L_2 and L_∞ Estimates of the Solutions for the Compressible Navier-Stokes Equations in a 3D Exterior Domain

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We consider the equation of the motion of compressible viscous fluid in a 3D exterior domain. The equation is given by the following system for the density $\rho(t,x)$ and the velocity $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$,

$$\begin{array}{cll} \rho_t + \nabla\cdot(\rho v) = 0 & \text{in} & (0,\infty)\times\Omega, \\ \rho(v_t + (v\cdot\nabla)v) + \nabla P(\rho) = \mu\Delta v + (\mu+\nu)\nabla(\nabla\cdot v) & \text{in} & (0,\infty)\times\Omega, \\ v|_{\partial\Omega} = 0 & \text{on} & (0,\infty)\times\partial\Omega, \\ \rho(0,x) = \rho_0(x), \quad v(0,x) = v_0(x) & \text{in} & \Omega, \end{array} \tag{1}$$

where Ω is an exterior domain in $\mathbf{R}^3$ with the compact smooth boundary $\partial\Omega$, $P = P(\rho)$ the pressure, $\mu \geq 0, \frac{2}{3}\mu + \nu > 0$ the viscosity coefficients. The unique existence of smooth solutions globally in time near constant state $(\bar\rho_0, 0)$, where $\bar\rho_0$ is a positive constant was proved by the employing the same argument as in Matsumura and Nishida [11, 12] for the Cauchy problem in $\mathbf{R}^3$; Matsumura and Nishida[13, 14, 15] for the exterior domain in $\mathbf{R}^3$. Concerning the decay property of solutions $(\rho(t,x), v(t,x))$, if the initial data $(\rho_0(x) - \bar\rho_0, v_0(x))$ belongs to H^4 and L_1, then as $t \to \infty$

$$\begin{aligned} \|(\rho(t,\cdot) - \bar\rho_0, v(t,\cdot))\|_{L_\infty} &= O(t^{-3/2}), \\ \|(\rho(t,\cdot) - \bar\rho_0, v(t,\cdot))\|_{L_2} &= O(t^{-3/4}), \\ \|(\rho(t,\cdot) - \bar\rho_0, v(t,\cdot))\|_{L_2} &= O(t^{1/2}). \end{aligned}$$

This fact was investigated by Hoff and Zumbrun [3, 4], Liu and Wang [9], Matsumura and Nishida [11, 12], Ponce [16] and Weike [17] for the Cauchy problem case ; Kobayashi [7], Kobayashi and Shibata [8] for the exterior domain case. On the other hand, if the initial data belongs to H^3 only, namely we do not assume that the initial data belongs to L_1, then, Deckelnick [1, 2] showed that as $t \to \infty$

$$\begin{aligned} \|(\rho_t(t,\cdot), v_t(t,\cdot))\|_{L_2(\Omega)} &= O(t^{-1/2}), \\ \|\partial_x(\rho(t,\cdot), v(t,\cdot))\|_{L_2(\Omega)} &= O(t^{-1/4}), \\ \|(v(t,\cdot))\|_{C^0(\bar\Omega)} &= O(t^{-1/4}), \\ \|\rho(t,\cdot) - \bar\rho_0\|_{C^0(\bar\Omega)} &= O(t^{-1/8}), \end{aligned}$$

in the exterior domain case ; Matsumura [10] showed that as $t \to \infty$

$$\|(\rho_t(t,\cdot), v_t(t,\cdot))\|_{L_2(\mathbf{R}^3)} = O(t^{-1/2}),$$
$$\|\partial_x(\rho(t,\cdot), v(t,\cdot))\|_{L_2(\mathbf{R}^3)} = O(t^{-1/2}),$$
$$\|\partial_x^2(\rho(t,\cdot), v(t,\cdot))\|_{L_2(\mathbf{R}^3)} = O(t^{-1}),$$
$$\|(\rho(t,\cdot) - \bar{\rho}_0, v(t,\cdot))\|_{L_\infty(\mathbf{R}^3)} = O(t^{-3/4}),$$

in the Cauchy problem case. In this lecture, we shall investigate the exterior problem of the system (1) and give the better decay rate than the rate obtained by Deckelnick [1, 2] in the case that the initial data belongs to H^3 or H^4 only. In particular, the L_2-decay rate of the first derivative with respect to the spacial variable x for the solutions corresponds to the rate obtained by Matsumura [10].

Now, in order to explain our main results, we shall introduce the notations and assumptions. Let L_p denotes the usual L_p space on Ω with norm $\|\cdot\|_{L_p}$. Put

$$W_p^m = \{u \in L_p \mid \|u\|_{W_p^m} < \infty\}, \quad \|u\|_{W_p^m} = \sum_{|\alpha|\le m} \|\partial_x^\alpha u\|_{L_p},$$
$$H^m = W_2^m, \quad W_p^0 = L_p, \quad H^0 = L_2.$$

Set

$$W_p^{k,m} = \{(\rho, v) = (\rho, v_1, v_2, v_3) \mid \rho \in W_p^k,\ v_j \in W_p^m,\ j = 1,2,3\},$$
$$\|(\rho,v)\|_{W_p^{k,m}} = \|\rho\|_{W_p^k} + \|v\|_{W_p^m},$$

and

$$H^{k,m} = W_2^{k,m}, \quad \|u\|_{H^{k,m}} = \|u\|_{W_2^{k,m}}.$$

Let $\bar{\rho}_0$ be a positive constant. We assume that

A1. P is a smooth function in a neighborhood of $\bar{\rho}_0$ and $\dfrac{\partial P}{\partial \rho} > 0$.

