Sharp Asymptotics of Small Solutions to Derivative Nonlinear Schrödinger Equations with Gauge Invariant Cubic Nonlinearity

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Abstract

In this talk, we consider the large time behavior of small solutions to the nonlinear Schrödinger equations in one space dimension. Our equations contain the gauge invariant cubic nonlinearities of derivative type. Since the nonlinear term is so-called the long range type, it is well-known that the nonlinear solution tends to the modified linear solution called the first asymptotic term. We present the higher order asymptotic expansion of the nonlinear solution in the L^2 and L^∞ spaces. In our theorem, the nonlinear interaction plays an explicit role in the higher order asymptotic terms as well as in the phase modification.

Key Words: large time behavior, gauge transformation, energy estimate, bilinear estimate.

1 Introduction

We consider the asymptotic behavior as $t \to \infty$ of the small solution to the nonlinear Schrödinger equations :

$$\begin{cases} iu_t &= -u_{xx} + \mathcal{N}(u, u_x), \\ u(0,x) &= u_0(x), \end{cases} \tag{1.1}$$

where u is a complex valued unknown function of $(t,x) \in \mathbf{R} \times \mathbf{R}$, $u_t = \partial_t u = \partial u/\partial t$, $u_x = \partial_x u = \partial u/\partial x$. The nonlinear term $\mathcal{N}(u,q)$ is a cubic polynomial of $u, \overline{u}, q$ and $\overline{q}$ defined by

$$\mathcal{N}(u,q) = \lambda_1|u|^2u + i\lambda_2|u|^2q + i\lambda_3u^2\overline{q} + \lambda_4u|q|^2 + \lambda_5\overline{u}q^2 + i\lambda_6|q|^2q, \tag{1.2}$$

where $\lambda_1, \lambda_6 \in \mathbf{R}$, $\lambda_j \in \mathbf{C}$ $(j = 2, \cdots, 5)$ satisfying $\lambda_2 - \lambda_3, \lambda_4 - \lambda_5 \in \mathbf{R}$ and $\overline{u}, \overline{q}$ denotes the complex conjugate of u, q. We note that $\mathcal{N}(u,q)$ satisfies the gauge invariant property, i.e., $e^{i\theta}\mathcal{N}(u,q) = \mathcal{N}(e^{i\theta}u, e^{i\theta}q)$ for $\theta \in \mathbf{R}$.

Motivation. Recently, Hayashi-Naumkin-Uchida [1] have studied the existence of the global small solution to (1.1) and given the explicit form of its asymptotic profile (we call this profile "the first asymptotic term"). Their work is considered in the general gauge invariant nonlinearities as in (1.2). They proved that, for $u_0 \in H^{3,0} \cap H^{2,1}$ with $\|u\|_{3,0} + \|u_0\|_{2,1}$ sufficiently small, there exist functions ϕ_0 and $\tilde{S}_0(t,\xi)$ such that

$$u(t) - U(t)\exp(-i\tilde{S}_0(t, -i\partial_x))\phi_0 = O(t^{-\alpha}) \quad \text{as } t \to \infty \text{ in } L^2, \tag{1.3}$$

where $\alpha \in [0, 1/4)$, $U(t) = \exp(it\partial_x^2)$ and $\exp(-i\tilde{S}_0(t, -i\partial_x)) = \mathcal{F}^{-1}\exp(-i\tilde{S}_0(t,\xi))\mathcal{F}$. In (1.3), it is not certain whether the time decay rate of the error term is sharp or not.

Our aim in this talk. Determine the sharp decay rate of the second asymptotic term and specify its form.

Our goal in this talk is the following.(The notations are shown in the last part of this section.)

Theorem 1.1 *Let $u_0 \in H^{5,0} \cap H^{1,4}$ and $\|u_0\|_{5,0} + \|u_0\|_{1,4} \le \epsilon$ with $\epsilon > 0$ sufficiently small. Then, there exists a unique global solution $u \in C(\mathbf{R}; H^{5,0} \cap H^{1,4})$ such that, for some $C_1, C_2 > 0$,*

$$\|u(t)\|_{5,0} + \|J^4u(t)\|_{1,0} \le C_2\epsilon\langle t\rangle^{C_1\epsilon}, \tag{1.4}$$

$$\|u(t)\|_\infty + \|u_{xx}(t)\|_\infty \le C_2\epsilon\langle t\rangle^{-1/2}. \tag{1.5}$$

