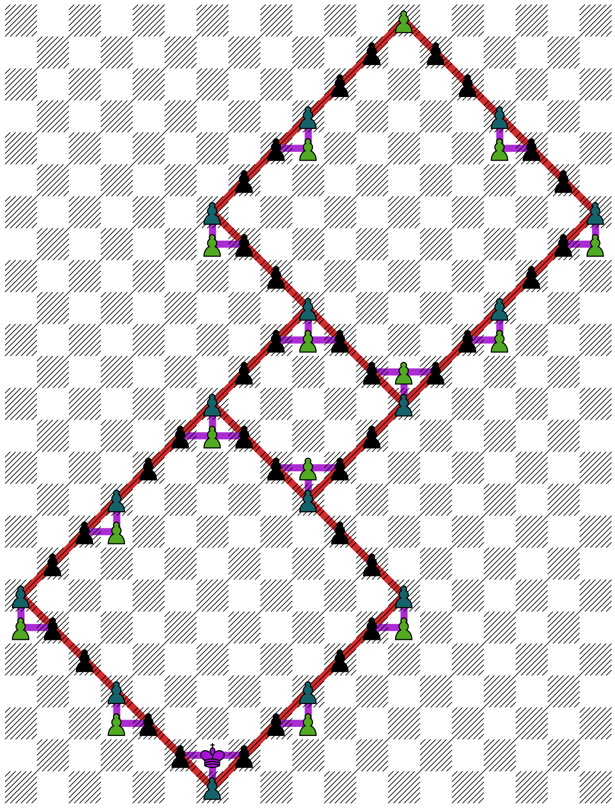

Figure 3: Placing pawns and one king to represent Hamiltonian path in a maximum-degree-3 grid graph with a specified start vertex. Blue pawns are at grid-graph vertices, while green pawns have no capture move.

Theorem 3.1. Solo-Chess({♞, ♟}) *and* Solo-Chess({♞, ♚}) *are NP-hard.*

Theorem 3.2. Solo-Chess(S) *is NP-hard for any S consisting of two distinct pieces from* {♝, ♚, ♟, ♞, ♛, ♜}, *at least one of which is in* {♝, ♛, ♜}.

See Figures 1 and 2 for proof sketches.

4 One-King Restriction

Next we study the one-king restriction, where only one king is allowed on the board and it must remain on the board (cannot be captured). We denote this restricted king piece by ♚^{1}. We prove three negative results, but a full characterization remains open.

Theorem 4.1. Solo-Chess({♚^{1}, ♟}), Solo-Chess({♚^{1}, ♝}), *and* Solo-Chess({♚^{1}, ♞}) *are NP-hard (under the one-king restriction).*

See Figure 3 for a proof sketch of the first case. The other two cases are reductions from 3SAT.

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This Game Is Not Going To Analyze Itself

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Abstract

We analyze the puzzle video game *This Game Is Not Going To Load Itself*, where the player routes data of three different colors from given sources to given sinks of the correct color. We prove that the game is NP-hard for three colors given data sources, sinks, and some previously placed arrow tiles. On the other hand, given three data sinks not too close to the boundary of an otherwise empty grid, we show how to place arrow tiles to route flow from arbitrary unknown sources, effectively solving the game as it is normally played.

1 Introduction

This Game Is Not Going To Load Itself (TGINGTLI) is a game created in 2015 by Roger "atiaxi" Ostrander for the Loading Screen Jam, a game jam hosted on `itch.io`, where it finished 7th overall out of 46 entries [Ost15]. This game jam was a celebration of the expiration of US Patent 5,718,632 [Hay98], which covered the act of including mini-games during video game loading screens. In this spirit, TGINGTLI is a real-time puzzle game themed around the player helping a game load three different resources of itself — save data, gameplay, and music, colored red, green, and blue — by placing arrows on the grid cells to route data entering the grid to a corresponding sink cell.

We formalize TGINGTLI as follows. You are given an $m \times n$ grid where each unit-square cell is either empty, contains a data sink, or contains an arrow pointing in one of the four cardinal directions. Each data sink and arrow has a color (resource), and there is exactly one data sink of each color in the grid. In the online version (as implemented), sources appear throughout the game; in the offline version considered here, all sources are present at the start (or are otherwise known a priori). Note that an outer edge of the grid may have multiple sources of different colors. Finally, there is an loading bar that starts at an integer k_0 and has a goal integer k^*.

During the game, each source produces data packets of its color/resource, which travel at a constant speed into the grid. If a packet enters the cell of an arrow of the same color/resource, then the packet will turn in that direction. (Arrows of other colors are ignored.) If a packet reaches the sink of its color/resource, then the loading bar increases by one unit of data. If a packet reaches a sink of the wrong color, or exits the grid entirely, then the loading bar decreases by one unit of data and the packet disappears. The player may at any time permanently replace an empty cell with an arrow, which may be of any color/resource and pointing in any of the four directions. If the loading bar hits the target amount k^*, then the player wins, but if the loading bar goes below zero, then the player loses.

In Section 2, we prove NP-hardness of the TGINGTLI decision problem: given a description of the initial grid configuration (including sources, sinks, and preplaced arrows), can the player place arrows to win? Conversely, in Section 3, we show that TGINGTLI has a winning strategy from a given initial grid that has sinks but no preplaced arrows (which is how each level starts in the implemented game). Notably, this solution works independent of the sources, so also works in the online setting.

2 NP-Hardness

We prove that TGINGTLI is NP-hard by reducing from a new problem called ***3-Dimensional SAT (3DSAT)***, defined by analogy to 3-Dimensional Matching (3DM). 3DSAT is a variation of 3SAT where, in addition to a 3CNF formula, the input specifies one of three colors (red, green, or blue) to each variable of the CNF formula, and the CNF formula is constrained to have trichromatic clauses, i.e., to have exactly one variable (possibly negated) of each color.

Lemma 2.1. *3DSAT is NP-complete.*

Theorem 2.2. *TGINGTLI is NP-hard.*

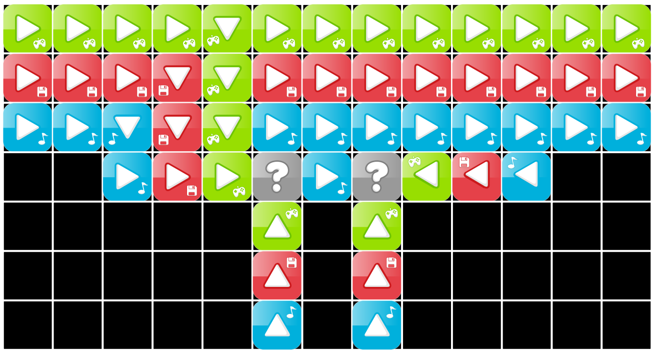

Figure 1: Clause Gadget. At most two streams of data, representing false literals, can pass through the gadget (by placing upwards arrows in the "?" cells) without getting lost. A left arrow instead will put the stream into a cycle, and a down or right arrow will cause damage.

