伝播速度の異なる非線型波動方程式系の大域解について

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この講演の内容は，横山和義氏との共同研究

Global existence of classical solutions to systems of
nonlinear wave equations with different speeds of propagation

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に基づいている．

Abstract. This paper is concerned with a system of nonlinear wave equations in three space dimensions

$$\partial_t^2 u^i - c_i^2 \Delta u^i = F^i(u, \partial u, \partial^2 u), \quad i = 1, 2, \cdots, m,$$

where $0 < c_1 < c_2 < \cdots < c_m$. We prove the global existence of classical solutions to the system with small initial data, provided F^i satisfy "Null conditon".

§1. Introduction.

This paper is a continuation of a previous paper [21] on global (in time) existence of small amplitude solutions to a system of quasilinear wave equations with different speeds of propagation. Indeed, the second author studied in it the system stated in the above abstract fot the case that F^i depend only on the first and second derivatives $\partial u, \partial^2 u$ of u, i.e., that $F^i = F^i(\partial u, \partial^2 u)$. The purpose of present paper is to extend the results of [21] to the case where $F^i = F^i(u, \partial u, \partial^2 u)$ depend explicitly on u as well as ∂u and $\partial^2 u$. As a result, we will refine the proof of the main theorem in Klainerman [14] which deals with the case of one speed of propagation, i.e., $m = 1$, in the sense that we do not need to make use of the vector fields $L_j = x_j \partial_t + t \partial_j$, $j = 1, 2, 3$, where $\partial_t = \partial / \partial_t$ and $\partial_j = \partial / \partial_{x_j}$.

We consider the following Cauchy problem

$$\Box_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(u, \partial u, \partial^2 u) \quad \text{in } (0,\infty)\times\mathbb{R}^3, \quad i = 1,2,\cdots,m, \tag{1.1}$$

$$u^i(0,x) = \varepsilon f^i(x), \ \partial_t u^i(0,x) = \varepsilon g^i(x) \quad \text{for } x \in \mathbb{R}^3, \ i = 1,2,\cdots,m, \tag{1.2}$$

where $c_1, c_2, \cdots, c_m$ are positive constants different each other, say, $0 < c_1 < c_2 < \cdots < c_m$, f^i and g^i are fixed functions in $C_0^\infty(\mathbb{R}^3)$ and ε is a positive parameter which serves to measure the *amplitude* of the initial values. Here $u^i = u^i(t,x) = (u_1^i(t,x), u_2^i(t,x), \cdots, u_{p_i}^i(t,x))$ is a vector of p_i real valued functions of $(t,x) \in [0,\infty)\times\mathbb{R}^3$, $u = (u^1, u^2, \cdots, u^m)$ a p-vector, where $p = p_1 + p_2 + \cdots + p_m$, $\partial = (\partial_t, \partial_x)$, $\partial_t = \partial_0 = \partial/\partial_t$, $\partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial_{x_j}$ for $j = 1,2,3$, $\Delta = \sum_{j=1}^3 \partial_j^2$ and $\partial^2 u$ stands for second derivatives of u. For convenience we will often denote ∂u and $\partial^2 u$ by u' and u'', respectively. Moreover $F^i(u,u',u'')$ is a vector of p_i real valued functions of (u,u',u''), belonging to $C^\infty(\mathbb{R}^{21p})$, such that

$$F^i(0,0,0) = 0, \quad i = 1,2,\cdots,m \tag{1.3.1}$$

and

$$(F^i)'(0,0,0) = 0, \quad i = 1,2,\cdots,m. \tag{1.3.2}$$

Here and in what follows we denote by $(F^i)'(u,u',u'')$ the first derivatives of F^i with respect to (u,u',u'') and by $\partial_u F^i(u,u',u'')$ the first derivatives of $F^i(u,u',u'')$ with respect to u, and so on.

Remark. The hypothesis (1.3.1) implies that the Cauchy problem (1.1)-(1.2) has a trivial solution $u = 0$, provided $\varepsilon = 0$. Moreover the condition (1.3.2) means that the *linear part* of (1.1) coincides with $\Box_i$. For the smoothness of F^i it is sufficient to assume that $F^i(u,u',u'')$ is of class C^∞ on a neighborhood of $(u,u',u'') = (0,0,0)$. However such a function F^i can be always extended to the whole space $\mathbb{R}^{21p}$ by multiplying a cutoff function. For the sake of simplicity of description we have therefore assumed that $F^i \in C^\infty(\mathbb{R}^{21p})$.

