ON A FRAMEWORK OF SCATTERING FOR DISSIPATIVE SYSTEMS

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1. Assumption and Result.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$.

$V(t) = e^{-itA}$ is a contraction semi-group in $\mathcal{H}$, where $t \geqq 0$.

$U_0(t) = e^{-itA_0}$ is a unitary group in $\mathcal{H}$, where $t \in \mathbf{R}$.

In this talk we assume that

(A1) $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbf{R}$ or $[0,\infty)$

(A2) $(A-i)^{-1} - (A_0-i)^{-1}$ defined as a form is extended to a compact operator K in $\mathcal{H}$.

(A3) There exist non-zero projection operators P_+ and P_- such that $P_+ + P_- = I_d$ and

(A3.1) $$\|KU_0(t)\psi(A_0)P_+\| \in L^1(\mathbf{R}_+),$$

(A3.2) $$\|K^*U_0(t)\psi(A_0)P_+\| \in L^1(\mathbf{R}_+),$$

(A3.3) $$\|K^*U_0(-t)\psi(A_0)P_-\| \in L^1(\mathbf{R}_+),$$

(A3.4) $$\mathrm{w}-\lim_{t\to+\infty} U_0(-t)\psi(A_0)P_- f_t = 0$$

for each $\psi \in C_0^\infty(\mathbf{R}\backslash 0)$ and $\{f_t\}_{t\in\mathbf{R}}$ satisfying $\sup_{t\in\mathbf{R}} \|f_t\|_{\mathcal{H}} < \infty$, where $\|\cdot\|$ is the operator norm of bounded operator from $\mathcal{H}$ to $\mathcal{H}$.

Let $\mathcal{H}_b$ be the space generated by the eignvectors of A with real eigenvalues.

We use the following fact (cf. Petkov[11]):

(F1) $\{(A-i)^{-2}Af : f \in D(A) \cap \mathcal{H}_b^\perp\}$ is dense in $\mathcal{H}_b^\perp$.

(F2) There exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and for any $f \in \mathcal{H}_b^\perp$
$$\mathrm{w}-\lim_{n\to\infty} V(t_n)f = 0.$$

Main result is

Theorem. *For any $f \in \mathcal{H}_b^\perp$, the wave opeartor*
$$Wf = \lim_{t\to\infty} U_0(-t)V(t)f$$
exists. Moreover W is not zero operator in $\mathcal{H}$.

Corollary. *Assume that (A1) $\sim$ (A3). Then there exist non-trivial initial datas $f \in \mathcal{H}_b^{\perp}$ and $f_+ \in \mathcal{H}$ such that*

$$\lim_{t\to\infty} \|V(t)f - U_0(t)f_+\|_{\mathcal{H}} = 0$$

and

$$\lim_{t\to\infty} \|V(t)f\|_{\mathcal{H}} \neq 0$$

.

In order to prove Theorem, we refer to Enss [2], Simon [12], Perry [10], Isozaki-Kitada [4] and Stefanov-Georgiev[13].

Related works (Scattering).

(1) Lax-Phillips [7](1973)
Wave equation with dissipative boundarly conditions in an exterior domain.

(2) Mochizuki [9](1976)
Wave equation with dissipative terms

(3) Simon [12](1979)
Schödinger equation with complex value potential

(4) Stefanov-Georgiev [13](1988)
Maxwell equation with a dissipative boundarly condition in an exterior domain.

Related works (Decay).

(1) Majda [8](1975)
Wave equation with dissipative term and boundarly conditions in an exterior domain.(Non existence of Disappering solution (?))

(2) Georgiev [3](1986)
Maxwell equation with a dissipative boundarly condition in an exterior domain.(Existence and Non existence of Disappering solution)

2. Example (Elastic wave with dissipative boundary condition in a half space of $\mathbf{R}^3_+$).

Let $x = (x_1, x_2, x_3) = (y, x_3) \in \mathbf{R}^2 \times \mathbf{R}_+$ and $\mu_0 > 0, \rho_0 > 0, \lambda_0 \in \mathbf{R}$ satistying $3\lambda_0 + 2\mu_0 > 0$.