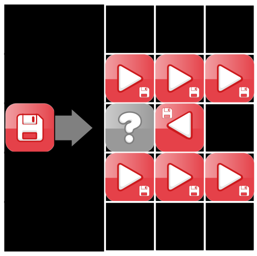

Figure 2: Variable Gadget lets the user route data from a source to one of two literal paths.

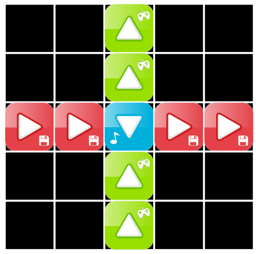

Figure 3: Crossover Gadget between two literal paths of same or different colors. (The center cell is colored different from both paths.)

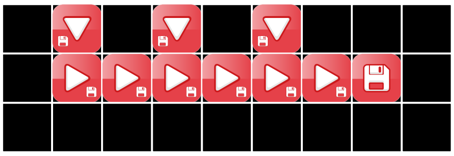

Figure 4: End Gadget loads data at a unit rate per literal path that has not been lost.

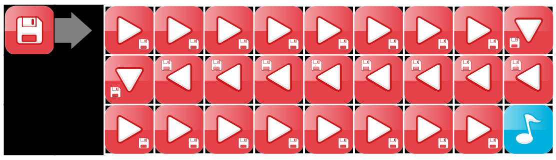

Figure 5: Damage Gadget forces damage at a unit rate after a desired start delay.

3 Guaranteed Victory Layouts

We show the existence of winning strategies when starting with a grid with no arrows, thus as a function of the location of the data sinks. Our layout where all sinks are at least three grid cells away from the boundary and no two data sinks are in the same row or column has two key components. First, we have a set of arrows of the appropriate color connected to the sink via a connected path of arrows of the same color and all pointed into the sink. We call these the *sink-adjacent* arrows. Then, we have a set of arrows on the boundaries which are pointed counterclockwise and routes packets of the same color along the boundary until they hit an arrow in the same row or column as a sink-adjacent arrow. These arrows are directed in the same direction as the sink-adjacent arrow in the same row/column and redirects the packet into a sink-adjacent arrow. Fig. 6 shows an example layout.

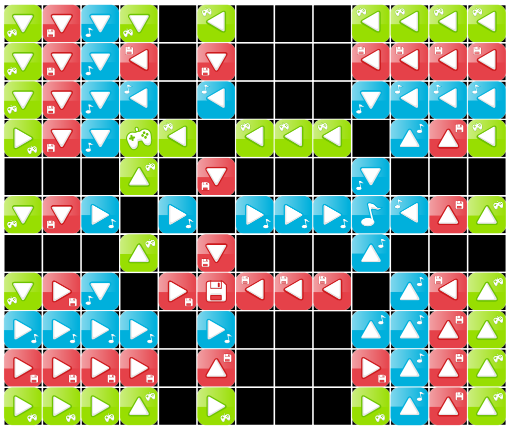

Figure 6: Example Guaranteed Victory Layout for Theorem 3.1. Any data packet that enters this rectangular region will be routed to its matching sink.

Theorem 3.1. *If all three data sinks in the grid are at least three grid cells away from the boundary, and no two data sinks share a row or column, then there is a layout of arrows guaranteeing every data packet from any data source will reach its packet's data sink.*

Theorem 3.2. *If a data sink is on the boundary or one cell away from the boundary of the grid, then there is not always a guaranteed victory layout.*

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Improved Bounds on Permutation Arrays for Chebyshev Metric

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July 19, 2022

In [7] an interesting study of permutation arrays under the Chebyshev metric was presented. This complemented many studies of permutation arrays under other metrics, such as the Hamming metric [1] [2] [4], Kendall τ metric [6] [3], and several others [5]. The use of the Chebyshev metric was motivated by applications of error correcting codes and recharging in flash memories [6].

Let σ and π be two permutations (or strings) over an alphabet $\Sigma \subseteq [1...n] = \{1, 2, ..., n\}$. The Chebyshev distance between σ and π, denoted by $d(\sigma, \pi)$, is $\max\{\ |\sigma(i) - \pi(i)| \mid i \in \Sigma\ \}$. For an array (set) A of permutations (strings), the pairwise Chebyshev distance of A, denoted by $d(A)$, is $\min\{\ d(\sigma, \pi) \mid \sigma, \pi \in A\ \}$. An array A of permutations on $[1...n]$ with $d(A) = d$ will be called an (n, d) PA. Note that this includes the case when A is a set of integers, *i.e.* a set of strings of length one, where $d(A)$ corresponds to the minimum difference between integers in the set. Let $P(n, d)$ denote the maximum cardinality of any (n, d) PA A. More generally, let $P_d(\Sigma)$ denote the maximum cardinality of any array of permutations over the alphabet $\Sigma \subseteq [1...n]$ with Chebyshev distance d. For example, $P_2(\{1, 3, 5, 7\}) = 4! = 24$, whereas $P(4, 2) = 6$.

Let A be a subset of $[1...(n+1)]$ such that $d(A) \geq d$, then, for all $i \in A$, $P_d([1..(n+1)] - \{i\}) \geq P(n, d)$. Observe that the set $\{1, d+1, 2d+1, ..., \lfloor \frac{n}{d} \rfloor d + 1\}$ is a subset of $[1...(n+1)]$ with $\lfloor \frac{n}{d} \rfloor + 1$ elements with Chebyshev distance d and was used in [7].

Theorem 1. *Let A be a subset of $[1...(n+1)]$ such that $d(A) \geq d$. If $n > d \geq 1$, then $P(n+1, d) \geq \sum_{i \in A} P_d([1..(n+1)] - \{i\})$.*

Theorem 1 by choosing $A = \{3, 6, 9\}$, gives the lower bound $P(11, 3) \geq 53,549$, as $P_3([1..11] - \{3\}) = P_3([1..11] - \{9\} \geq 17,573$ and $P_3([1..11] - \{6\}) \geq 18,403$.

Theorem 2. $P(n, d) \geq max\{P(n_1, d_1) * P(n_2, d_2) \mid d_1 + d_2 = d$ *and* $n_1 + n_2 = n$ *and, for some constant* a, $n_1 = ad_1 + r_1$ *and* $n_2 = ad_2 + r_2$, *with* $0 \leq\ r_1 \leq\ d_1$ *and with* $0 \leq\ r_2 \leq\ d_2\}$, *where the maximum is taken over all possible values of* n_1, n_2, d_1, d_2.

Theorem 2 gives the lower bound $P(16, 9) \geq P(9, 5) * P(7, 4) \geq 3,399$, where $a = 1$, $9 = 1*5+4$, $7 = 1*4+3$, and the best lower bounds known for $P(9, 5)$ and $P(7, 4)$ are 103 and 33, respectively.

We improve another result in [7] and give an exact equation for $P(n, 2)$.

Theorem 3. $P(n, 2) = \frac{n!}{2^{\lfloor n/2 \rfloor}}$.

We also give a generalization of Theorem 1 using strings of more than one symbol. Let A be a set of length m strings with no repeated symbols (permutations) over $[1..(n+m)]$ with $d(A) \geq d$.

