The Cauchy problem of a semilinear integro-differential equation interporating heat and wave equations

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In this talk, we study the Cauchy problem for the nonlinear integro-differential equation

$$(1) \quad \frac{\partial u(t)}{\partial t} = \int_0^t R_\alpha(t-s)\Delta u(s)ds + \int_0^t R_\alpha(t-s)f(u(s))ds, \quad u(0) = \varphi,$$

where $u(t) = u(x,t)$, $\varphi = \varphi(x)$, $(x,t) \in \mathbf{R}^n \times I$, I is an interval in $(0,\infty)$, $0 < \alpha < 1$, $R_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $f(u) \in C^1(\mathbf{R},\mathbf{R})$ is a nonlinear function such that

$$(2) \quad f(0) = 0 \quad \text{and} \quad |f'(u)| \le C(1+|u|^\sigma), \quad \sigma > 0.$$

Formally, (1) is written as a fractional order evolution equation

$$\frac{\partial^{1+\alpha} u}{\partial t^{1+\alpha}} = \Delta u + f(u), \quad u(0) = \varphi,$$

which interpolates nonlinear heat and wave equations.

On the linear case ($f(u) = 0$) for $n = 1$, (1) discribes the heat conduction with memory [4, 7], and many authors studied this problem ([1,2,4,7,8,9,10]). Schneider, Wyss [8] and Y. Fujita [1] obtained the representation formula of fundamental solution for this linear equation for $n = 1$. Furthermore, Y. Fujita [1] proved that the fundamental solution of this linear equation has some properties which are similar to heat equation's one and other properties which are similar to wave equation's one. For example, the points where the fundamental solution takes its maximum propagate with finite speed. Y. Fujita

[2] also obtained the representation of the solution of linear integro-differential equation including wave equation by probability methods. On the other hand, there is no study for the nonlinear equations of this type. In the heat equation case $\alpha = 0$, the nonlinear term $f(u)$ represents the heat source, and the wave equation case $\alpha = 1$, the interaction term. In our case $0 < \alpha < 1$, we can consider $f(u)$ as the source with the memory. The typical form of nonlinear term is the power type: $f(u) = au^\sigma$. The purpose of this talk is devoted to the study of the Cauchy problem for nonlinear integro-differential equation (1).

For stating main results, we first define "admissible triplet".

Definition 1. We call (p, q, r) as admissible triplet with respect to α if

$$\frac{1}{q} = \frac{n(1+\alpha)}{2}\Big(\frac{1}{r} - \frac{1}{p}\Big), \quad 1 < r \le p < \begin{cases} \frac{n(1+\alpha)r}{n(1+\alpha)-2}, & \text{if } n \ge 2, \\ \infty, & \text{if } n = 1. \end{cases}$$

Our main results can be stated as follows.

Theorem 2. Assume that $f(u)$ satisfies condition (2). Let $r \ge r_0 = \frac{n(1+\alpha)(1+\sigma)}{2} > 1$, or $r > 1$ when $r_0 \le 1$, $\varphi \in L^r(\mathbf{R}^n)$, and fix arbitrary (p, q) as (p, q, r) form an admissible triplet. Then, there is a maximal existence time $T^* > 0$ and a function u satisfying (1), satisfying following properties.

(a) $u \in C((0, T^*); L^r(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$. In particular, for any $p \ge r$ and $t \in (0, T^*)$, $u(t) \in L^p(\mathbf{R}^n)$.

(b) This solution u is unique in the class

$$\Big\{ v \in C((0, T^*); L^p(\mathbf{R}^n)) \;\Big|\; \sup_{t \in (0, T^*)} t^{1/q} \|v(t)\|_p < \infty \Big\}$$

for any $p \ge r$.

(c) For any $p > r$, u satisfies $t^{\frac{1}{q}} \|u(t)\|_p \to 0$, as $t \to 0$. Here q is determined as (p, q, r) form an admissible triplet.

(d) $u \in L^q((0, T^* - \varepsilon); L^p(\mathbf{R}^n))$ for any $\varepsilon > 0$.

(e) If $T^* < \infty$, for any $p \ge r$, $\lim_{t \to T^*} \|u(t)\|_p = \infty$, $r \le p \le \infty$, In addition, we have

$$\|u(t)\|_p \ge C(T^* - t)^{(1+\alpha)\left(\frac{n}{2p} - \frac{1}{\sigma}\right)}.$$

Outline of the Proof of the Theorem 2. First, we consider the linear inhomogeneous problem corresponding to (1),

$$(3)\quad \frac{\partial u(t)}{\partial t} = \int_0^t R_\alpha(t-s)\Delta u(s)ds + \int_0^t R_\alpha(t-s)f(\cdot,s)ds, \quad u(0)=\varphi.$$

By using Fourier and Laplace transforms, we can show that (3) admits a unique solution

$$u(x,t) = (\tilde{E}_{1+\alpha} * \varphi)(x,t) + \left(\int_0^t \tilde{E}_{1+\alpha}(\cdot, t-s) * \int_0^s R_\alpha(s-\tau)f(\cdot,\tau)d\tau ds\right)(x),$$

where

$$\tilde{E}_{1+\alpha}(x,t) = \left(\frac{1}{2\pi}\right)^{n/2} \mathcal{F}^{-1}(E_{1+\alpha}(-|\cdot|^2 t^{1+\alpha}))(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} e^{ix\xi} E_{1+\alpha}(-|\xi|^2 t^{1+\alpha})d\xi.$$