We assume that $B(y)$ belongs to $L^\infty(\mathbf{R}^3_+, M_{3\times 3})$ and satisfies

$$O_{3\times 3} \leqq B(y) \leqq C\varphi(|y|)I_{3\times 3},$$

where $\varphi(r)$ is a non-increasing function and belongs to $L^1(\mathbf{R}_+)$. $M_{3\times 3}$ is the class of 3×3 matrix.

We set

$$\varepsilon_{i,j}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

and

$$\sigma^0_{i,j}(u(x)) = \lambda_0(\nabla\cdot u)\delta_{i,j} + 2\mu_0\varepsilon_{i,j}(u)$$

where $i, j = 1, 2, 3$ and $u(x) = {}^t(u_1(x), u_2(x), u_3(x))$.

We define an operator

$$(\tilde{L_0}u)_i = -\sum_{j=1}^{3} \frac{1}{\rho_0} \frac{\partial \sigma_{i,j}(u(x))}{\partial x_j} \quad (i = 1, 2, 3),$$

respectively. We consider two elastic wave equations as follows:

$$(2.1) \qquad \begin{cases} u_{tt}(x,t) + \tilde{L_0}u(x,t) = 0, (x,t) \in \mathbf{R}^3_+ \times [0, \infty), \\ {}^t(\sigma_{13}(u), \sigma_{23}(u), \sigma_{33}(u)) \mid_{x_3=0} = B(y)u_t \mid_{x_3=0} \end{cases}$$

and

$$(2.2) \qquad \begin{cases} u_{tt}(x,t) + \tilde{L_0}u(x,t) = 0, (x,t) \in \mathbf{R}^3_+ \times \mathbf{R}, \\ \sigma_{i3}(u) \mid_{x_3=0} = 0 (i = 1, 2, 3) \end{cases}$$

The following operator L_0 in $\mathcal{G}_0 = L^2(\mathbf{R}^3_+, \mathbf{C}^3; \rho_0 dx)$:

$$L_0 u = \tilde{L_0} u$$

and

$$D(L_0) = \{u \in H^1(\mathbf{R}^3_+, \mathbf{C}^3); L_0 u \in \mathcal{G}_0, \sigma_{i3}(u) \mid_{x_3=0} = 0 (i = 1, 2, 3)\}.$$

is a positive self-adjoint operator. Let $\mathcal{H}$ be Hilbert space with inner product :

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbf{R}^3_+} \Big(\sum_{i,j,k,h=1}^{3} a^0_{ijkh} \varepsilon_{k,h}(f_1) \overline{\varepsilon_{i,j}(g_1)} + f_2 \overline{g_2} \rho_0 \Big) dx,$$

respectively, where $a^0_{ijkh} = \lambda_0 \delta_{ij}\delta_{kh} + \mu_0(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})$ and $f = {}^t(f_1, f_2), g = {}^t(g_1, g_2)$. By Korn's inequality (cf. Ito [5]) we note that $\mathcal{H}$ is equivalent to $\dot{H}^1(\mathbf{R}^3_+) \times L^2(\mathbf{R}^3_+)$ as Banach space.

We set $f = (u(x,t), u_t(x,t))$, where $u(x,t)$ is the solution to (2.1) (resp. (2.2)) with a initial data $(u(x,0), u_t(x,0)) = (f_1, f_2) \in \mathcal{H}$. Then (2.1) (resp. (2.2)) can be written as

$$\partial_t f = -iAf \qquad (\text{resp.} \quad \partial_t f = -iA_0 f),$$

where

$$A = i \begin{pmatrix} 0 & I_{3\times3} \\ -\tilde{L_0} & 0 \end{pmatrix}, \qquad A_0 = i \begin{pmatrix} 0 & I_{3\times3} \\ -\tilde{L_0} & 0 \end{pmatrix},$$

$$D(A) = \{f \in \mathcal{H}; \tilde{L_0} f_1 \in L^2(\mathbf{R}^3_+; \mathbf{C}^3), f_2 \in H^1(\mathbf{R}^3_+; \mathbf{C}^3), \\ {}^t(\sigma_{13}(f_1), \sigma_{23}(f_1), \sigma_{33}(f_1)) \mid_{x_3=0} = B(y) f_2 \mid_{x_3=0}\}$$

and

$$\begin{aligned} D(A_0) = \{f \in \mathcal{H}_0; \tilde{L_0} f_1 \in L^2(\mathbf{R}^3_+;\mathbf{C}^3), f_2 \in H^1(\mathbf{R}^3_+;\mathbf{C}^3), \\ \sigma_{i3}(f_1)\mid_{x_3=0} = 0 (i=1,2,3)\} \end{aligned}$$

