# Quasi-linear Klein-Gordon equations in one space dimension I, II

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The aim of our talk will be to give a proof of global existence for solutions to the quasi-linear Klein-Gordon equation with small, smooth, compactly supported Cauchy data, in one space dimension, when the nonlinearity satisfies a convenient compatibility condition :

$$(1)\qquad \begin{aligned} (\partial_t^2 - \partial_x^2 + 1)u &= F(u, \partial_t u, \partial_x u, \partial_t\partial_x u, \partial_x^2 u) \\ u|_{t=0} &= \epsilon u_0 \\ \partial_t u|_{t=0} &= \epsilon u_1 \end{aligned}$$

where $\epsilon > 0$ is a small parameter and where $u_0$, $u_1$ are in $C_0^\infty$.

Let us recall first that the quasi-linear Klein-Gordon equation has global solutions for small, smooth, compactly supported Cauchy data, in $d$-space dimension, if $d \geq 2$ for any quasilinear nonlinearity $F$ vanihing at least at order 2 at 0. This comes from the fact that such problems are essentially short range ones.
More precisely, the solution of the linear equation $(\partial_t^2 - \Delta_x + 1)u = 0$, with smooth compactly supported Cauchy data, has $L^\infty$ norm decaying like $t^{-d/2}$ when $t \to +\infty$. If one writes the nonlinear term $F$ in (1) as a nonlinear potential $V(u, \partial_t u, \partial_x u)$ times $u$ or one of its derivatives, and if one expects that $\|u(t,\cdot)\|_{L^\infty}$ will behave when $t$ goes to $+\infty$ as in the case of linear solutions, one sees that $\|V(t)\|_{L^\infty} = O(t^{-k\frac{d}{2}})$ when $t \to +\infty$, where $k$ is the order of vanishing of $V$ at 0. In particular, if $d \geq 3$ and if $F$ is at least quadratic, i.e. if $k \geq 1$, $t \to \|V(t)\|_{L^\infty}$ is integrable close to $+\infty$. In other words, the potential $V$ is a *short range* nonlinear perturbation of the operator $\partial_t^2 - \partial_x^2 + 1$. This behaviour plays a key role in the proof of global existence for small Cauchy data with compact support given by Klainerman [6] and Shatah [10], for space dimensions larger or equal to 3. Another consequence of this short range character of the nonlinear perturbation is the fact that when $d \geq 3$, the solutions to (1) have for small $\epsilon$ the same long time asymptotics as linear solutions, namely

$$(2)\qquad u(t,x) \sim \frac{\epsilon}{t^{d/2}} A(x/t) e^{i\sqrt{t^2-x^2}} + \frac{\epsilon}{t^{d/2}} B(x/t) e^{-i\sqrt{t^2-x^2}}$$

where $A$ and $B$ are smooth functions supported in the closed unit ball.
The same short range character holds true in $d = 2$ space dimension for cubic nonlinearities. In case of quadratic nonlinearities, i.e. when the potential $V$ vanishes at order $k = 1$, and still for $d = 2$ space dimensions, one has $\|V(t)\|_{L^\infty} = O(t^{-1})$, which is not integrable

close to infinity. Anyway, it turns out that using a method of normal forms, Ozawa, Tsutaya and Tsutsumi [9] succeeded, at least for semilinear equations, to reduce the case of quadratic nonlinearities to cubic ones, and so to a short range problem. Because of that, the linear type asymptotics (2) hold true also in two space dimension.
For the one dimensional problem, the nonlinear perturbation is genuinely a *long range* one. As a consequence, for general nonlinearities, the solution *does not* exist globally in time. More precisely, Moriyama, Tonegawa and Tsutsumi [8] proved that the time of existence $T^*(\epsilon)$ of a smooth solution satisfies $T^*(\epsilon) \geq ce^{c/\epsilon^2}$ for a positive $c$. Moreover Yordanov [11], and Keel and Tao [5], constructed examples of nonlinearities for which the solution does blow up in finite time of order $c'e^{c'/\epsilon^2}$. In [4], Hörmander raised the question of determining the set of all nonlinearities for which the solution is global. In [2], constructing asymptotic solutions, we conjectured the form of these nonlinearities. More precisely, if one considers the quasi-linear Klein-Gordon equation in one space dimension (1) where $F$ is the sum of a quadratic polynomial $Q$, and of a cubic one $P$, $P$ and $Q$ being of order less or equal to 1 relatively to second order derivatives, we conjectured an explicit condition (called a "null condition") in terms of the coefficients of $P$ and $Q$ under which (1) should have global solutions for small Cauchy data in $C_0^\infty$. We do not reproduce here the (quite technical) form of this condition. Let us just mention that when this condition *is not* satisfied, one can construct *formal solutions* to (1) blowing up in finite time. This does not prove that true solutions do blow up in finite times, but makes very reasonnable the conjecture that this "null condition" should hold true if and only if (1) has global solutions for small smooth compactly supported Cauchy data. The aim of paper [3] that we will expose in our talks was to prove one half of this conjecture, namely the following theorem:

