On the Cauchy problem for hyperbolic operators with non-regular coefficients.

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We present some new results on the well–posedness of the Cauchy problem for a class of hyperbolic operators. Let $T > 0$. We are concerned with the equation

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} + \sum_{i=1}^{n} b_i(t) u_{x_i} + c(t)u = 0 \quad \text{in } [0,T] \times \mathbf{R}^n, \tag{1}$$

with initial data

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad \text{in } \mathbf{R}^n, \tag{2}$$

where the coefficients b_i and c are measurable and bounded and (a_{ij}) is a real symmetric matrix such that

$$a(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j/|\xi|^2 \geq \lambda_0 > 0, \qquad \text{(strictly hyperbolic case)}$$

or

$$a(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j/|\xi|^2 \geq 0, \qquad \text{(weakly hyperbolic case)}$$

for all t and for all $\xi \neq 0$.

Our main hypothesis is the following: we suppose that there exist $\bar{t} \in [0,T]$ and there exist q, $C > 0$ such that $a_{ij} \in \mathcal{C}^1([0,T] \setminus \{\bar{t}\})$ for all i, $j = 1, \ldots, n$, and

$$|a'(t,\xi)| \leq C|t - \bar{t}|^{-q}, \tag{3}$$

for all $(t,\xi) \in ([0,T] \setminus \{\bar{t}\}) \times (\mathbf{R}^n \setminus \{0\})$ (here $'$ denotes the derivative with respect to the variable t).

Theorem 1 *Consider the equation (1) and suppose that the condition (3) holds with* $q = 1$.

Then the Cauchy problem (1), (2) is $\mathcal{C}^\infty$*–well–posed in the strictly hyperbolic case and it is* $\gamma^{(s)}$*–well–posed for all* $s < 3/2$ *in the weakly hyperbolic case.*

Our second result is the following.

Theorem 2 *Let $q > 1$ and $0 \leq p < 1$, with $p \leq q - 1$. Consider the equation (1) and suppose that the condition (3) holds. Suppose moreover that there exists $C' > 0$ such that*

$$|a(t, \xi)| \leq C' |t - \bar{t}|^{-k}, \tag{4}$$

for all $(t, \xi) \in ([0, T] \setminus \{\bar{t}\}) \times (\mathbf{R}^n \setminus \{0\})$.

Then (1), (2) is $\gamma^{(s)}$–well–posed for all $s < (q - k)/(q - 1)$ in the strictly hyperbolic case and it is $\gamma^{(s)}$–well–posed for all $s < (q - 3k + 2)/(q - 2k + 1)$ in the weakly hyperbolic case.

Our third theorem gives a link between the results for operators having Hölder–continuous coefficients in the principal part

Theorem 3 *Let $q > 1$ and $0 < \alpha < 1$. Consider the equation (1) and suppose that $a_{ij} \in \mathcal{C}^{0,\alpha}([0, T])$ for all i, $j = 1, \ldots, n$. Suppose moreover that the condition (3) holds.*

Then (1), (2) is $\gamma^{(s)}$–well–posed for all $s < (q/(q - 1))(1/(1 - \alpha))$ in the strictly hyperbolic case and it is $\gamma^{(s)}$–well–posed for all $s < (q + 2)(\alpha + 2)/(2(q + \alpha + 2))$ in the weakly hyperbolic case.

The results of the Theorems 1, 2 and 3 in the case of the strictly hyperbolicity are in some sense optimal. In fact we can construct some sharp counter examples.

References

[1] F. Colombini, D. Del Santo and T. Kinoshita, *On the Cauchy problem for hyperbolic operators with non-regular coefficients*, to appear.

[2] F. Colombini, D. Del Santo and T. Kinoshita, *Well–posedness of the Cauchy problem for a hyperbolic equation with non–Lipschitz coefficients*, to appear.

[3] F. Colombini, D. Del Santo and T. Kinoshita, *Gevrey-well-posedness for weakly hyperbolic operators with non-regular coefficients*, to appear.