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By an abuse of notation, for each $\sigma \in A$, let σ^C denote the complement in $[1..(n+m)]$ of the set of symbols used in σ. We show that $P(n+m,d) \geq \sum_{\sigma \in A} P_d(\sigma^C)$. Let $Q((n+m),m,d)$ denote the collection of all sets A of permutations on a m symbol subset of $[1..(n+m)]$ with $d(A) \geq d$. Maximizing the sum over all such sets A yields the following.

Theorem 4. *For any $n \geq d \geq 1$, $P(n+m,d) \geq \max_{A \in Q((n+m),m,d)} \sum_{\sigma \in A} P_d(\sigma^C)$.*

Kløve *et al.* [7] gave the following Gilbert-Varshamov type upper bound, when d is even and $2d \geq n \geq d \geq 2$,

$$P(n,d) \leq \frac{(n+1)!}{V(n+1,d/2)},$$

where $V(n+1,d/2)$ is the number of permutations on $\{1,2,\ldots,(n+1)\}$ within distance $d/2$ of the identity permutation.

In Theorem 5 we give a better upper bound. Using Theorem 5 we show, for example, $P(11,6) \leq 462$.

Theorem 5. *For $1 \leq k \leq d < n$,*

$$P(n,d) \leq P(n-k,d) \cdot \binom{n}{k}.$$

Theorem 6. *Suppose that $P(n_0, n_0-k) \leq m$ such that*

$$2k(m+1) < (n_0+1)(1+\lfloor n_0/(2k-1) \rfloor). \tag{1}$$

Then $P(n,n-k) \leq m$, for all $n \geq n_0$.

In addition, we give tables to show our improved lower bounds and upper bounds for $P(n,d)$.

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Improved Permutation Arrays for Kendall Tau Metric

Sergey Bereg* William Bumpass* Mohammadreza Haghpanah*, Brian Malouf* I. Hal Sudborough*

In [9], [6], [3], [8] permutation arrays under the Kendall -τ metric were studied. This complemented many studies of permutation arrays under other metrics, such as the Hamming metric [1] [2] [4], Chebyshev metric [7] and several others [5]. The use of the Kendall-τ metric was motivated by applications of error correcting codes and rank modulation in flash memories [6].

Let σ and π be two permutations (or strings) over an alphabet $\Sigma \subseteq [1...n] = \{1, 2, ..., n\}$. The *Kendall-$\tau$ distance* between σ and π, denoted by $d(\sigma, \pi)$, is the minimum number of adjacent transpositions (bubble sort operations) required to transform σ into π. For an array (set) A of permutation (strings), the pairwise Kendall-τ distance of A, denoted by $d(A)$, is $\min\{\ d(\sigma, \pi) \mid \sigma, \pi \in A\ \}$. An array A of permutations on $[1...n]$ with $d(A) = d$ will be called an (n, d) PA. Let $P(n, d)$ denote the maximum cardinality of any (n, d) PA A.

Theorem 1. *[9] Let $m = ((n\text{-}2)^{t+1}\text{-}1)/(n\text{-}3)$, where n-2 is a prime power. Then*

$$P(n, 2t + 1) \geq \frac{n!}{(2t + 1)m}.$$

For example, by choosing t=1 and n=11, one obtains, by Theorem 1, $P(11, 3) \geq 1, 330, 560$. Theorem 1 applies only when n is two greater than a power of a prime.

Theorem 2. *[6] For all $n, d > 1$ we have $P(n + 1, d) \geq \lceil \frac{n+1}{d} \rceil P(n, d)$.*

For example, to compute a lower bound for $P(17, 3)$ one can use, iteratively, Theorem 2 to obtain $P(17, 3) \geq \lceil \frac{16}{3} \rceil \cdot \lceil \frac{17}{3} \rceil \cdot P(15, 3) = 36 \cdot P(15, 3)$. We give a generalization and an improvement. Let A be a set of permutations on $[1..n]$ with the restriction that the first $n - m$ symbols are in sorted order, for a given $m < n$. Such a set with Kendall-τ distance d is called an (n, m, d) PA. Let $P(n, m, d)$ denote the maximum cardinality of any (n, m, d) PA A.

Theorem 3. *For any $m < n$ and d, $P(n, d) \geq P(n, m, d) \cdot P(n - m, d)$.*

In [6] Theorem 2 was proved using the set $\{1, d+1, 2d+1, \ldots, \lceil \frac{n+1}{d} \rceil d + 1\}$, which corresponds to a $(n+1, 1, d)$ PA. In general, a (n, m, d) PA can be much larger than one obtained by the iterative use of Theorem 2. For example, for all n, we give $(n, 3, 3)$ PAs with $\frac{n(n+1)}{6}$ permutations, when $n-1$ is not divisible by 3. This means that, for $n = 24$ we give a $(24, 2, 3)$ PA with 100 permutations, whereas the two fold iterative use of Theorem 2 gives a $(24, 2, 3)$ PA with 64 permutations. Thus, using Theorem 3 we obtain $P(24, 3) \geq 100 \cdot P(22, 3)$, an improved lower bound.

It should be noted that one can improve on Theorem 3. For each permutation, say τ in a (n, m, d) set A, one can generally find a larger set of permutations than in the best $(n - m, d)$ PA. Let $P_\tau(n, d)$ denote the maximum cardinality of any (n, d) PA with the highest m symbols in the same positions as in τ, but where the other n-m symbols can be in any order.

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Theorem 4. *For any n, d, $P(n,d) \geq max_{(n,m,d)-PA\ A} \sum_{\tau \in A} P_\tau(n,d)$.*

Let us, temporarily, describe strings (permutations) in $P(n,2,3)$ and $P(n,3,3)$, by replacing the symbols $[1..n-2]$ ($[1..n-3]$, respectively), which are in order, by blank symbols, *i.e.* ".". Then consider $\pi_1(a,b,c) = .., n-1, .., n, ..$ and $\pi_2(a,b,c) = .., n, .., n-1, ..$, where a, b, c denotes the number of symbols in the 3 gaps represented by the "..". Specifically, we use:
$\pi_1(a,b,c)$ for $a = 0,2,4,..$ and $b = 0,3,6,..$, and $\pi_2(a,b,c)$ for $a = 1,3,5,..$ and $b = 0,3,6,...$
For example, π_1, for $n = 7$ and $a = b = 0$, is "6 7 - - - - -" and π_2, for $n = 7$ and $a = 1, b = 0$, is "- 7 6 - - - -" . Using π_1 and π_2:

Theorem 5. $P(n,2,3) \geq \frac{n(n+1)}{6}$*, for* $n \neq 1$ *mod 3 and* $P(n,2,3) \geq \frac{(n+2)(n-1)}{6}$ *for* $n = 1$ *mod 3.*

Similarly, for Kendall-τ distance 4 and for $n = 2k+1$, use $\pi_1(a,b,c)$ for $a = 0,2,4,..$ and $b = 0,4,8,..$; $\pi_2(a,b,c)$ for $a = 0,2,4,..$ and $b = 3,7,11,...$

Theorem 6. $P(4k+1,2,4) \geq 2k^2 + k$ *for* $k \geq 1$ *and* $P(4k+3,2,4) \geq 2k^2 + 3k + 1$ *for* $k \geq 0$.