A (resp. A_0) generates a contraction semi-group $\{V(t)\}_{t \geqq 0}$ (resp. a unitary group $\{U_0(t)\}_{t\in\mathbf{R}}$) in $\mathcal{H}$. Using $\{V(t)\}_{t \geqq 0}$ (resp. $\{U_0(t)\}_{t\in\mathbf{R}}$) we solve $\partial_t f = -iAf$ (resp. $\partial_t f = -iA_0 f$) as follows :

$$f = V(t) f_0 \quad (\text{resp.} f = U_0(t) f_0),$$

where $f_0 = (f_1, f_2) \in \mathcal{H}$.

Making a check on Assumption (A1),(A2) and (A3).

We state a result which follows from Dermenjian - Guillot[1].

Let $k = (p, p_3) \in \mathbf{R}^2 \times \mathbf{R}_+ = \mathbf{R}^3_+$ be the deal variable of x. By the polar coordiates we write k and p as

$$k = |k|\omega = |k|(\overline{\omega}, \omega_3) = |k|(\omega_1, \omega_2, \omega_3)$$

and

$$p = |p|\nu = |p|(\nu_1, \nu_2).$$

There exist partially isometric operators F_P, F_S, F_{SH} and F_R, from $\mathcal{G}_0$ onto $L^2(\mathbf{R}^3_+;\mathbf{C}^3)$ and $L^2(\mathbf{R}^2;\mathbf{C}^3)$, respectively. We define the operator F as follows :

$$Fu = (F_P u, F_S u, F_{SH} u, F_R u) \quad \text{for} \quad u \in \mathcal{G}_0.$$

Then we have by Theorem 3.6 of [1]

Lemma D-G. *F is unitary operator from $\mathcal{G}_0$ to*

$$\hat{\mathcal{H}} = \bigoplus_{j=1}^{3} L^2(\mathbf{R}^3_+;\mathbf{C}^3) \bigoplus L^2(\mathbf{R}^2;\mathbf{C}^3)$$

and for every $u \in D(L_0)$

$$F L_0 u = (c_P^2 |k|^2 F_P u, c_S^2 |k|^2 F_S u, c_S^2 |k|^2 F_{SH} u, c_R^2 |p|^2 F_R u),$$

where

$$c_P^2 = \frac{\lambda_0 + 2\mu_0}{\rho_0}, \quad c_S^2 = \frac{\mu_0}{\rho_0}$$

and c_R^2 is the unique solution in $(0, \frac{\mu_0}{\rho_0})$ of the following implicit equation :

$$(1 - \frac{\alpha^2 \rho_0}{2\mu_0})^{\frac{1}{2}} - (1 - \frac{\alpha^2 \rho_0}{\mu_0})^{\frac{1}{2}} (\frac{\alpha^2 \rho_0}{\lambda_0 + 2\mu_0})^{\frac{1}{2}} = 0.$$

Lemma D-G implies $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbf{R}$ (see [1]). Therefore we have (A1).