Moreover, there exists a unique function $\widehat{\phi}_0 \in H^{3,1} \cap H^{1,3}$ such that

$$e^{i\tilde{S}}\mathcal{F}U(-t)u(t) - (\widehat{\phi}_0 + \sum_{j=0}^{1}\widehat{\phi}_{1,j}t^{-1}(\log t)^j) = o(t^{-1}) \quad \text{as } t \to \infty \text{ in } H^{1,0}, \tag{1.6}$$

where

$$\widehat{\phi}_{1,1}(\xi) = -(i/8)\widetilde{\mathcal{N}}|\widehat{\phi}_0|^2\widehat{\phi}_0\partial_\xi^2(\widetilde{\mathcal{N}}|\widehat{\phi}_0|^2),$$

$$\begin{aligned}
\widehat{\phi}_{1,0}(\xi) &= \widehat{\phi}_{1,1} - (1/8)\Big\{2\widetilde{\mathcal{N}}(\widehat{\phi}_0^2\partial_\xi^2\overline{\widehat{\phi}}_0 + \overline{\widehat{\phi}}_0(\partial_\xi\widehat{\phi}_0)^2 + 2\widehat{\phi}_0|\widehat{\phi}_0|^2) \\
&\qquad +2i(\widetilde{\mathcal{N}}_q - 2\widetilde{\mathcal{N}}_{\overline{q}})\widehat{\phi}_0\partial_\xi(|\widehat{\phi}_0|^2) - (\partial_\xi^2\widetilde{\mathcal{N}})|\widehat{\phi}_0|^2\widehat{\phi}_0\Big\}, \\
\tilde{S}(t,\xi) &= (1/2)\widetilde{\mathcal{N}}|\widehat{\phi}_0|^2\log t - \widetilde{\mathcal{N}}Re(\overline{\widehat{\phi}}_0\widehat{\phi}_{1,0})t^{-1},
\end{aligned}$$

with $\widetilde{\mathcal{N}} = \mathcal{N}(1, i\xi)$, $\widetilde{\mathcal{N}}_q = \partial_q\mathcal{N}(1, i\xi)$ *and* $\widetilde{\mathcal{N}}_{\overline{q}} = \partial_{\overline{q}}\mathcal{N}(1, i\xi)$.

Theorem 1.1 suggests that the second asymptotic term sharply decays like $t^{-1}\log t$ in L^2. As a simple corollary of Theorem 1.1, we obtain the pointwise asymptotic behavior of the solution.

Corollary 1.2 *Under the same assumptions as in Theorem 1.1, we have*

$$\begin{aligned}
u(t) &= (2it)^{-1/2}\exp(ix^2/4t - i\tilde{S}(t, x/2t))\left(\widehat{\phi}_0(x/2t) + \sum_{j=0}^{2}\widehat{\psi}_{1,j}(x/2t)t^{-1}(\log t)^j\right) \\
&\qquad +o(t^{-3/2}) \quad \textit{as } t\to\infty \textit{ in } L^\infty, \qquad (1.7)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\psi}_{1,2}(x) &= (i/16)(\partial_x(\widetilde{\mathcal{N}}|\widehat{\phi}_0|^2))^2\widehat{\phi}_0, \\
\widehat{\psi}_{1,1}(x) &= \widehat{\phi}_{1,1} - (1/8)(2\partial_x\widehat{\phi}_0\partial_x(\widetilde{\mathcal{N}}|\widehat{\phi}_0|^2) + \widehat{\phi}_0\partial_x^2(\widetilde{\mathcal{N}}|\widehat{\phi}_0|^2)), \\
\widehat{\psi}_{1,0}(x) &= \widehat{\phi}_{1,0} - (i/4)\partial_x^2\widehat{\phi}_0.
\end{aligned}$$

Notations. $\mathcal{F}\phi$ and $\widehat{\phi}$ denote the Fourier transform of ϕ defined by $(2\pi)^{-1/2}\int e^{-ix\xi}\phi(x)dx$. We define two operators $M(t)$ and $D(t)$ by $M(t)\phi(x) = \exp(ix^2/4t)\phi(x)$ and $D(t)\phi(x) = (2it)^{-1/2}\phi(x/2t)$, respectively. Then, the free Schrödinger group $U(t)$ satisfies

$$\begin{aligned}
U(t) &= M(t)D(t)\mathcal{F}M(t), \\
U(-t) &= M(t)^{-1}\mathcal{F}^{-1}D(t)^{-1}M(t)^{-1}. \qquad (1.8)
\end{aligned}$$

Let $J(t) \equiv U(t)xU(-t) = x + 2it\partial_x$. Then, by (1.8), we have $J(t) = M(t)(2it\partial_x)M(-t)$. We remark that this expression of $J(t)$ is very useful when we apply it to the gauge invariant nonlinearity. Note that $J(t)$ satisfies

$$[J(t), \partial_x] = -1, \quad \left[J(t), (i\partial_t + \partial_x^2)\right] = 0, \qquad (1.9)$$

where $[A, B] = AB - BA$. We sometimes use brief notations M, D and J instead of $M(t), D(t), J(t)$.