In addition, we give tables to show computed bounds for $P(n,m,d)$. We also give improved lower and upper bounds obtained for sporadic values of $P(n,d)$, which were obtained by the use of automorphisms (as in [3]), maximum cliques, the use of greedy programs with randomness, and Theorem 4.

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Two Strong Avoider-Avoider Games

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Abstract

Given an increasing graph property $\mathcal{F}$, the strong Avoider-Avoider $\mathcal{F}$ game is played on the edge set of a complete graph. Two players, Red and Blue, take turns in claiming previously unclaimed edges with Red going first, and the player whose graph possesses $\mathcal{F}$ first loses the game. If the property $\mathcal{F}$ is "containing a fixed graph H", we refer to the game as the H game.

We prove that Blue has a winning strategy in two strong Avoider-Avoider games, P_4 game and $\mathcal{CC}_{>3}$ game, where $\mathcal{CC}_{>3}$ is the property of having at least one connected component on more than three vertices.

We also study a variant, the strong CAvoider-CAvoider games, with additional requirement that the graph of each of the players must stay connected throughout the game. We prove that Blue has a winning strategy in the strong CAvoider-CAvoider games $K_{1,3}$ and P_4, as well as in the *Cycle* game, where the players aim at avoiding all cycles.

1 Introduction

A positional game is a pair $(X, \mathcal{F})$, where X is a finite set called a *board*, and $\mathcal{F}$ is the family of *target sets*. The game is played by two players who alternately claim previously unclaimed elements of X until all the elements of the board are claimed. Our interest lies with games whose board is the edge set of the complete graph K_n.

When it comes to the rules for determining the game winner in a positional game, there are several variants. In a *Maker-Breaker positional game* two players are called Maker and Breaker. Maker wins the game if by the end of the game he claims all elements of a winning set, otherwise Breaker wins the game. *Avoider-Enforcer games* are the *misère* version of Maker-Breaker games, with two players Avoider and Enforcer. Enforcer wins the game if, by the end of the game, Avoider claimed all elements of a losing set, otherwise Avoider wins.

In contrast to the weak positional games where the first player is given a goal while the second one just tries to prevent the first player from achieving his goal, in the *strong games* the two players compete for achieving the same goal. In the *strong Maker-Maker game* $(X, \mathcal{F})$, two players called Red and Blue take turns in claiming previously unclaimed elements of X, with Red going first. The player who first fully occupies some $F \in \mathcal{F}$ is the winner. If neither of the players wins and all the elements of the board are claimed, the game is declared a *draw*.

The *Strong Avoider-Avoider game* $(X, \mathcal{F})$ is again played by Red and Blue, but now the player who first fully occupies some $F \in \mathcal{F}$ *loses the game*. One such game, widely known as *Sim*, is played on the edge set of K_6, and a player who first claims a triangle loses. Even though it is immediate that draw is impossible, and the board is reasonably small, analyzing it is challenging, and the proof that Blue wins is performed with the help of a computer.

It may seem that in strong Avoider-Avoider games, in contrast to the strong Maker-Maker games, Blue always has an upper edge, and Red as the first player cannot expect to win under

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optimal play? This, however, turns out not to be true! For example, in d-dimensional Tic-Tac-Toe game n^d, where n is odd, Red has an explicit drawing strategy, and 3^3 Tic-Tac-Toe is Red's win.

In the last few years several variants of positional games have emerged, like the *PrimMaker-Breaker game* introduced in [3] where the subgraph induced by Maker's edges must be connected throughout the game. We study *Strong CAvoider-CAvoider games* in which the graph of each player must stay connected throughout the game. The board is still the edge set of K_n, and the players should not claim a copy of the forbidden graph. This is a natural extension of the strong Avoider-Avoider games, with a connectedness constraint analogue to the one mentioned above.

2 Our results

We are interested in Strong Avoider-Avoider games played on the edges of the complete graph K_n. These games are known to be notoriously hard to analyze, not much is known, while there are many open problems. In [2] it was shown that Blue has a winning strategy in the P_3 game, where the forbidden graph is the path with just two edges. Recently, Beker [1] generalized this result to all stars, proving that for each fixed k the Strong Avoider-Avoider star $K_{1,k+1}$ game is a win for the second player for all sufficiently large n. Finally, Malekshahian [4] studied the possibility of Blue's win in the triangle game with assumption that the game starts on several special mid-game positions, without any definite implications on the outcome of the triangle game itself. Hence, the only non-trivial Strong Avoider-Avoider game played on $E(K_n)$ for which the outcome is previously known is the star game.

The abbreviation $\mathcal{CC}_{>3}$ stands for the collection of inclusion-minimal connected graphs on more than three vertices, P_4 represents a path on four vertices and $K_{1,3}$ a star with three leaves.

Theorem 2.1. *Blue has a winning strategy in the Strong Avoider-Avoider P_4 game, played on K_n, where $n \geq 8$.*

Theorem 2.2. *Blue has a winning strategy in the Strong Avoider-Avoider $\mathcal{CC}_{>3}$ game, played on K_n, where $n \geq 5$.*

We prove that Blue can win in three different strong CAvoider-CAvoider games.

Theorem 2.3. *Blue has a winning strategy in the Strong CAvoider-CAvoider $K_{1,3}$ game, played on K_n, where $n \geq 7$.*

Theorem 2.4. *Blue has a winning strategy in the Strong CAvoider-CAvoider P_4 game, played on K_n, where $n \geq 5$.*

Theorem 2.5. *Blue has a winning strategy in the Strong CAvoider-CAvoider Cycle game, played on K_n, where $n \geq 6$.*

More details and all of the proofs can be found in the full version of our paper [5].

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MULTISTAGE POSITIONAL GAMES

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Abstract. We initiate the study of a new variant of the Maker-Breaker positional game, which we call multistage game. Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{F})$ and a bias $b \geqslant 1$, the $(1 : b)$ multistage Maker-Breaker game on $\mathcal{H}$ is played in several stages as follows. Each stage is played as a usual $(1 : b)$ Maker-Breaker game, until all the elements of the board get claimed by one of the players, with the first stage being played on $\mathcal{H}$. In every subsequent stage, the game is played on the board reduced to the elements that Maker claimed in the previous stage, and with the winning sets reduced to those fully contained in the new board. The game proceeds until no winning sets remain, and the goal of Maker is to prolong the duration of the game for as many stages as possible. Here we estimate the maximum duration of the $(1 : b)$ multistage Maker-Breaker game, for biases b subpolynomial in n, for some standard graph games played on the edge set of K_n: the connectivity game, the Hamilton cycle game, the non-k-colorability game, the pancyclicity game and the H-game. While the first three games exhibit a probabilistic intuition, it turns out that the last two games fail to do so.