**Theorem 1** *Assume that the nonlinearity $F = Q + P$ in (1) satisfies the null condition. Fix numbers $B > 0$, $\sigma > 7/2$. There is $\epsilon_0 > 0$ such that for any $\epsilon \in ]0, \epsilon_0[$ and any couple $(u_0, u_1)$ in the unit ball of $H^\sigma \times H^{\sigma-1}$, supported in the interval $[-B, B]$, equation (1) with Cauchy data $u|_{t=0} = \epsilon u_0$, $\partial_t u|_{t=0} = \epsilon u_1$ has a unique global solution $u \in C^0(\mathbb{R}, H^\sigma) \cap C^1(\mathbb{R}, H^{\sigma-1})$.*

The proof of the above theorem, which contains as a special case the older results of Moriyama [7], gives also the asymptotic behaviour of the solutions:

**Theorem 2** *Let us keep the above notations and assume moreover that $u_0, u_1$ are in $C_0^\infty$. There is a smooth function $\Psi(y)$ on $]-1,1[$, which depends on $Q, P$ but not on the Cauchy data, and a function $a(y)$ $C^\infty$ on $\mathbb{R}$, supported in $[-1,1]$, such that the solution $u$ has the following asymptotic expansion when $t \to +\infty$*

$$u(t,x) \sim \frac{\epsilon}{\sqrt{t}} a(x/t) \exp i \left[ \sqrt{t^2 - x^2} - \frac{1}{2}\epsilon^2 \log t \sqrt{1 - \frac{x^2}{t^2}} \Psi(x/t) |a(x/t)|^2 \right] \tag{3}$$
$$+ \text{conjugated quantity} + r(t,x)$$

*where $r(t,x)$ is a remainder satisfying for any $\delta > 0$, $M > 0$*

$$|r(t,x)| \leq C t^{-1+\delta} \left[ \frac{1}{t} + \left( 1 - \left| \frac{x}{t} \right| \right)_+ \right]^M .$$

One remarks immediately that the phase in the expansion (3) is given by the phase of oscillation of linear solutions $\sqrt{t^2-x^2}$ modified by a new $\epsilon^2 \log t$ term. This reflects the long range character of the problem, and is at the origin of all the difficulties of the proof. The standart methods, used in higher space dimension (i.e. for short range problems), rely on an inequality of type

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{d/2}} \sum_{|\alpha|\leq M} \|X^\alpha u(t,\cdot)\|_{L^2} \tag{4}$$

where $X^\alpha$ are iterates of convenient vector fields (in one space dimension one can take for $X$ the usual derivatives and $t\partial_x + x\partial_t$). In higher space dimensions, energy inequalities allow one to get a bound for $\|X^\alpha u(t,\cdot)\|_{L^2}$ *uniformly in time*. The optimal decay estimate $\|u(t,\cdot)\|_{L^\infty} \leq Ct^{-d/2}$ follows from (4) and this uniform energy bound.
In one space dimension, the asymptotic behaviour (3) shows that one cannot expect a uniform bound for $\|X^\alpha u(t,\cdot)\|_{L^2}$: actually, because of the $\epsilon^2 \log t$ term in the phase, this quantity will blow up like a power of $\epsilon^2 \log t$ when $t \to +\infty$. The idea of our proof is to obtain the optimal $L^\infty$ estimate for $u(t,\cdot)$ in a different way. Writing the equation in convenient hyperbolic coordinates, one is reduced to an ODE perturbed by $\partial_x$ derivatives terms, all of whose contain expressions of type $\frac{1}{t}\frac{\partial_x}{t}u$. The fact that Sobolev norms of $u(t,\cdot)$ blow up only at some logarithmic rate when $t$ goes to infinity implies that

$$\left\| \frac{1}{t}\frac{\partial_x}{t} u \right\|_{L^\infty} = O(t^{-2+\delta}),\ t \to +\infty$$

for any positive $\delta$, and so is integrable. In that way, one sees that the $\partial_x$ perturbations of the ODE can be considered as remainders, as long as one is interested only in $L^\infty$ bounds, and as long as one has proved a convenient Sobolev estimate. Consequently, one is reduced to the study of global existence and boundedness for solutions of an *ordinary differential equation* with small Cauchy data. The null condition (i.e. the structural assumption made on the nonlinearity) is exactly what is needed to achieve that.

Our first talk will be devoted to the description to the general strategy of proof described above. As we just mentionned, the main idea is to reduce oneself to an ODE through the use of convenient Sobolev estimates. Our second talk will discuss these energy estimates. They rely on the definition of convenient spaces, well adapted to the expected behaviour of the solutions, and on systematic use of Bony's paradifferential calculus [1] to get estimates in these spaces.

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