We shall assume without loss of generality that the system (1.1) is quasilinear, i.e., that $F^i(u,u',u'')$ is linear in u'', hence one can write

$$F^i(u,u',u'') = \sum_{j=1}^m \sum_{a,b=0}^3 A_{ij}^{ab}(u,u')\partial_a\partial_b u^j + B^i(u,u') \quad \text{for } i = 1,2,\cdots,m, \tag{1.4.1}$$

where $A_{ij}^{ab}(u,u')$ is a $p_i \times p_j$ matrix function of $(u,u') \in \mathbb{R}^p \times \mathbb{R}^{4p}$ and u^j are regarded as column vectors. (See for instance Courant and Hilbert [4], chapter I, §7). We suppose that

$$A_{ij}^{ba}(u,u') = A_{ij}^{ab}(u,u') \quad \text{for } a,b = 0,1,2,3 \text{ and } i,j = 1,2,\cdots,m, \tag{1.4.2}$$

since only classical solutions are considered in this paper. Clearly, conditions (1.3) and (1.4.1) imply that

$$A_{ij}^{ab}(0,0) = 0 \quad \text{for } i,j = 1,2,\cdots,m, \ \ a,b = 0,1,2,3 \tag{1.4.3}$$

and

$$B^i(0,0) = 0, \ \ (B^i)'(0,0) = 0 \quad \text{for } i = 1,2,\cdots,m. \tag{1.4.4}$$

In order to assure the existence of local (in time) solutions, we also require that the system (1.1) can be reduced to a symmetric hyperbolic system of the first order. Namely, we assume that the $p \times p$ matrices

$$(A_{ij}^{ab}(u,u'); \ i,j = 1,2,\cdots,m)$$

are (real) symmetric for $a,b = 0,1,2,3$ where $A_{ij}^{ab}(u,u')$ are given by (1.4.1), i.e., that

$$A_{ij}^{ab}(u,u') = {}^t A_{ij}^{ab}(u,u') \quad \text{for } i,j = 1,2,\cdots,m, \ \ a,b = 0,1,2,3, \tag{1.5}$$

where ${}^tA$ stands for the transposed matrix of A.

Let $F^{i,2}(u,u',u'')$ be the quadratic part of Taylor's expansion for $F^i(u,u',u'')$ about $(u,u',u'') = (0,0,0)$. Then by the assumptions (1.3) and (1.4.1) one can write

$$F^i(u,u',u'') = F^{i,2}(u,u',u'') + O((|u|+|u'|)^2(|u|+|u'|+|u''|)) \quad \text{for } (u,u',u'') \text{ near } (0,0,0). \tag{1.6}$$

We shall devide all terms of the homogeneous polynomial $F^{i,2}(u,u',u'')$ into two group as follows.

$$\begin{aligned} &F^{i,2}(u,\partial u,\partial^2 u) \\ &= \sum_{k=1}^{m} \{N_k^i(u^k,\partial u^k,\partial^2 u^k) + R_k^i(u,\partial u,\partial^2 u)\} \\ &\qquad \text{for } i = 1,2,\cdots,m, \end{aligned} \tag{1.7}$$

where $N_k^i(u^k,\partial u^k,\partial^2 u^k)$ are homogeneous polynomials in $(u^k,\partial u^k,\partial^2 u^k)$ of degree 2, and $R_k^i(u,\partial u,\partial^2 u)$ are homogeneous polynomials in $(u,\partial u,\partial^2 u)$ of degree 2 which are linear in $(u^k,\partial u^k,\partial^2 u^k)$. Then we assume that for each $i,k = 1,2,\cdots,m$ the polynomial

$N_k^i(u^k, \partial u^k, \partial^2 u^k)$ satisfies the null condition in the sense of Christodoulou [3]. Besides, we suppose that for each $i, k = 1, 2, \cdots, m$ the polynomial $R_k^i(u, \partial u, \partial^2 u)$ does not explicitly depend on u, i.e., that

$$\partial_u R_k^i(u, u', u'') = 0 \quad \text{for } i, k = 1, 2, \cdots, m. \tag{1.8}$$

Then we have the following.

Theorem 1. *Suppose that* (1.4), (1.5) *and* (1.8) *hold. Assume that* $N_k^i(u^k, \partial u^k, \partial^2 u^k)$ *satisfies the null condition for each* $i, k = 1, 2, \cdots, m$. *Let* k *be a positive integer with* $k \geq 14$ *and let* ν *be a positive number with* $\nu < 1/2$. *Then there are positive constants* ε_0 *and* C *such that for any* $0 < \varepsilon \leq \varepsilon_0$ *there exists uniquely a solution* $u(t, x) \in C^\infty([0, \infty) \times \mathbb{R}^3)$ *of* (1.1)-(1.2) *satisfying*

$$\begin{aligned} &|u^i(t, x)|(1 + \tau + t)(1 + |\tau - c_i t|)^\nu + \sum_{1 \leq |\alpha| \leq k} |\partial^\alpha u^i(t, x)|(1 + \tau) \\ &\times (1 + |\tau - c_i t|)^{1+\nu} \leq C\varepsilon \quad \textit{for } (t, x) \in [0, \infty) \times \mathbb{R}^3,\ i = 1, 2, \cdots, m \end{aligned} \tag{1.9}$$

and

$$(1 + t)^{-1} \|u(t)\|_{\mathbf{L}^2} + \sum_{1 \leq |\alpha| \leq k+6} \|\partial^\alpha u(t)\|_{\mathbf{L}^2} \leq C\varepsilon \quad \textit{for } t \geq 0, \tag{1.10}$$

where $\tau = |x|$ *and*

$$\|v(t)\|_{\mathbf{L}^2} = \left\{ \int_{\mathbb{R}^3} |v(t, x)|^2 dx \right\}^{1/2}.$$

Here the constants ε_0 *and* C *depend only on* k, ν, F^i, f^i *and* g^i.

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