Global existence for wave equations with quadratic terms in two space dimensions

By Hideo Kubo

In this talk, I would like to present a joint work with Prof. A. Hoshiga. It is related with some decay properties of the wave equation:

$$\partial_t^2 u - \Delta u = F \quad \text{in } \mathbf{R}^n \times [0,\infty). \tag{1}$$

Especially, in lower space dimensional cases, one can estimate the L^∞–norm of a solution $u(x,t) = L(F)(x,t)$ of (1) with zero initial data in terms of the right hand member $F(x,t)$ without loss of derivatives. Namely, we have

Theorem 1 *Let $n = 2$ or $n = 3$. Let $F \in C(\mathbf{R}^n \times [0,\infty))$ and let ν and μ be positive numbers. When $n = 2$, we further assume $0 < \nu < 1/2$. Then we have*

$$|L(F)(x,t)|(1+|x|+t)^{\frac{n-1}{2}}(1+||x|-t|)^{\nu} \le CM_{\nu,\mu}(F) \tag{2}$$

for $(x,t) \in \mathbf{R}^n \times [0,\infty)$. Here C is a constant depending only on ν and μ, and we have set

$$M_{\nu,\mu}(F) = \sup_{(y,s)\in\mathbf{R}^n\times[0,\infty)} \{(1+|y|)^{\frac{n-1}{2}}(1+|y|+s)^{1+\nu+\mu}(1+||y|-s|)^{1-\mu}|F(y,s)|\}.$$

As an application of Theorem 1, let us consider a system of nonlinear wave equations with one speed:

$$\partial_t^2 u^i - \Delta u^i = F^i(u, \partial u, \partial^2 u) \quad \text{in } \mathbf{R}^n \times [0,\infty), \tag{3}$$

where $u^i(x,t)$ is a scalar unknown, $i = 1,\cdots,m$ with an integer $m \ge 1$, $\partial = (\partial_0, \partial_1, \partial_2)$, $\partial^2 = (\partial_\alpha \partial_\beta)_{\alpha,\beta=0,1,2}$, and $u = {}^t(u^1,\cdots,u^m)$.

We assume that $F^i(u, \partial u, \partial^2 u)$ is a smooth function and satisfies

$$F^i(u, \partial u, \partial^2 u) = O(|u|^2 + |\partial u|^2 + |\partial^2 u|^2) \tag{4}$$

near $(u, \partial u, \partial^2 u) = (0,0,0)$.

When $n = 3$, it is shown by S. Klainerman [4] that the initial value problem to (3) with sufficiently small and smooth initial data adomits a unique smooth solution, provided the second order terms $F_2^i(u, \partial u, \partial^2 u)$ in the nonlinearity satisfy, so called, "null condition". The null condition is defined as follows. (See also [2], [4]).

Definition. *Let* $u = {}^t(u_1, \cdots, u_m)$, $v = {}^t(v_1, \cdots, v_m)$, $w = {}^t(w_1, \cdots, w_m) \in \mathbf{R}^m$ *and let* $X_0 = -1$, $X = (X_1, X_2) \in \mathbf{S}^1$. *By* $G(u, Xv, X^2w)$ *we denote* $G(u, \partial u, \partial^2 u)$ *with* $\partial_\alpha u$, $\partial_\beta \partial_\gamma u$ *replaced by* $X_\alpha v$ *and* $X_\beta X_\gamma w$, *respectively. Then if*

$$G(u, Xv, X^2w) \equiv 0 \tag{5}$$

holds for any $X \in \mathbf{S}^1$ *and for any* u, v *and* w, *we say that* $G(u, \partial u, \partial^2 u)$ *satisfies "null condition".*

Now we turn our attention to the two space dimensional case. S. Katayama [3] proved that when $n = 2$, the same conclusion as the above mentioned work due to S. Klainerman holds, if we suppose the second order terms are identically zero and the third order terms satisfy the null condition. Moreover, only when $m = 1$, i.e., for the sigle case, it is shown that the golbal existence theorem is still valid, if the second order term has the following form

$$F_2^i(u, \partial u, \partial^2 u) = A\{(\partial_t u)^2 - |\nabla u|^2\}, \quad A : \text{constant},$$

and the third order terms satisfy the null condition, by making use of a transformation.

Recently, S. Alinhac [1] obtained the golbal existence theorem in two space dimentions for the quasilinear case, i.e.,

$$F^i(u, \partial u, \partial^2 u) = \sum_{j=1}^{m} \sum_{\gamma,\delta=0}^{2} H_{\gamma\delta}^{ij}(\partial u) \partial_\gamma \partial_\delta u^j,$$

under the assumption that the second and third order terms satisfy the null condition, without using the transformation.

Our goal is to show that the result established by S. Alinhac is also true for the fully nonliear case. To obtain decay estimates, it is helpful to use the Lorentz invariance of the wave equation. Namely, we introduce vector fields Γ that consist of

$$\partial_\alpha \ (\alpha = 0, 1, 2), \quad \Omega = x_1\partial_2 - x_2\partial_1, \quad S = t\partial_t + x \cdot \nabla, \quad L_j = t\partial_j + x_j\partial_t \ (j = 1, 2). \tag{6}$$

Now we are in a position to state our main result.

Theorem 2 *Let the second and third order terms satisfy the null condition and let the initial data be in* $C_0^\infty(\mathbf{R}^2)$. *Then there exists a global smooth solution of the initial value problem to* (3) *with sufficiently small initial data. Moreover the solution* $u(x, t)$ *satisfies*

$$|\Gamma^\alpha u(x,t)| \leq C\varepsilon(1 + |x| + t)^{-\frac{1}{2}}(1 + ||x| - t|)^{-\nu}, \tag{7}$$

$$|\Gamma^\alpha \partial_{t,x} u(x,t)| \leq C\varepsilon(1 + |x| + t)^{-\frac{1}{2}}(1 + ||x| - t|)^{-1-\nu} \tag{8}$$

for some ν *with* $0 < \nu < 1/2$, *where* $\varepsilon > 0$ *stands for a size of the initial data.*

References

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