RESOLVENT ESTIMATES AND SCATTERING STATES FOR DISSIPATIVE SYSTEMS

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1. Introduction.

Mochizuki[7] has studied the scattering problem for wave equations with dissipative terms. In this talk we shall study the similar problem for wave equations with dissipative terms in stratifed media.

To explain our wave equations, we state acoustic operators in stratified media.

Let $n \geqq 1$ and $(x, y) \in \mathbf{R}^n \times \mathbf{R}$. We set

$$c_0(y) = \begin{cases} c_+ & (y \geqq h) \\ c_h & (0 < y < h) \\ c_- & (y \leqq 0), \end{cases}$$

for some positive constants h and c_+, c_-, c_h.

Acoustic operators are

$$L_0 = -c_0(y)^2 \triangle,$$

where

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y^2}.$$

Considering the case $c_h < \min(c_+, c_-)$ we find the guided waves (cf. Wilcox [13] or Weder [12]). But we do not restrict ourselves to such case.

L_0 is a non-negative self-adjoint operator in $\mathcal{G} = L^2(\mathbf{R}^{n+1}; c_0(y)^{-2} dxdy)$. $D(L_0)$ is given by $H^2(\mathbf{R}^{n+1})$, $H^s(\mathbf{R}^{n+1})$ being Sobolev space of order s over $\mathbf{R}^{n+1}$.

We deal with the following dissipative wave equations :

$$\partial_t^2 u(x, y, t) + b(x, y)\partial_t u(x, y, t) + L_0 u(x, y, t) = 0 \tag{1.1}$$

and

$$\partial_t^2 u(x, y, t) + \langle \partial_t u, \varphi \rangle_{\mathcal{G}} \varphi(x, y) + L_0 u(x, y, t) = 0, \tag{1.2}$$

where $(x, y, t) \in \mathbf{R}^n \times \mathbf{R} \times [0, \infty)$ and $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ is the inner-product of $\mathcal{G}$.

We assume that $b(x, y)$ and $\varphi(x, y)$ are measurable functions which satisfy

$$0 \leqq b(x, y) \leqq C(1 + |x|^2 + y^2)^{-\frac{\theta}{2}}$$

and

$$\varphi(x, y) \in L^2(\mathbf{R}^{n+1}; (1 + |x|^2 + y^2)^{\frac{\theta}{2}} dxdy)$$

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for some $\theta > 1$ and $C > 0$.

We shall show the existence of the scattering states for (1.1) and (1.2) which are considered as the pertubed systems of

$$\partial_t^2 u(x,y,t) + L_0 u(x,y,t) = 0, \quad (x,y,t) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \tag{1.3}$$

[7] has already dealt with the case $c_h = c_+ = c_- = 1$ and $n \geqq 2$ of (1.1). His proof has been baesd on Kato's smooth pertubation theory [5] and global resolvent estimates for L_0(see also Kuroda[6] Theorem 4.4.1) : there exists positive constants C and η such that

$$\sup_{0<|\mathrm{Im}\zeta|<\eta} |\zeta| \| X_{\frac{\theta}{2}} (L_0 - \zeta^2) X_{\frac{\theta}{2}} \|_{L^2(\mathbf{R}^{n+1}) \to L^2(\mathbf{R}^{n+1})} \leqq C, \tag{1.4}$$

where $\zeta \in \mathbf{C}, \gamma \in \mathbf{R}, X_\gamma = (1+|x|^2+y^2)^{-\frac{\gamma}{2}}$ and $\| \cdot \|_{L^2(\mathbf{R}^{n+1}) \to L^2(\mathbf{R}^{n+1})}$ is the norm of the bounded operator in $L^2(\mathbf{R}^{n+1})$.

Kadowaki[3] has established (1.4) for the case $c_h < c_+ = c_-$. The condition $c_+ = c_-$ has needed to get the estimate of the high energy part ($|\zeta| \to \infty$). It seems that this condition is a technical one.

In Weder[12] and Zhang[14], we can find local resolvent estimates as follows : for any $\beta > \alpha > 0$, there exists positive constants $C_{\alpha,\beta}$ and η such that

$$\sup_{\alpha \leqq |\mathrm{Re}\zeta| \leqq \beta, 0<|\mathrm{Im}\zeta|<\eta} \| X_{\frac{\theta}{2}} (L_0 - \zeta^2)^{-1} X_{\frac{\theta}{2}} \|_{L^2(\mathbf{R}^{n+1}) \to L^2(\mathbf{R}^{n+1})} \leqq C_{\alpha,\beta}. \tag{1.5}$$

We shall give an abstract framework which is used the idea of [13]. Applying our framework to (1.1) and (1.2) we show the existence of the scattering states for these equations. Then (1.4) and (1.5) will be used.