1. Introduction

Maker-Breaker games are combinatorial games of perfect information and no chance moves, played by two players who alternately claim previously unclaimed elements of the given board. Formally stated, given any positive integer b and any hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{F})$, in a biased $(1 : b)$ Maker-Breaker game on $\mathcal{H}$ Maker and Breaker alternate in moves, usually with Maker being the first player and in each move, Maker is allowed to claim one previously unclaimed element from the *board* $\mathcal{X}$, and Breaker is allowed to claim up to b such elements. If, throughout the game, Maker manages to occupy all the elements of a *winning set* $F \in \mathcal{F}$, she is declared the winner of the game. Otherwise, Breaker wins. Note that these rules imply monotonicity in Breaker's bias [2]. In particular, there must be a *threshold bias* $b_{\mathcal{H}}$ such that Breaker wins if and only if $b \geqslant b_{\mathcal{H}}$. The size of such threshold biases has been investigated for many standard graph games in recent years, and complete studies can be found in [1, 3].

We look at the Maker-Breaker games played on $E(K_n)$, i.e. the edge set of the complete graph on n vertices, precisely at the games defined by the hypergraphs $\mathcal{C}_n$, $\mathcal{HAM}_n$, $\mathcal{H}_{H,n}$, $\mathcal{PAN}_n$ and $\mathcal{COL}_{n,k}$ on the vertex set $\mathcal{X} = E(K_n)$, with the hyperedges being the edge sets of all spanning trees of K_n, all Hamilton cycles of K_n, all copies of a fixed graph H in K_n, all pancyclic spanning subgraphs of K_n and all subgraphs of K_n with chromatic number larger than k, respectively.

There is a strong connection between these games and the so-called *random graph intuition*, as observed by Chvátal and Erdős [2], which roughly says that the more likely winner in the game between two random players is the same as the winner between two perfect players when played on $E(K_n)$. If this is true, then the threshold bias $b_{\mathcal{H}}$ should asymptotically be the inverse of the threshold probability $p_{\mathcal{H}}$ for the property that $G(n, p)$, where $p = \frac{1}{b+1}$, contains an element of $\mathcal{H}$.

Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{F})$ and a bias $b \geqslant 1$, we define the *multistage* $(1 : b)$ *Maker-Breaker game* on $\mathcal{H}$ as follows. The game proceeds in several stages, with each stage being played as a usual $(1 : b)$ Maker-Breaker game. For convenience we define $\mathcal{X}_0 := \mathcal{X}$, $\mathcal{F}_0 := \mathcal{F}$ and $\mathcal{H}_0 := \mathcal{H}$. Then, for $i \geqslant 1$, in the stage i, Maker and Breaker play on the board $\mathcal{X}_{i-1}$, consider the hypergraph $\mathcal{H}_{i-1} = (\mathcal{X}_{i-1}, \mathcal{F}_{i-1})$, and alternate in turns in which Maker occupies exactly one unclaimed element of $\mathcal{X}_{i-1}$, and afterwards Breaker occupies up to b unclaimed elements of $\mathcal{X}_{i-1}$, with Maker moving first. Once all the elements of $\mathcal{X}_{i-1}$ have been distributed among both players, we let $\mathcal{X}_i \subset \mathcal{X}_{i-1}$ be the set of all elements claimed by Maker in stage i, and we let $\mathcal{F}_i = \{F \in \mathcal{F}_{i-1} : F \subset \mathcal{X}_i\}$ be the set of all remaining winnings sets (from $\mathcal{F}$) that Maker managed to fully occupy in stage i. Observe that this defines a new hypergraph $\mathcal{H}_i := (\mathcal{X}_i, \mathcal{F}_i)$. We stop the game the first time that there are no winning sets left anymore, and our main question is how long Maker can delay the stop of a given game. For that, we define the threshold parameter $\tau(\mathcal{H}, b)$ as the largest number s such that, in the $(1 : b)$ multistage Maker-Breaker game on hypergraph $\mathcal{H}$, Maker has a strategy to ensure $\mathcal{F}_s \neq \varnothing$ and thus to play at least s stages.

2. Results

A random graph intuition in this setting corresponds to the assumption that a random game is likely to last as long as a perfectly played game. Therefore, if true, the random graph intuition would suggest that Maker can maintain a spanning subgraph, a Hamilton cycle, or a non-k-colorable graph for roughly $\log_{b+1}(n) - \log_{b+1}(\ln(n))$ stages. We show this is indeed asymptotically the best that Maker can do.

Note that in our results, we use $m_2(H) = \max_{\substack{G \subseteq H \\ v(G) \geqslant 3}} \frac{e(G)-1}{v(G)-2}$ to denote the *maximum 2-density* of H and $m(H) = \max_{\substack{G \subseteq H \\ v(G) \geqslant 1}} \frac{e(G)}{v(G)}$ to denote the *maximum density* of H.

Theorem 2.1 (Multistage Hamilton cycle game)**.** *If b is subpolynomial in n, then*

$$\tau(\mathcal{HAM}_n, b) = (1 + o(1)) \log_{b+1}(n) .$$

Corollary 2.2 (Multistage connectivity game)**.** *If b is subpolynomial in n, then*

$$\tau(\mathcal{C}_n, b) = (1 + o(1)) \log_{b+1}(n) .$$

Theorem 2.3 (Multistage non-k-colorability game)**.** *If b and k are subpolynomial in n, and $k \geqslant 2$, then*

$$\tau(\mathcal{COL}_{n,k}, b) = (1 + o(1)) \log_{b+1}(n) .$$

However the random graph intuition fails for both the multistage H-game and the multistage pancyclicity game, as, if played randomly, they would typically last $\left(\frac{1}{m(H)} + o(1)\right) \log_{b+1}(n)$ and $(1 + o(1)) \log_{b+1}(n)$ stages, respectively, while we can show the following.

Theorem 2.4 (Multistage H-game)**.** *If b is subpolynomial in n, then*

$$\tau(\mathcal{H}_{H,n}, b) = \left(\frac{1}{m_2(H)} + o(1)\right) \log_{b+1}(n) .$$

Theorem 2.5 (Multistage pancyclicity game)**.** *If b is subpolynomial in n, then*

$$\tau(\mathcal{PAN}_n, b) = \left(\frac{1}{2} + o(1)\right) \log_{b+1}(n) .$$

Observe that in all our results, it happens that the threshold $\tau(\mathcal{H}, b)$ is asymptotically the same as $\log_{b+1}(b_{\mathcal{H}})$, where $b_{\mathcal{H}}$ denotes the threshold bias discussed earlier. We believe that this is not a coincidence.

Using Lehman's Theorem [4], we can even provide a precise result for the $(1:1)$ multistage connectivity game, showing that Maker can do slightly better than what the above random graph argument suggests.

Theorem 2.6 (Unbiased connectivity game)**.**

$$\tau(\mathcal{C}_n, 1) = \lfloor \log_2(n) - 1 \rfloor .$$

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Abstract

Plain Weavings on Polyhedra

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An *m-fold weaving on a polyhedron* is a wrapping of set of rings interlaced with each other such that every point on the polyhedron either belongs to (the interior of) exactly m rings or lies on the boundaries of rings. In a weaving on a polyhedron, a *ring* is a closed strip of constant width which can be thought of as a rectangular strip of paper having zero thickness connected end to end to form a cylinder. At each point Q (on the polyhedron) which does not lie on the boundaries of the rings, the rings containing Q have stated ranking and this ranking is the same for each point Q contained in the rings. The *ranking* indicates that one ring is higher (passes over) or lower (passes under) than other rings at a particular section of the ring.