A2. The initial data (ρ_0, v_0) satisfies the compatibility condition and regularity, namely $(\rho_0 - \bar{\rho}_0, v_0) \in H^3$, $v_0|_{\partial\Omega} = 0$ and $(\rho_1, v_1) = (\rho_t, v_t)|_{t=0}$ satisfies

$$\rho_1 = -\nabla \cdot (\rho_0 v_0),$$
$$v_1 = -(v_0 \cdot \nabla) v_0 + \frac{\mu}{\rho_0}\Delta v_0 + \frac{\nu}{\rho_0}\nabla(\nabla \cdot v_0) - \frac{\nabla P(\rho_0)}{\rho_0},$$

and

$$\rho_1 \in H^2, \quad v_1 \in H^1, \quad v_1|_{\partial\Omega} = 0.$$

Put

$$\begin{aligned} X(0,\infty) \quad = \quad & \{U = (\rho, v) \mid \rho - \bar{\rho}_0 \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{3-j}), \\ & \partial_x \rho \in L_2((0,\infty); H^2), \quad \rho_t,\ v_t \in L_2((0,\infty); H^2), \\ & v \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{3-2j}), \quad \partial_x v \in L_2((0,\infty); H^3)\}, \end{aligned}$$

and

$$N(0,\infty)^2 = \sup_{0\le t<\infty} \left(\|U(t)-\bar U_0\|^2_{H^3} + \|U_t(t)\|^2_{H^{2,1}}\right) \quad + \int_0^\infty \left(\|\partial_x U(s)\|^2_{H^{2,3}} + \|U_s(s)\|^2_{H^2}\right) ds,$$

where $\bar U_0 = (\bar\rho_0, 0)$. Then, we have

Proposition 1 *(Matsumura and Nishida [13, 14, 15]) Assume that the assumptions A.1 and A.2 hold. Then, there exists an ϵ_0 such that if $\|(\rho_0-\bar\rho_0, v_0)\|_{H^3} \le \epsilon_0$, then (1) admits a unique solution $(\rho, v) \in X(0,\infty)$.*

Moreover, there exists a constant C such that

$$N(0,\infty) \le C\|(\rho_0-\bar\rho_0, v_0)\|_{H^3}.$$

Remark. If the initial data $(\rho_0-\bar\rho_0, v_0) \in H^4$ and satisfies the second order compatibility condition and regularity, namely $(\rho_2, v_2) = (\rho_{tt}, v_{tt})|_{t=0}$ is determined successively by the initial data (ρ_0, v_0) through the system (1), then we have

$$\begin{aligned} \tilde N(0,\infty)^2 &= \sup_{0\le t<\infty} \left(\|U(t)-\bar U_0\|^2_{H^4} + \|U_t(t)\|^2_{H^{3,2}}\right) \\ &\quad + \int_0^\infty \left(\|\partial_x U(s)\|^2_{H^{3,4}} + \|U_s(s)\|^2_{H^3}\right) ds, \\ &\le C\|(\rho_0-\bar\rho_0, v_0)\|^2_{H^4}. \end{aligned}$$

Now, we shall state our main results.

Theorem 1 *Assume that the assumptions A.1 and A.2 hold. Then, there exists an ϵ_1 such that if $\|(\rho_0-\bar\rho_0, v_0)\|_{3,2} \le \epsilon_1$, the solution (ρ, v) of the system (1) has the following asymptotic behavior as $t\to\infty$:*

$$\begin{aligned} \|(\rho_t(t,\cdot), v_t(t,\cdot))\|_{L_2} &= O(t^{-1/2}), \\ \|\partial_x v(t,\cdot)\|_{H^1} &= O(t^{-1/2}), \\ \|\partial_x \rho(t,\cdot)\|_{L_2} &= O(t^{-1/2}), \\ \|\partial_x^2 \rho(t,\cdot)\|_{L_2} &= O(t^{-3/4} log t), \\ \|(\rho(t,\cdot)-\bar\rho_0, v(t,\cdot))\|_{L_\infty} &= O(t^{-3/4} log t). \end{aligned}$$

Corollary 1 *The assumptions in Theorem 2.1 hold. Moreover, if the initial data $(\rho_0 - \bar\rho_0, v_0) \in H^4$ satisfies the second order compatibility condition and regularity in Remark, then we have*

$$\|(\rho(t,\cdot)-\bar\rho_0, v(t,\cdot))\|_{L_\infty} = O(t^{-3/4}) \quad as \quad t\to\infty.$$

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