Next we show (A2). Note that

$$\begin{aligned}&\langle((A-i)^{-1} - (A_0-i)^{-1})f, g\rangle_{\mathcal{H}} \\ =&\langle (A-i)^{-1}f, A_0(A_0+i)^{-1}g\rangle_{\mathcal{H}} - \langle A(A-i)^{-1}f, (A_0+i)^{-1}g\rangle_{\mathcal{H}}\end{aligned}$$

for $f, g \in \mathcal{H}$. By easy caluculation we have

$$\begin{aligned}&\langle((A-i)^{-1} - (A_0-i)^{-1})f, g\rangle_{\mathcal{H}} \\ =&i\int_{\mathbf{R}^2} \Gamma_0((A-i)^{-1}f)_2 \overline{B(y)\Gamma_0((A_0-i)^{-1}g)_2}dy,\end{aligned}$$

where Γ_0 is a tarce operator which is defined by

$$(\Gamma_0 u)(y) = u(y, 0).$$

For $s > 1/2$, since $\Gamma_0((A_0+i)^{-1}g)_2 \in H^{2-s}(\mathbf{R}^2)$,

$$B(y)\Gamma_0\Pi_2(A_0+i)^{-1}$$

is a compact operator from $\mathcal{H}$ to $L^2(\mathbf{R}^2;\mathbf{C}^3)$. Moreover, noting the domain of A, we have that

$$\Gamma_0((A-i)^{-1}f)_2 \in L^2(\mathbf{R}^2;\mathbf{C}^3).$$

Therefore the form $(A-i)^{-1} - (A_0-i)^{-1}$ can be extended to a compact operator in $\mathcal{H}$.

Finally we show (A3). Using $F_j (j = P, S, SH, R)$, we construst $P_\pm$ as follows : for $f \in \mathcal{H}$,

$$P_\pm f = T^{-1}\{\sum_{j=P,S,SH} F_j^* \begin{pmatrix} P_\mp^{(3)} I_{3\times3} & O_{3\times3} \\ O_{3\times3} & P_\pm^{(3)} I_{3\times3}\end{pmatrix} F_j + F_R^* \begin{pmatrix} P_\mp^{(2)} I_{3\times3} & O_{3\times3} \\ O_{3\times3} & P_\pm^{(2)} I_{3\times3}\end{pmatrix} F_R\}T \tag{2.3}$$

where

$$T = \begin{pmatrix} L_0^{\frac{1}{2}} & iI_{3\times3} \\ L_0^{\frac{1}{2}} & -iI_{3\times3}\end{pmatrix}$$

and $P_\mp^{(3)}$ and $P_\mp^{(2)}$ are negative(positive) projection of

$$D^{(3)} = \frac{1}{2i}(k\cdot\nabla_k + \nabla_k\cdot k) \quad \text{and} \quad D^{(2)} = \frac{1}{2i}(p\cdot\nabla_p + \nabla_p\cdot p), \quad \text{respectively.}$$

Proposition 2.1. *$P_\pm$ as in (2.3) satisfy (A3).*

To show Proposition 2.1 we prepare

Lemma 2.2. *Let $\psi(\lambda)$ be same as in (A3) and $0 < \delta < c_R$. Then for any positive integer N and $t \in \mathbf{R}_\pm$, there exists a positive constant $C_{N,\psi}$ which is independent of t such that*

$$\|\Gamma_0(e^{-itA_0}\psi(A_0)P_\pm f)_2\|_{L^2(\mathbf{R}^2;\mathbf{C}^3)}^{|y|\leqq\delta|t|} \leqq C_{N,\psi}(1+|t|)^{-N}\|f\|_{\mathcal{H}_0}$$

for any $f \in \mathcal{H}$.

This lemma is the key lemma to show (A3). Using the representation of the generalized eigenfunction of L_0 and the Mellin transformation we show Lemma 2.2 (cf. Perry[10] and Kadowaki[6]). The Mellin transformations for $D^{(3)}, D^{(2)}$ are given as

$$(M^{(3)}u)(\lambda,\omega) = (2\pi)^{-1/2}\int_0^{+\infty} r^{1/2-i\lambda}u(r\omega)dr$$

and

$$(M^{(2)}u)(\lambda,\nu) = (2\pi)^{-1/2}\int_0^{+\infty} r^{-i\lambda}u(r\nu)dr$$

Then $M^{(3)}$ (resp. $M^{(2)}$) is a unitary operator from $L^2(\mathbf{R}^3_+)$ (resp. $L^2(\mathbf{R}^2)$) to $L^2(\mathbf{R}\times S^2_+)$(resp. $L^2(\mathbf{R}\times S^1)$.

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