Below we state our framework.

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. This norm is denoted by $\| \cdot \|_{\mathcal{H}}$. Let A_0 and B be self-adjoint operators in $\mathcal{H}$. We suppose that B is non-negative bounded operator.

First we consider the following assumptions :

(A1) $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbf{R}$ or $[0, \infty)$.

(A2) $B(A_0 - i)^{-1}$ is a compact operator in $\mathcal{H}$.

(A3) For any $\beta > \alpha > 0$, there exist positive constants $C_{\alpha,\beta}$ and η such that

$$\sup_{\alpha \leqq |Re\zeta| \leqq \beta, 0<|\mathrm{Im}\zeta|<\eta} \| \sqrt{B} (A_0 - \zeta)^{-1} \sqrt{B} \| \leqq C_{\alpha,\beta}, \tag{1.6}$$

or

$$\sup_{\alpha \leqq |Re\zeta| \leqq \beta, 0<|\mathrm{Im}\zeta|<\eta} |\mathrm{Im}\zeta| \| \sqrt{B} (A_0 - \zeta)^{-1} \|^2 \leqq C_{\alpha,\beta}, \tag{1.7}$$

where $\zeta \in \mathbf{C}$ and $\| \cdot \|$ is the norm of the bounded operator from $\mathcal{H}$ to $\mathcal{H}$.

(1.6) implies (1.7)(cf. [5])

We set $A = A_0 - iB$, $D(A) = D(A_0)$. Then Reed-Simon [10] (see Theorem X-50) implies that A generates a contraction semi-group ,$\{V(t)\}_{t \geqq 0} (V(t) = e^{-itA})$. A unitary group generated by A_0 is denoted by $\{U_0(t)\}_{t \in \mathbf{R}} (U_0(t) = e^{-itA_0})$.

Abstract result is

Theorem 1. *Assume that (A1) ∼ (A3). Then*

(1) *A has no real eigenvalues.*

(2) *The wave operator*

$$W = s - \lim_{t\to\infty} U_0(-t)V(t)$$

exists. Moreover W is not zero as an operator in $\mathcal{H}$.

We shall use (1.5) to get (1.6) or (1.7)(see §3).

Theorem 1(2) implies that there exists scattering states of $\frac{dV(t)g}{dt} = -iAV(t)g$, $g \in D(A)$ as follows:

Corollary 2. *Assume that (A1) ∼ (A3). Then there exist non-trivial initial data* $f \in \mathcal{H}$ *and* $f_+ \in \mathcal{H}$ *such that*

$$\lim_{t\to\infty} \|V(t)(A-i)^{-1}f - U_0(t)(A_0-i)^{-1}f_+\|_{\mathcal{H}} = 0.$$

Abstractly, Mochizuki's result[7] can be explained as follows. If we suppose that

(A4) There exist positive constants C and η such that

$$\sup_{0<|\mathrm{Im}\zeta|<\eta} \|\sqrt{B}(A_0-\zeta)^{-1}\sqrt{B}\| \leqq C \tag{1.8}$$

or

$$\sup_{0<|\mathrm{Im}\zeta|<\eta} |Im\zeta|\|\sqrt{B}(A_0-\zeta)^{-1}\|^2 \leqq C \tag{1.9}$$

instead of (A1), (A2) and (A3), we have the same conclusion of Theorem 1.

Previous framework [4] using the idea of Simon[21] and Perry[8] has not been applied to acoustic wave equations with dissipative terms in stratifed media. Since generalized eigenfunctions of acoustic wave propagation in stratifed media are not smooth at thresholds(cf. Wilcox [13] or Weder [12]), the key estimates of [4] has not been obtained in the neighborhood of each threshold (cf. Kadowaki[2]).

Moreover we have the following theorem.

Theorem 3. *Assume that (A4) and B is a compact operator in* $\mathcal{H}$. *Then one has the same conclusions of Theorem 1, Corollary 2 and*

$$dim\ KerW < \infty \quad or \quad KerW = \{0\}.$$

Note that

$$f \in \mathrm{Ker}W \Longleftrightarrow \lim_{t\to\infty} \|V(t)f\|_{\mathcal{H}} = 0.$$

Therefore the last assertion of Theorem 3 means that energy decay states are not much. It seems that there is little literature which study dissipative systems with energy scattering and decay states, for instance, Georgiev [1] and Stefanov-Georgiev[11]. They have dealt with Maxwell equations with dissipative boundary conditions in exterior domains. In §3 we shall given an example of (1.2) which has energy decay and scattering states.