For $m, s \in \mathbf{R}$, we define $H^{m,s}(\mathbf{R}) = \{\phi \in \mathcal{S}'(\mathbf{R}); \|\phi\|_{m,s} < \infty\}$, where $\|\phi\|_{m,s} = \|(1+x^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\|_2$ with $\|\cdot\|_p = \|\cdot\|_{L^p}$. For $m \in \mathbf{N}$, $W^{m,p}(\mathbf{R})$ denotes the usual

Sobolev space with the norm $\|\phi\|_{W^{m,p}} = \sum_{j=0}^{m} \|\partial_x^j \phi\|_p$. We note that $H^{m,0} = W^{m,2}$. Let ϵ be the size of initial data. We estimate the solution in X_T defined by

$$X_T = \{u(t,x); |||u|||_{X_T} < \infty\},$$

where $|||u|||_{X_T} = \sup_{-T \le t \le T} \{\langle t \rangle^{-C_1 \epsilon} (\|u(t)\|_{5,0} + \|J^4 u(t)\|_{1,0})\}$ with $\langle t \rangle = (1+t^2)^{1/2}$ and $C_1 > 0$.

2 Formal Argument for Sharp Asymptotics

In this section, we give a brief outline of our idea. Let $v(t) = U(-t)u(t)$. Then, the equation (1.1) is replaced by

$$\begin{aligned} iv_t &= U(-t)\mathcal{N}(U(t)v, U(t)v_x) \\ &= M^{-1}\mathcal{F}^{-1}D^{-1}M^{-1}\mathcal{N}(MD\mathcal{F}Mv, MD\mathcal{F}Mv_x) \\ &= (2t)^{-1}M^{-1}\mathcal{F}^{-1}\mathcal{N}(\mathcal{F}Mv, \mathcal{F}Mv_x), \end{aligned}$$

where we used the gauge invariance of $\mathcal{N}$. Since $M(t) \sim 1$ as $t \to \infty$, we have

$$i\widehat{v}_t = (2t)^{-1}\mathcal{N}(\widehat{v}, \widehat{v_x}) + I_1(t) + I_2(t), \tag{2.1}$$

where

$$\begin{aligned} I_1(t) &= (2t)^{-1}(\mathcal{N}(\widehat{Mv}, \widehat{Mv_x}) - \mathcal{N}(\widehat{v}, \widehat{v_x})), \\ I_2(t) &= (2t)^{-1}\mathcal{F}(M^{-1} - 1)\mathcal{F}^{-1}\mathcal{N}(\widehat{Mv}, \widehat{Mv_x}). \end{aligned}$$

We note that $\mathcal{N}(\widehat{v}, \widehat{v_x}) = \mathcal{N}(1, i\xi)|\widehat{v}|^2\widehat{v}$ and multiply $e^{iS(t,\xi)} = \exp(\int_1^t \mathcal{N}(1, i\xi)|\widehat{v}(\tau,\xi)|^2 \frac{d\tau}{2\tau})$ to the both sides of (2.1). Then, we have

$$iw_t = e^{iS}(I_1(t) + I_2(t)), \tag{2.2}$$

where $w(t) = e^{iS}\widehat{v}(t)$. Since the right hand side of (2.2) decays rapidly enough in time, it is integrable around $t = \infty$ and we can find a limit $\lim_{t\to\infty} w(t) = w_0$ in some weighted Sobolev space. By the definition of $S(t,\xi)$, we also have the asymptotics of the phase function like $S(t) \sim (1/2)\mathcal{N}(1, i\xi)|w_0(\xi)|^2 \log t$.

To obtain the sharp asymptotics of the solution, we require to know the large time behavior of $I_j(t)$ $(j = 1, 2)$ in detail. We explain this only for $I_2(t)$. Since $M(t) - 1 \sim ix^2/4t$, we can show that

$$\begin{aligned} I_2(t) &\sim -(i/8t^2)\partial_\xi^2 \mathcal{N}(\widehat{v}, \widehat{v_x}) \\ &= -(i/8t^2)\partial(e^{-iS}\mathcal{N}(1, i\xi)|w|^2 w). \end{aligned} \tag{2.3}$$

By substituting the asymptotic profiles of $S(t)$ and $w(t)$ into (2.3), we obtain the sharp asymptotics of the nonlinear solution.