A *plain weaving on a polyhedron* is a two-fold weaving where each ring passes alternately over and under at crossings or overlaps. In this presentation, we discuss a necessary condition for the existence of a plain weaving on a polyhedron. Given a regular or semi-regular polyhedron, a systematic method to arrive at a polyhedron admissible to plain weavings will also be presented.

An example of a plain weaving on polyhedron A_{3^4} is shown in Figure 1(a). Polyhedron A_{3^4} is obtained from regular octahedron P_{3^4} by replacing each face of P_{3^4} with a baseless pyramid having right isosceles triangle faces. We say that A_{3^4} is admissible to plain weaving with 5 rings in each direction where rings in the same direction are given the same color. This plain weaving is represented by its plane diagram in Figure 1(b). This diagram is obtained from the net diagram of P_{3^4} where vertices with the same labels correspond to a single vertex of P_{3^4} and where the sides of an appended pyramid are projected to a face of P_{3^4}. In this diagram, the rankings of the rings at each crossing can be determined. For instance, in polygon $a_1v_3a_2'v_2$, at yellow squares the yellow rings passes over red rings, and at red squares the yellow rings passes under red rings.

Keywords: weaving, plain weaving, polyhedra

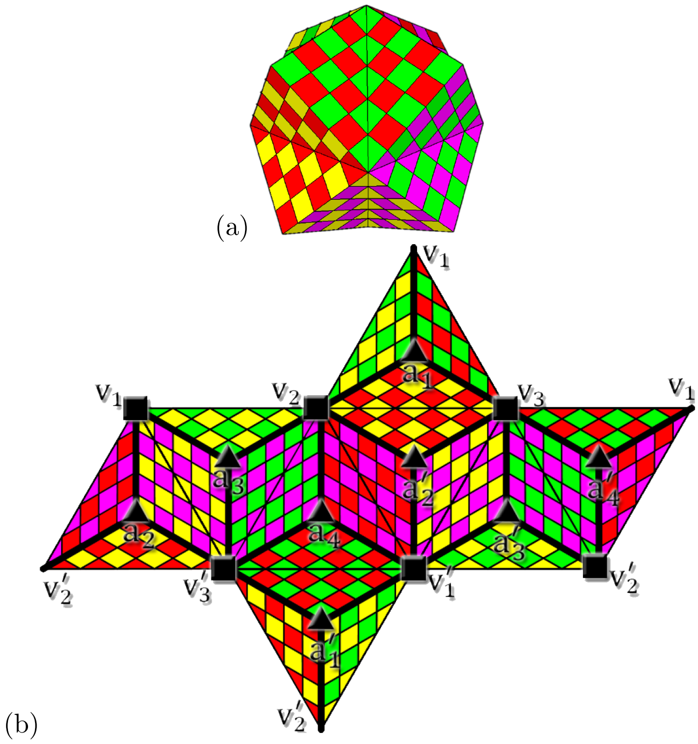

Figure 1: (a) A plain weaving on polyhedron A_{3^4} and (b) its plane diagram.

Finite Automata Encoding Functions: An Approach Using Computational Geometry

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Abstract

Finite automata are used to encode geometric figures, functions and can be used for image compression and processing. The most common approach is to represent each point of a figure in $\mathbb{R}^n$ as a convolution of its coordinates written in some base $b \geqslant 2$. Then a figure is said to be encoded as a finite automaton if the collection of all such convolutions is a regular ω–language. Similarly, a function is said to be encoded as a finite automaton if its graph can be encoded as a finite automaton. Jürgensen, Staiger and Yamasaki [1] showed that the only continuously differentiable functions which can be encoded as a finite automaton are linear. In this work we propose an alternative approach to encode functions using tools from computational geometry which enables us to extend a family of functions that can be encoded as finite automata. These tools are B–splines and hierarchical tensor product B–splines which are used in computer–aided design and computer graphics. With our approach we show that not only linear function can be encoded as finite automata but a large family of piecewise polynomial functions of an arbitrary degree of smoothness. Furthermore, we demonstrate that our encoding fits well in the framework of hierarchical tensor product B–splines by providing algorithms for basic problems arising in this framework. Our approach works not only with standard repesentations in base $b \geqslant 2$, but with other (nonstandard) representations of numbers.

1 Introduction

The idea to encode subsets of $\mathbb{R}^n$ as finite automata is not new. It was originally proposed by Büchi in the 1960s as an approach to prove decidability results in arithmetic. This idea later emerged in the 1990s–2000s but as a tool for image compression and processing see, e.g., [2,3] and for handling linear arithmetic over integers and reals [4]. To encode figures of zero size such as points, lines or planes the following approach is used [3,4]. A point $\overline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is represented as a convolution of infinite strings written in a base $b \geqslant 2$. For example, a point $\left(4\pi, \frac{\pi}{6}\right) \in \mathbb{R}^2$ can be represented by the convolution of two infinite strings $12.5663706144\dots$ and $0.5235987756\dots$ representing 4π and $\frac{\pi}{6}$, respectively, in the decimal representation ($b = 10$): $\begin{array}{ccccccccccccc} 1 & 2 & . & 5 & 6 & 6 & 3 & 7 & 0 & 6 & 1 & 4 & 4 \\ 0 & 0 & . & 5 & 2 & 3 & 5 & 9 & 8 & 7 & 7 & 5 & 6 \end{array} \dots$.

A figure $U \subseteq \mathbb{R}^n$ is given by its set of points $\overline{x} \in U$ which defines an ω–language L (a set of infinite strings) of all possible convolutions representing $\overline{x} \in U$ in some base $b \geqslant 2$. The figure U is said to be encoded as a finite automaton* if L is recognized by a finite automaton $\mathcal{A} = (\Sigma, S, s_0, \delta)$ with an input alphabet Σ, a set of states S, the initial state s_0 and the transition function $\delta : S \times \Sigma \to S \cup \{\perp\}$. If $\delta(s, \sigma) = \perp$, it means that $\delta(s, \sigma)$ is undefined. An automaton $\mathcal{A}$ accepts† an infinite string w if $\delta(s_0, u) \neq \perp$ for all prefixes u of w, where $\delta(s_0, u)$ is defined in a usual way; in other words the automaton does not get stuck while reading the string w. An ω–language is said to be recognized by $\mathcal{A}$ if it consists of all infinite strings accepted by $\mathcal{A}$.