2. Applications.

In order to apply our framework to (1.3) and (1.1), we set $f(t) = {}^t(u(t), \partial_t u(t))$. Then (1.3) and (1.1) can be written as $\partial_t f = -iA_0 f$ and $\partial_t f = -iAf$ respectively, where

$$A_0 = i\begin{pmatrix} 0 & 1 \\ -L_0 & 0 \end{pmatrix}, \qquad A = i\begin{pmatrix} 0 & 1 \\ -L_0 & -b(x,y) \end{pmatrix}.$$

Let $\mathcal{H}$ be Hilbert spaces with inner product

$$\langle f, g\rangle_{\mathcal{H}} = \int_{\mathbf{R}^{n+1}} (\nabla f_1(x,y)\overline{\nabla g_1(x,y)} + f_2(x,y)\overline{g_2(x,y)}c_0^{-2}(y))dxdy,$$

and $\|\cdot\|_{\mathcal{H}}$ is the corresponding norm, where $f = {}^t(f_1, f_2), g = {}^t(g_1, g_2)$.

The domains of A_0 is

$$D(A_0) = \{f \in \mathcal{H}; \triangle f_1 \in L^2(\mathbf{R}^{n+1}), f_2 \in H^1(\mathbf{R}^{n+1})\}.$$

Then A_0 is a self-adjoint operator in $\mathcal{H}$ and generates a unitary group $\{U_0(t)\}_{t\in\mathbf{R}}$ in $\mathcal{H}$. Below we make a cheak on (A1), (A2) and (A3).

Since

$$B = \begin{pmatrix} 0 & 0 \\ 0 & b(x,y) \end{pmatrix}$$

is A_0-compact by Rellich's theorem, we have (A2). Therefore A generates a contraction semi-group $\{V(t)\}_{t\geqq 0}$ in $\mathcal{H}$.

In the same argument as in Mochizuki[7]§3 we can show

$$\|\sqrt{B}(A_0 - \zeta)^{-1}\sqrt{B}g\|_{\mathcal{H}} = |\zeta|\|\sqrt{b(x,y)}(L_0 - \zeta^2)^{-1}\sqrt{b(x,y)}g_2\|_{\mathcal{G}}. \tag{2.1}$$

(1.4) and (2.1) imply (A3).

To deal with (1.2) we set

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \langle\cdot, \varphi\rangle_{\mathcal{G}}\varphi \end{pmatrix}$$

Then B is a compact operator in $\mathcal{H}$. We shall show (1.7). Note that

$$\begin{aligned} &|\mathrm{Im}\zeta|\|\sqrt{B}(A_0 - \zeta)^{-1}f\|_{\mathcal{H}}^2 \\ &\leqq \|\sqrt{B_1}\{(A_0 - \zeta)^{-1} - (A_0 - \overline{\zeta})^{-1}\}\sqrt{B_1}\|\|f\|_{\mathcal{H}}^2 \times \|X_{-\frac{\theta}{2}}\varphi\|_{\mathcal{G}}^2 \end{aligned} \tag{2.2}$$

for any $f \in \mathcal{H}$, where

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & X_\theta \end{pmatrix}.$$

We can get (1.7) by (2.1) which is changed B and $b(x,y)$ to B_1 and X_θ, respectively. Therefore we have the conclusion of Theorem 1 for (1.2).

Restricting ourselves to $c_h = c_+ = c_- = 1$ or $c_h < c_+ = c_-$, we have by (1.4) the conclusion of Theorem 3.

3. Energy decay states for (1.2) (cf. Perry[8] P47 and P92).

In this section we give a simple example of (1.2) which has energy decay states. By choosing $\varphi(x,y)$ we construct complex eigenvalues and eigenvectors for

$$A = i\begin{pmatrix} 0 & 1 \\ -L_0 & -\langle\cdot,\varphi\rangle_{\mathcal{G}}\varphi \end{pmatrix}.$$

Indeed, for any non-trivial $v \in C_0^\infty(\mathbf{R}^{n+1})$, considering $\kappa < 0$ which satisfies

$$\kappa^3\|v\|_{\mathcal{G}}^2 + \kappa\|\nabla v\|_{L^2(\mathbf{R}^{n+1})}^2 + 1 = 0$$

and setting

$$\varphi = (L_0 + \kappa^2)v,$$

we have

$$A\begin{pmatrix} \frac{1}{i\kappa}v \\ v \end{pmatrix} = i\kappa\begin{pmatrix} \frac{1}{i\kappa}v \\ v \end{pmatrix} \quad \text{and} \quad V(t)\begin{pmatrix} \frac{1}{i\kappa}v \\ v \end{pmatrix} = e^{\kappa t}\begin{pmatrix} \frac{1}{i\kappa}v \\ v \end{pmatrix},$$

where $V(t) = e^{-itA}$.

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