Decay estimates of solutions for the dissipative wave equations

RYO IKEHATA

1. Introduction. In this talk, I would like to introduce the recent results concerning the following linear and semilinear wave equations with linear dissipative terms:

$$u_{tt} - \Delta u + \delta u_t = 0, \quad (t,x) \in (0,+\infty) \times \Omega, \tag{1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \quad x \in \Omega, \tag{2}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,+\infty), \tag{3}$$

and

$$u_{tt} - \Delta u + \delta u_t = |u|^{p-1}u, \quad (t,x) \in (0,+\infty) \times \Omega, \tag{4}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \quad x \in \Omega, \tag{5}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,+\infty). \tag{6}$$

Here, $\Omega \subset R^N (N \geq 2)$ is a smooth exterior domain with compact boundary $\partial\Omega$, $\delta > 0$ is a constant and $p > 1$.

First, let us define some notations used throughout this lecture. The total energy $E(t)$ to the equation (1) or (4) is defined as follows.

$$E(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla u(t,\cdot)\|^2,$$

where in this case, $\|\cdot\|$ means the usual $L^2(\Omega)$-norm. Furthermore, we set $(f,g) = \int_\Omega f(x)g(x)dx$.

In this lecture, first we will introduce the following results.

Theorem 1 *For each* $[u_0,u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ *satisfying* $\|d(\cdot)(u_1 + \delta u_0)\| < +\infty$, *there exists a unique solution* $u \in C([0,+\infty);H_0^1(\Omega)) \cap C^1([0,+\infty);L^2(\Omega))$ *to the linear problem (1)-(3) such that*

$$E(t) \leq CI_0^2(1+t)^{-2},$$

and further,

$$\|u(t,\cdot)\|^2 \leq CI_0^2(1+t)^{-1},$$

where we set

$$I_0 = \|u_0\|_{H^1} + \|u_1\| + \|d(\cdot)(u_1 + \delta u_0)\|,$$

$$d(x) = |x|, \quad N \geq 3,$$
$$d(x) = |x| \log(B|x|), \quad N = 2,$$

with some constant $B > 0$.

Next, by relying on the linear estimate above we have the fast decay estimates of the solutions to the semilinear problem (4)-(6). Our assumptions are as follows:

(A.1) $1 + \dfrac{6}{N+2} < p \leq \dfrac{N}{N-2}$, if $N = 3$,

(A.2) $1 + \dfrac{6}{N+2} < p < +\infty$, if $N = 2$,

(A.3) $suppu_0 \cup suppu_1 \subset \Omega \cap B_R(0)$,

with $B_R(0) = \{x \in R^N | \, |x| < R\}$.

Then, our main second result reads as follows.

Theorem 2 *Assume* (A.1) *and* (A.3) *or* (A.2) *and* (A.3), *and let* $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$. *Then there exists a constant* $\epsilon_0 > 0$ *such that if* $I_0 < \epsilon_0$, *the problem (4)-(6) has a unique solution* $u \in C([0, +\infty); H_0^1(\Omega)) \cap C^1([0, +\infty); L^2(\Omega))$ *satisfying*

$$\|u(t, \cdot)\|^2 \leq C I_0^2 (1+t)^{-1},$$
$$E(t) \leq C I_0^2 (1+t)^{-2}.$$

with some constant $C > 0$.

We will also introduce the various recent results concerning the other types of wave equations.

References

[1] W. Dan and Y. Shibata, *On a local energy decay of solutions of a dissipative wave equation*, Funkcial. Ekvac. **38**, 545-568 (1995).

[2] R. Ikehata and T. Matsuyama, *L^2-behaviour of solutions for the linear heat and wave equations in exterior domains*, submitted (1999).

[3] R. Ikehata, *Energy decay of solutions for the semilinear dissipative wave equations in an exterior domain*, submitted (2000).

[4] R. Ikehata, *Decay estimates of solutions for the wave equations with strong damping terms in unbounded domains*, to appear in Math. Meth. Appl. Sci..

[5] R. Ikehata, *Energy decay of solutions for the linear wave equations with localized dissipation in an exterior domain*, preprint (2000).

[6] R. Ikehata, T. Kobayashi and T. Matsuyama, *Remark on the L_2 estimates of the density for the compressible Navier-Stokes flow in R^3*, submitted (2000).

[7] S. Kawashima, M. Nakao and K. Ono, *On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term*, J. Math. Soc. Japan **47**, 617-653 (1995).
[8] K. Mochizuki and H. Nakazawa, *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. RIMS, Kyoto Univ. **32**, 401-414 (1996).
[9] C. Morawetz, *The limitting amplitude principle*, Comm. Pure Appl. Math. **15**, 349-361 (1962).
[10] M. Nakao and K. Ono, *Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations*, Math. Z. **214**, 325-342 (1993).
[11] M. Nakao, *Existence of global decaying solutions in exterior domains for the semilinear wave equations with some localized dissipations*, to appear in Math. Z..
[12] A. Saeki and R. Ikehata, Remarks on the decay rate for the energy of the dissipative linear wave equations in exterior domains, submitted (2000).
[13] Y. Shibata, *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain*, Tsukuba J. Math. **7**, 1-68 (1983).

Department of Mathematics, Faculty of Education, Hiroshima University, Higashi-Hiroshima 739-8524, Japan

E-mail address: ikehatar@hiroshima-u.ac.jp