Let $f : \mathbb{R}^k \to \mathbb{R}$ be a continuously differentiable function. It is said to be encoded as a finite automaton if the graph $\Gamma(f) = \{(\overline{v}, f(\overline{v})) \,|\, \overline{v} \in \mathbb{R}^k\} \subseteq \mathbb{R}^{k+1}$ is encoded as a finite automaton. Jürgensen, Staiger and Yamasaki showed that a continuously differentiable function of one variable with non–constant derivative is not encodable as a finite automaton [1]. Therefore, if f is encoded as a finite automaton, it can only be linear. An alternative representation which enables encoding of continuously differentiable functions as finite automata other than linear ones comes from a branch of computational geometry – geometric modeling. In this representation f is written as a sum $f = \sum_{\xi \in \Xi} \lambda_\xi B_\xi$, where $\lambda_\xi \in \mathbb{R}$, B_ξ are tensor

*In their original paper Jürgensen, Staiger and Yamasaki [1] define figures encoded as a finite automaton only for subsets of the unit cube $[0,1]^n$. However their definition can be straightforwardly applied to subsets in $\mathbb{R}^n$. Also, in their definition they use a more general notion – regular ω–languages which extend ω–languages recognized by finite automata by applying operations of union, intersection, set–theoretical difference, projections and their inverse mappings.

†The reader can notice that $\mathcal{A}$ is equivalent to a certain deterministic Büchi automaton. For practical reasons some researchers prefer to use more powerful nondeterministic Büchi automata to define figures encoded as finite automata, see, e.g. [3]. However, all examples which appear in the literature can be obtained as sets of solutions to systems of linear inequalities with rational coefficients and their infinite copies, see, e.g., [4].

product B–splines and Ξ is a countable set. Tensor product B–splines are products of univariate B–splines $N[x_0, \ldots, x_{m+1}](x)$ defined by knots $x_0 \leqslant \cdots \leqslant x_{m+1}$ which are concrete piecewise polynomials of degree m having support in the interval $[x_0, x_{m+1}]$; if $x_0 < \cdots < x_{m+1}$, then they are of differentiability class C^{m-1}. Thus, instead of representing every single point of $\Gamma(f)$ as a convolution of strings, we can represent each pair (λ_ξ, B_ξ) as a convolution of strings presenting λ_ξ and the knots defining B_ξ. The numbers λ_ξ and the knots defining B_ξ can be chosen from some countable subset dense in $\mathbb{R}$ and can be presented by finite strings instead of infinite ones. Since Ξ is countable, all this enables us to use classical finite automata instead of their Büchi counterparts to encode functions.

In geometric modeling knots defining B_ξ are not chosen arbitrarily but in a systematic way to guarantee linear independence of $B_\xi, \xi \in \Xi$ and their completeness[‡]. In this work we use a mainstream approach which guarantees both (linear independence and completeness) – hierarchical tensor product B–splines. In brief, for a d–dimensional mesh defined by a nested sequence of domains $\mathbb{R}^d \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N-1} \supset \Omega^N = \varnothing$ hierarchical tensor product B–splines are obtained through the process which is referred to as Kraft's selection mechanism [5]. They are linearly independent. Moreover, they form a complete basis if a certain geometric condition on the shapes of the domains $\Omega^1, \ldots, \Omega^{N-1}$ is satisfied [6]. In this work we show in particular that the proposed encoding of functions as finite automata fits well in the classical framework of hierarchical tensor product B–splines. Moreover, it enables to have a finite representation for infinite meshes and functions with unbounded domains which is suitable for computing.

2 Main Results

We carefully introduce our construction for encoding C^{m-1} piecewise polynomial functions of degree m over a hierarchical mesh as finite automata. Then we show that for our encoding linear functions along with the traditional approach can also be encoded as finite automata for each m. After that, using our encoding, we describe algorithms for specific problems which occur in the framework of hierarchical B–splines. In brief, these problems and algorithms are as follows.

Let us first be given a finite automaton encoding a hierarchical mesh. This mesh is defined by a nested sequence of domains $\mathbb{R}^d \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N-1} \supset \Omega^N = \varnothing$. We describe a verification procedure for this nestedness condition. Kraft's selection mechanism [5] generates a complete basis consisting of tensor product B–splines if a certain geometric condition on the shapes of the domains $\Omega^1, \ldots, \Omega^{N-1}$ is satisfied [6]. We describe a verification procedure for this geometric condition. Furthermore, we describe an algorithm for constructing a finite automaton which encodes a collection of the basis generated by Kraft's selection mechanism. Now let us be given a finite automaton encoding a function f. We describe an algorithm for computing the value of this function f at any given point. Moreover, suppose that the mesh is refined by selecting a nonempty subdomain $\Omega \subseteq \Omega^{N-1}$. Then the automaton encoding f must be updated for the refined mesh. We describe an algorithm for constructing a finite automaton encoding f with respect to the refined mesh.

Finally we discuss how our encoding can be adapted to the use of other (nonstandard) representation of numbers, see, e.g., [7].

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[‡]Informally speaking, completeness means that $B_\xi, \xi \in \Xi$ span a space of functions of some degree of smoothness over a mesh which are polynomials in each cell of this mesh. We give all necessary technical details in a full paper.

Multipartite Ramsey Numbers of Double Stars

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Abstract

Let s be a positive integer with $k \geq 2$ and G_1, G_2 be simple graphs. The set multipartite Ramsey number, $M_s(G_1, G_2)$, is the smallest positive integer c such that any 2-coloring of the edges of $K_{c\times s}$ contains a monochromatic copy of G_i in color i for some $i \in \{1, 2\}$. Let t be a positive integer with $t \geq 2$. The size multipartite Ramsey number, $m_t(G_1, G_2)$, is the smallest positive integer ζ such that any 2-coloring of the edges of $K_{t\times\zeta}$ contains a monochromatic copy of G_i in color i for some $i \in \{1, 2\}$. For $a \geq b \geq 0$, the double star $S(a, b)$ is the graph on the vertex-set $\{v_0, v_1, \ldots, v_a, w_0, w_1, \ldots, w_b\}$ with edge-set $\{v_0w_0, v_0v_i, w_0w_j | 1 \leq i \leq a, 1 \leq j \leq b\}$. In this talk, we utilize $(-1, 1)$-matrices and strongly regular graphs to establish lower bounds, upper bounds, and some exact values of multipartite Ramsey numbers of double stars. Our main result is the following.

Theorem 1. *Let m, n, t_1, t_2 be integers where $m \geq 4$ and $n \geq 6$. If m is even and n is not a prime number, then we have the following.*

- *If $m \geq 2\max\{t_1, t_2\}+1$, then $M_m(S(m(n-1)/2, t_1), S(m(n-1)/2, t_2)) = n+1$.*
- *If $n \geq 2\max\{t_1, t_2\}+2$, then $m_n(S(m(n-1)/2, t_1), S(m(n-1)/2, t_2)) = m+1$.*

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Trees with certain total vertex irregularity strength