The notation E_α denotes Mittag-Leffler's function, that is,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}.$$

By using the integral representation of Mittag-Leffler's function ([1])

$$E_{1+\alpha}(-|\xi|^2) = \frac{1}{1+\alpha}\exp\{a_{1+\alpha}(\xi)\} + \frac{1}{1+\alpha}\exp\{b_{1+\alpha}(\xi)\} + f_{1+\alpha}(\xi),$$

where

$$a_{1+\alpha}(\xi) \equiv |\xi|^{\frac{2}{1+\alpha}} e^{\frac{i\pi}{1+\alpha}}, \quad b_{1+\alpha}(\xi) \equiv |\xi|^{\frac{2}{1+\alpha}} e^{-\frac{i\pi}{1+\alpha}},$$

and

$$f_{1+\alpha}(\xi) \equiv \begin{cases} -\dfrac{\sin(\alpha\pi)}{\pi}\displaystyle\int_0^\infty \frac{|\xi|^2 s^\alpha e^{-s}}{s^{2+2\alpha} - 2|\xi|^2 s^{1+\alpha}\cos(\alpha\pi) + |\xi|^4} ds, & |\xi| \neq 0, \\ 1 - \frac{2}{1+\alpha}, & |\xi| = 0, \end{cases}$$

we obtain

$$|E_{1+\alpha}(-|\xi|^2)| \leq C(1+|\xi|^2)^{-1} \text{ for all } \xi \in \mathbf{R}^n.$$

Since

$$\mathrm{Re}(e^{\pm\frac{i\pi}{1+\alpha}}) = \cos\left(\frac{\pi}{1+\alpha}\right) < 0, \text{for } 0<\alpha<1,$$

and

$$|\partial_\xi^\beta f_{1+\alpha}(\xi)| \leq C_\beta |\xi|^{-2-|\beta|},$$

we can estimate

$$\sum_{|\beta|\le[\frac{n}{2}]+1}\int_{\frac{R}{2}<|\xi|<2R}|R^{|\beta|}\partial_\xi^\beta f_{1+\alpha}(\xi)|^2d\xi \le C\sum_{|\beta|\le[\frac{n}{2}]+1}\int_{\frac{R}{2}}^{2R}R^{2|\beta|}r^{-4-2|\beta|}r^{n-1}dr \le CR^n.$$

So, we obtain

$$\sum_{|\beta|\le[\frac{n}{2}]+1}\int_{\frac{R}{2}<|\xi|<2R}|R^{|\beta|}\partial_\xi^\beta E_{1+\alpha}(-|\xi|^2)|^2d\xi \le CR^n$$

for some constant C, independent of R. By Mihlin-Hörmander's theorem ([6, Theorem 7.9.5]), we have $E_{1+\alpha}(-|\cdot|^2)$ belongs to the Hörmander spacs $\mathcal{M}_p$, that is,

$$\|\tilde{E}_{1+\alpha}(\cdot,t)*\phi\|_p \le C\|\phi\|_p$$

for any $p\in(1,\infty)$. Moreover, since $E_{1+\alpha}(-|\cdot|^2)\in L^r(\mathbf{R}^n)$ for any $r>n/2$, we can show

$$\|\tilde{E}_{1+\alpha}(\cdot,t)*\varphi\|_p \le Ct^{-\frac{n(1+\alpha)}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\varphi\|_r$$

for any $p>r>n/2$, where C is a constant independent of t and φ. By virtue of these estimates, we can obtain floowing key lemma.

Lemma 3. Let (p,q,r) be any admissible triplet such that $r>n/2$ and $\varphi\in L^r(\mathbf{R}^n)$, then

$$\tilde{E}_{1+\alpha}*\varphi \in C_b(I;L^r)\cap L^q(I;L^p),$$

and

$$\|\tilde{E}_{1+\alpha}*\varphi\|_{L^q(I;L^p)} \le C\|\varphi\|_r,$$

where $I=[0,T)$ or $I=[0,\infty)$ and C is a constant independent of I.

For managing Duhamel term, we use following lemma, obtained from above estimates.

Lemma 4. Let (p,q,r) be any admissible triplet and σ satisfy

$$\max(n(1+\sigma),n(1+\alpha)\sigma)<2p \quad\text{and}\quad n\sigma<2r.$$

Then, for any $f\in L^{\frac{q}{1+\sigma}}(0,T;L^{\frac{p}{1+\sigma}})$, $Gf\in L^q(0,T;L^p)$ and

$$\|Gf\|_{L^q(0,T;L^p)} \le CT^{(1+\alpha)(1-\frac{n\sigma}{2r})}\|f\|_{L^{\frac{q}{1+\sigma}}\left(0,T;L^{\frac{p}{1+\sigma}}\right)},$$

where C is a constant independent of f and T.

By virue of these two lemmas, we can construct a unique time local solution of (1) in the function spaces

$$X_{p,q}(0,T) = \{u \in C((0,T];L^p) \mid \|u\|_{X_{p,q}(0,T)} = \sup_{0<t\leq T} t^{\frac{1}{q}}\|u(t)\|_p < \infty\},$$

and

$$Y_{p,q}(0,T) = L^q(0,T;L^p).$$

By the standard continuation argument for time local solution, we obtain (a), (b), (d) and (e). And by applicating Giga's idea ([3]), we also obtain (c).

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