The lifespan of smooth solutions to the systems of quasilinear wave equations with different propagation speeds

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We are concerned with the Cauchy problem for systems of quasilinear wave equations of the form

$$\Box_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = \sum_{l=1}^{m} \sum_{\alpha,\beta=0}^{2} a_{il}^{\alpha\beta}(\partial u)\partial_\alpha \partial_\beta u^l \quad \text{in } \mathbf{R}^2 \times (0,\infty), \tag{1}$$

$$u^i(x,0) = \varepsilon f^i(x), \quad \partial_t u^i(x,0) = \varepsilon g^i(x) \qquad \text{in } \mathbf{R}^2, \tag{2}$$

where $i = 1, \cdots, m$, $u = {}^t(u^1, \cdots, u^m)$ and $\varepsilon > 0$ is a small parameter. We also denote $\partial = (\partial_0, \partial_1, \partial_2)$ with $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ $(i = 1, 2)$. We assume that f^i, $g^i \in C_0^\infty(\mathbf{R}^2)$, $a_{il}^{\alpha\beta} \in C^\infty(\mathbf{R}^{3m})$,

$$a_{il}^{\alpha\beta}(\partial u) = a_{li}^{\alpha\beta}(\partial u) = a_{li}^{\beta\alpha}(\partial u)$$

and

$$a_{il}^{\alpha\beta}(\partial u) = O(|\partial u|^p) \quad \text{near} \quad \partial u = 0$$

hold for any $i, l (= 1, \cdots, m)$ and $\alpha, \beta (= 0, 1, 2)$. Here p is a positive integer. Furthermore, we assume that the system (1) is strictly hyperbolic, namely,

$$0 < c_1 < c_2 < \cdots < c_m.$$

The main topic in this talk is the existence and nonexistence of smooth solutions of the Cauchy problem (1) and (2). Our aim is to obtain the optimal estimate of the lifespan T_ε which is defined by

$$T_\varepsilon \equiv \sup\{\ T > 0 \ : \ \text{there exists a solution of the Cauchy problem (1) and (2) in } C^\infty(\mathbf{R}^2 \times [0,T))\ \}.$$

In [4], R. Agemi and K. Yokoyama proved that if $p \geq 3$, then $T_\varepsilon = \infty$ for sufficiently small $\varepsilon > 0$. Moreover, when $p = 2$, I proved that there exists an almost global solution of the Cauchy problem (1) and (2), in [6]. Thus we restrict ourselves in the case $p = 1$. In this case, we can write

$$a_{il}^{\alpha\beta}(\partial u) = \sum_{j=1}^{m} \sum_{\gamma=0}^{2} Z_{ijl}^{\alpha\beta\gamma} \partial_\gamma u^j + O(|\partial u|^2) \quad \text{near} \quad \partial u = 0,$$

where each $Z_{ijl}^{\alpha\beta\gamma}$ is a constant.

To state our results, we need some notations.

Definition 1. For any $i = 1, \cdots, m$, we define

$$\Phi_i(X) = \sum_{\alpha,\beta,\gamma=0}^{2} Z_{iii}^{\alpha\beta\gamma} X_\alpha X_\beta X_\gamma$$

for $X = (X_0, X_1, X_2)$.

Definition 2. Fix $i = 1, \cdots, m$. Let U^i be the solution of the Cauchy probrem

$$\begin{aligned} &\Box_i U^i = 0 \qquad \text{in } \mathbf{R}^2 \times (0, \infty), \\ &U^i(x,0) = f^i(x), \quad \partial_t U^i(x,0) = g^i(x) \quad \text{in } \mathbf{R}^2. \end{aligned}$$

Then we define the Friedlander radiation field $\mathcal{F}_i$ by

$$\mathcal{F}^i(\rho,\omega) = \lim_{\substack{r\to\infty \\ r-c_i t=\rho}} r^{\frac{1}{2}} U^i(x,t) \quad \text{with} \quad x = r\omega \ (\omega \in S^1), \ \rho = r - c_i t.$$

$\mathcal{F}_i$ is expressed by

$$\mathcal{F}_i(\rho,\omega) = \frac{1}{2\sqrt{2}\pi} \int_\rho^\infty (s-\rho)^{-\frac{1}{2}} \{R_{g^i}(\omega,s) - \partial_s R_{f^i}(\omega,s)\} ds,$$

where R_h is the Radon transform of $h \in C_0^\infty(\mathbf{R}^2)$, *i.e.*,

$$R_h(\omega,s) = \int_{\omega\cdot y = s} h(y) dS_y.$$

Definition 3. For any $i = 1, \cdots, m$, we define a nonnegative constant H_i by

$$H_i = \max_{\rho\in\mathbf{R},\ \omega\in S^1} \left\{ -c_i^{-\frac{3}{2}} \Phi_i(-c_i,\omega) \partial_\rho^2 \mathcal{F}_i(\rho,\omega) \right\}.$$

Also we define

$$H = \max\{H_1, \cdots, H_m\}.$$

Then we have the following.

Theorem 1. If $H > 0$, then we have

$$\liminf_{\varepsilon\to 0} \varepsilon \sqrt{T_\varepsilon} \geq \frac{1}{H}.$$

The above estimate of T_ε is optimal in the following sense.

Let us consider the following Cauchy problem

$$\partial_t^2 u^i - \kappa_i^2(\partial_t u)\Delta u^i = 0 \qquad \text{in} \quad \mathbf{R}^2 \times (0,\infty), \tag{3}$$

$$u^i(x,0) = \varepsilon\phi^i(|x|), \quad \partial_t u^i(x,0) = \varepsilon\psi^i(|x|) \quad \text{in} \quad \mathbf{R}^2, \tag{4}$$

where

$$\kappa_i^2(\partial_t u) = c_i^2 + \sum_{j=1}^m a_{ij}\partial_t u^j + O(|\partial_t u|^2) \quad \text{near} \quad \partial_t u = 0$$

and $\phi^i(|x|), \psi^i(|x|) \in C_0^\infty(\mathbf{R}^2)$. Then the Cauchy problem (3) and (4) is the special case of the Cauchy problem (1) and (2).

Definition 4. For any $i = 1, \cdots, m$, we define

$$\hat{H}_i = \max_{\rho\in\mathbf{R}} \left\{ c_i^{-\frac{1}{2}} a_{ii} \mathcal{F}_i''(\rho) \right\}$$

and also define

$$\hat{H} = \max\{\hat{H}_1, \cdots, \hat{H}_m\}.$$

Then we obtain the following.

Theorem 2. If $\hat{H} > 0$, then we have

$$\limsup_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} \leq \frac{1}{\hat{H}}.$$

By the definition of H and $\hat{H}$, we can easily see that $H = \hat{H}$. Therefore we find that

$$\lim_{\varepsilon \to 0} \varepsilon \sqrt{T_\varepsilon} = \frac{1}{\hat{H}}.$$

When $m = 1$, the similar results have already been obtaind by S. Alinhac [2] and A. Hoshiga [6]. However, since we are concerned with the system of wave equations, the problem is more complex. In this talk, I will explain what is the difficulty of deriving the nonexistence theorem for the system.

References

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