Setting of the function spaces. By seeing (2.3), we know that the limit of $w(t)$ should be taken in $H^{3,1} \cap H^{1,3}$. This is because we estimate $I_j(t)$ in $H^{1,0}$. Therefore, (2.2) suggests that $v(t) \in H^{5,0} \cap H^{1,4}$ is sufficient. This is why we assume that $u_0 \in H^{5,0} \cap H^{1,4}$.

3 Global Estimates of the Solution

In the previous section, we could know that our interest is in the global estimate of $\|v(t)\|_{H^{5,0}\cap H^{1,4}}$, i.e., $\|u(t)\|_{5,0}$ and $\|J^4\partial_x u(t)\|_2$. We can show that these quantities increase in time and its increasing order depends on the size of initial data. This is why we require the smallness of u_0 for the estimates of error terms in our argument.

Proposition 3.1 *For $\epsilon > 0$ suficiently small, there exists a unique global solution $u \in C(\mathbf{R}; H^{5,0} \cap H^{1,4})$ and it satisfies*

$$\|u(t)\|_{5,0} + \|J^4 u(t)\|_{1,0} \le C_2 \epsilon \langle t \rangle^{C_1 \epsilon} \quad \text{for } t \in \mathbf{R}. \tag{3.1}$$

Proposition 3.1 follows from the local solvability result and the energy estimate. We note that the higher order application of J makes the energy estimate so complicated. We explain this in more detail by taking the concrete case $\mathcal{N}(u, u_x) = \mathcal{N}(u_x) = i\lambda_6 |u_x|^2 u_x$. We write

$$\begin{aligned} J^4 \partial_x \mathcal{N}(u_x) &= i\lambda_6 \sum_{\ell=0}^{4} \binom{4}{\ell} \left\{ (2it\partial_x)^{4-\ell} |u_x|^2 \cdot J^\ell u_{xx} + (2it\partial_x)^{4-\ell} (\partial_x |u_x|^2) \cdot J^\ell u_x \right\} \\ &= \partial_q \mathcal{N}(u, u_x) \partial_x (J^4 u_x) + \partial_{\overline{q}} \mathcal{N}(u, u_x) \partial_x (\overline{J^4 u_x}) \\ &\quad + (\text{remainder terms}). \end{aligned} \tag{3.2}$$

The top two terms in (3.2) are controled by the energy estimate after taking the suitable transformation of unknown functions [2].

Our concern is to obtain the decay estimate of remainder terms in (3.2). We note that the unfavorable contribution of oscillating factors in u_x (or u) is canceled in $|u_x|^2$ (or $|u|^2, u\overline{u_x}$). Therefore, we employ the calculating method as in (3.2) and require the bilinear estimates. The combination of the following two lemmas is a key to prove Proposition 3.1.

Lemma 3.2 (bilinear estimate) *Let $\alpha \in [0, 1/4)$ and $v_j(t) = U(-t)u_j(t)$ for $j = 1, 2$. Then, we have*

$$\|(2it\partial_x)(u_1 \overline{u_2})\|_\infty \le Ct^{-1} (\|\partial_\xi (\widehat{v}_1 \overline{\widehat{v}}_2)\|_\infty + t^{-\alpha} \|v_1\|_{0,2} \|v_2\|_{0,2}), \tag{3.3}$$

$$\|(2it\partial_x)^4 (u_1 \overline{u_2})\|_2 \le C(\|u_1\|_\infty + \|u_2\|_\infty)(\|J^4 u_1\|_2 + \|J^4 u_2\|_2) \tag{3.4}$$

Lemma 3.3 *Let u be the local solution to (1.1) with ϵ sufficiently small and let $v(t) = U(-t)u(t)$. Then, there exists $C > 0$ independent of T such that, for any $t \in [-T, T]$,*

$$\|\partial_\xi |\widehat{v}|^2(t)\|_\infty + \|\partial_\xi |\widehat{v_x}|^2(t)\|_\infty + \|\partial_\xi (\widehat{v}\overline{\widehat{v_x}})(t)\|_\infty \le C \|\!|u|\!\|_{X_T}^4, \tag{3.5}$$

$$\|u(t)\|_{W^{2,\infty}} \le C\langle t\rangle^{-1/2}(\|\!|u|\!\|_{X_T} + \|\!|u|\!\|_{X_T}^3). \tag{3.6}$$

References

[1] N. Hayashi, P.I. Naumkin and H. Uchida, *Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations*, Publ. Res. Inst. Math. Sci. **35** (1999), 501–513.

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