LARGE TIME BEHAVIOR OF SOLUTIONS OF THE KORTEWEG-DE VRIES-BURGERS EQUATION ON HALF-LINE

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1. Introduction

We consider the initial-boundary value problem on a half-line for the Korteweg-de Vries-Burgers equation. The number of the boundary data for this problem depends on the direction of the half-line: $x \geq 0$ or $x \leq 0$, so we are interested in the following two initial-boundary value problems

$$\begin{cases} u_t + uu_x - au_{xx} + bu_{xxx} = 0, \quad t > 0, x > 0, \\ u(x,0) = u_0(x),\ x > 0;\ u(0,t) = 0,\ t > 0, \end{cases}$$

and

$$\begin{cases} u_t + uu_x - au_{xx} + bu_{xxx} = 0, \quad t > 0, x < 0, \\ u(x,0) = u_0(x),\ x < 0;\ u(0,t) = u_x(0,t) = 0,\ t > 0, \end{cases}$$

where $a, b > 0$. By the change of variables and introducing a parameter $\alpha = 0,\ 1$ we can combine these two problems as one problem on the half-line $x > 0$

$$\begin{cases} u_t + (-1)^\alpha uu_x - u_{xx} + (-1)^\alpha u_{xxx} = 0,\ t > 0, x > 0, \\ u(x,0) = u_0(x),\ x > 0,\ \partial_x^j u(0,t) = 0,\ t > 0,\ j = 0, \alpha. \end{cases} \tag{1.1}$$

The Korteweg-de Vries-Burgers equation (1.1) is a simple universal model equation which appears as the first approximation in the description of the dispersive dissipative nonlinear waves.

The main goal of this paper is to find the large time asymptotics of solutions to the problem (1.1). Some results on the decay estimates of the solutions in different norms to the Cauchy problems for some nonlinear evolution equations with strongly dissipative operator were obtained in papers [1], [2], [3]. In papers [4], [7], [8], [9] it was found that the large time asymptotic behavior of solutions to the Korteweg-de Vries-Burgers type equations on the line. In paper [5] the problem

$$\begin{cases} u_t + u_x^2 - u_{xx} + u_{xxx} = 0, \quad t > 0, x > 0, \\ u(x,0) = u_0(x),\ x > 0;\ u(0,t) = 0,\ t > 0, \end{cases}$$

was studied and asymptotic behavior of solutions was obtained under more weak condition on the data since the $\mathbf{L}^2$ norm of the nonlinear term decays faster than the case of this paper. This problem corresponds to the case $\alpha = 0$ in this paper. As far as we know there is a few results on the asymptotics of solutions to the initial-boundary value problem (1.1) when $\alpha = 1$ which is different from the case

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$\alpha = 0$. For the nonlinear nonlocal Schrödinger equation on a half-line the initial-boundary value problem was studied in paper [6], where it was introduced the pseudodifferential operator $\mathbb{K}$ with homogeneous symbol $K(p) = Cp^\beta$ through the Laplace transformation and it was shown that the number of the boundary data which is necessary to show the well-posedness of the probelm is equal to $\left[\frac{\beta}{2}\right]$ except the case, when β is an odd integer. In the case of equation (1.1) we have the operator $\mathbb{K}u = -u_{xx} + (-1)^\alpha u_{xxx}$, i.e. the symbol $K_\alpha(p) = -p^2 + (-1)^\alpha p^3$ is not homogeneous and also the order of p in the second term is an odd integer. Thus we consider the critical case that the number of the boundary data also depends on the sign of the highest derivative in the equation. Note that in the case of the Cauchy problem for the Korteweg-de Vries-Burgers equation the nonlinearity uu_x of the shallow water type is critical from the point of view of the large time asymptotic behavior of solutions since the nonlinearity in the equation has the same decay rate as the linear terms. In the case of the initial-boundary value problem due to the homogeneous boundary data the solution obtains an additional decay and as a result the nonlinear term in the boundary-value problem (1.1) appears to be super critical in the contrary to the corresponding Cauchy problem. The main difficulty in the boundary value problem (1.1) is in the evaluation of the contribution of the boundary data to the large time asymptotic formulas of the solutions. Our approach here is based on the $\mathbf{L}^p$ estimates of the Green function.

We consider the generalized solutions of the initial-boundary value problem (1.1), that is, we multiply equation (1.1) by any function $\phi \in \mathbf{C}^2([0,T] \times (0,\infty))$, such that $\phi(x,T) = 0$ and $\phi(0,t) = 0$ and integrate by parts in the domain $[0,T] \times (0,\infty)$.

Let us denote $\mathbf{H}^{s,k} = \left\{ f \in \mathbf{L}^2; \|f\|_{\mathbf{H}^{s,k}} = \|(1+x)^k (1-\partial_x^2)^{s/2} f\|_{\mathbf{L}^2} < \infty \right\}$ is the weighted Sobolev space. The symbol of the operator $\mathbb{K}$ has the form $K_\alpha(p) = -p^2 + (-1)^\alpha p^3$, so that it satisfies the dissipation condition $\Re K_\alpha(p) > 0$ for all p on the imaginary axis $\Re p = 0, p \neq 0$. By the same letter C we denote different positive constants. First of all we formulate the local existence of the solutions of the initial-boundary value problem (1.1).

Now we state our results.

Theorem 1.1. *Suppose that the initial data $u_0(x) \in \mathbf{H}^{0,\omega} \cap \mathbf{H}^{1,0}$, where $\omega \in \left(\frac{1}{2}, \frac{3}{2}\right)$. Then for some $T > 0$ there exists a unique solution $u(x,t) \in \mathbf{C}([0,T], \mathbf{H}^{0,\varkappa}) \cap \mathbf{C}((0,T], \mathbf{H}^{1,\omega})$ of the initial boundary value problem (1.1), where $\varkappa \in \left(0, \frac{1}{2}\right)$.*

Now we give sufficient conditions for the global existence of solutions.

Theorem 1.2. *Suppose that the initial data $u_0 \in \mathbf{H}^{0,\omega} \cap \mathbf{H}^{1,0}$, where $\omega \in \left(1, \frac{3}{2}\right)$ and the norm $\|u_0\|_{\mathbf{H}^{0,\omega}} + \|u_0\|_{\mathbf{H}^{1,0}} \leq \epsilon$, where $\epsilon > 0$ is small enough. Then there exists a unique solution $u \in \mathbf{C}([0,\infty), \mathbf{H}^{0,\varkappa}) \cap \mathbf{C}\left((0,\infty), \mathbf{H}^{1,\omega}\right)$ of the initial-boundary value problem (1.1), where $\varkappa \in \left(0, \frac{1}{2}\right)$. Moreover if the initial data are such that $x^{1+\mu} u_0(x) \in \mathbf{L}^1$, $\mu = \omega - \frac{1}{2}$, then there exists a constant A such that the solution has the following asymptotics*

$$u(x,t) = \frac{A}{t} \Phi_\alpha \left(\frac{x}{2\sqrt{t}}, t \right) + O\left(\min\left(\frac{x}{\sqrt{t}}, 1 \right) t^{-1-\frac{\mu}{2}} \right)$$

for $t \to \infty$ uniformly with respect to $x > 0$, where $\alpha = 0, 1$, $\Phi_0(q,t) = \frac{q}{\sqrt{\pi}} e^{-q^2}$, $\Phi_1(q,t) = \frac{1}{2\sqrt{\pi}\sqrt{t}} \left(e^{-q^2} \left(2q\sqrt{t} - 1 \right) + e^{-2q\sqrt{t}} \right)$.

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