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Abstract

For a simple graph $G(V,E)$ and positive integer k, a vertex irregular total k-labeling of G is a mapping $\varphi : V \cup E \to \{1,2,\ldots,k\}$ such that $wt(x) \neq wt(y)$ for any two distinct vertices $x, y \in V$, where $wt(x) = \varphi(x) + \sum_{xz\in E} \varphi(xz)$. The minimum k for which G has a vertex irregular total labeling is called the total vertex irregularity strength of G and is denoted by $\mathrm{tvs}(G)$. Finding the total vertex irregularity strength for general trees is a difficult and a challenging problem. Nurdin et al. [1] conjectured that for every tree T, then $\mathrm{tvs}(T) = \max\{\lceil (n_1+1)/2\rceil, \lceil (n_1+n_2+1)/3\rceil, \lceil (n_1+n_2+n_3+1)/4\rceil\}$, where n_i $(i = 1,2,3)$ denote the number of vertices of degree i in T. The truth of the conjecture has been verified for any tree with maximum degree four or five [2, 3]. However, for trees with maximum degree at least six, the conjecture is still open. Let T be a tree. A *leaf* is a vertex of degree 1. A vertex is called an exterior vertex if it is adjacent to a leaf. In this talk, we consider trees with large number of exterior vertices. In particular, we prove that any tree T containing n_2^e exterior exterior of degree 2 and at least $\lceil (n_1+1)/2\rceil - 2n_2^e - 1$ exterior vertices of degree $d > 2$ such that $3n_3 - n_1 - 1 \leq n_2 \leq (n_1+1)/2$ or $n_2 \leq \min\{3n_3 - n_1 - 2, n_1 - n_3 + 1\}$ then $\mathrm{tvs}(T) = \lceil (n_1+1)/2\rceil$. This result further supports the above-mentioned conjecture of Nurdin et al. [1].

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H-SUPERMAGIC LABELING OF EDGE-COMB PRODUCT OF GRAPHS RELATED TO PATH AND CYCLE

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Abstract

We use the definition of graph based on [1] where $E \subset [V]^2$. Also, the graph that we talk here is a nontrivial and connected graph. Suppose that G and H two graphs where H is a subgraph of G. We define G admits H-covering if for every $e \in E(G)$, there exist H' as subgraph of G isomorphic to H such that $e \in E(H')$. Let us see the figure below. We can see that for every edge e in graph

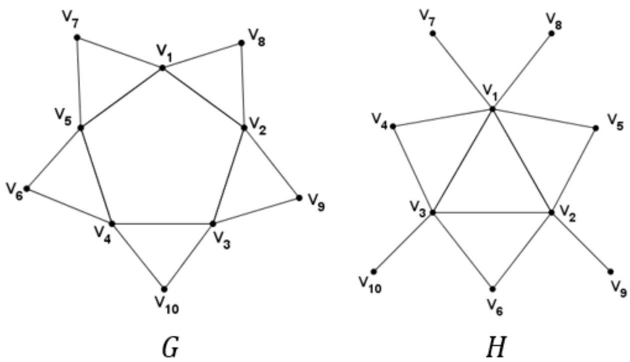

Figure 1: **Example of C_3-covering**

G, there exist graph C_3 such that $e \in E(C_3)$. But, it's a different case for graph H. For example, let us take $v_1v_7 \in E(H)$. We see that there is no subgraph isomorphic to C_3 in H (let's say $H^{'}$) such that $v_1v_7 \in E(H^{'})$. So, we say that G admits C_3-covering but graph H is not admit C_3-covering. Suppose that graph G admits H-covering. Such a graph G is called H-magic labeling if there is a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |E(G)| + |V(G)|\}$ such that for every subgraph H' of G isomorphic to H, the sum of every vertex and edge label is equal to k, where k is the magic number. Furthermore, it is called H-supermagic labeling if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ [2]. Here is an example of C_3-supermagic labeling of Fan graph F_3. We see from figure 2 that the sum of every vertex and edge label of any C_3 is equal to 27. That number is called the magic number.

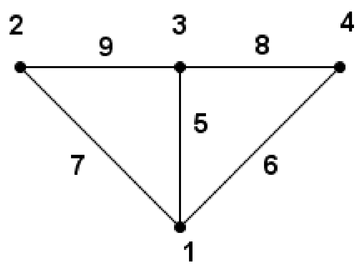

Figure 2: **C_3-supermagic labeling of fan graph** F_3

We then give the edge comb product definition which introduced by [3]. Suppose G and H be two graphs and e is an edge of graph H. Edge comb product of graph between G and H, denoted by $G \trianglerighteq_e H$, is a graph obtained by taking a copy of G and $|E(G)|$ copies of graph H, then identifying i-th copy of H at the edge e to the i-th edge of G. By the example at figure 3 and the definition, we conclude that edge comb product of a graph is not unique operation. Furthermore, it is also not

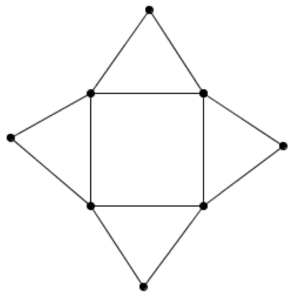

Figure 3: $C_4 \trianglerighteq_e C_3$ **graph**

commute. Actually, the definition is quite similar to the paper introduced by [4] where the calendula graph is an example of edge comb product between two cycles. Now, we enter the main topic of this paper. In this paper, we proof the existence of H-supermagic labeling of graph $C_n \trianglerighteq_e H$ and $P_n \trianglerighteq_e H$ for any nontrivial graph H where $|E(H)| \geq 2$ using the balanced multi-set method.

Keywords: Edge-comb product, path, cycle, balanced multi-set method

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An old problem of Erdős

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1 Abstract

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, P. Erdős raised the problem of determining $f(n)$ (see [1], p.247, Problem 11). Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths. Let M_n be the set of simple graphs on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. A natural question is what is the numbers of $mc(n)$. Shi[12] proved that

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$ and $f(n) = f^*(n-1)+3$ for $n \geq 3$. Lai[5] proved that for $n \geq e^{2m}(2m+3)/4$,

$$f(n) < n - 2 + \sqrt{nln(4n/(2m+3)) + 2n} + log_2(n+6).$$

Chen, Lehel, Jacobson and Shreve[3] gave a quick proof of this result.

Lai[7] proved that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

and Lai[6] conjectured that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

Boros, Caro, Füredi and Yuster[2] proved that

$$f(n) \leq n + 1.98\sqrt{n}(1+o(1)).$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

In 1988, Shi[12] proved that

For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n-15}-3)]$.

In 1998, G. Chen, J. Lehel, M. S. Jacobson, and W. E. Shreve [3] proved that

$f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] improved this lower bound significantly:

$f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.

and conjectured that

$\lim \frac{f_2(n)-n}{\sqrt{n}} = 1$.

J. Ma, T. Yang [11] proved that

Any n-vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.

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H. Lin, M. Zhai,Y. Zhao [10] proved that

Let G be a graph of order $n \geq 26$. If $\rho(G) \geq \rho(K_{1,n-1}^{+})$, then G contains two cycles of the same length unless $G \cong K_{1,n-1}^{+}$.

and asked the following problem.

What is the maximum spectral radius among all 2-connected n-vertex graphs without two cycles of the same length?

C. Lai [4] proved that

$mc(n) \leq n-1$ for $n \geq \sum_{i=1}^{71} i - 8 \times 18$.

Lai[8] proved that

For n sufficiently large,

$$mc(n) \leq \frac{15}{16}n.$$

and conjectured that

$$\lim_{n\to\infty} \frac{mc(n)}{n} = 0.$$

Survey papers on this problem can be found in Tian[13], Zhang[14], Lai and Liu[9].

We present the problems, conjectures related to this problem and we summarize the know results.

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