Small data scattering for nonlinear Klein-Gordon equations in the Sobolev spaces

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In this talk we consider the small data global solvability (SDGS) for nonlinear Klein-Gordon equations in the Sobolev space $H^s \equiv H^s(\mathbb{R}^n)$.

$$\textbf{NLKG} \qquad \begin{array}{l} \partial_t^2 u - \Delta u + u = f(u), \quad u : \mathbb{R} \times \mathbb{R}^n \longmapsto \mathbb{C}, \ n \geq 2, \\ (u(0), \partial_t u(0)) \in \mathcal{H}^s \equiv H^s \oplus H^{s-1}, \ s \geq 1/2. \end{array}$$

The nonlinearities considered are typically given by

$$f(u) = c_1|u|^{p(0)} + c_2|u|^{p(s)}, \quad c_1, c_2 \in \mathbb{C}, \quad p(s) \equiv 1 + 4/(n-2s), \tag{0.1}$$

where $p(s)$ is expected as the critical value for SDGS in the class H^s by the scaling argument. When $f(u)$ is given by the single term such as $c_1|u|^{p(0)}$ or $c_2|u|^{p(s)}$, SDGS has been already shown. However, the sum of the such powers as above has not been shown.

Known results [6, Wang, 1999] SDGS holds with f satisfying (a) or (b).

(a) $f \in C^1$, $|\partial f(z)| \lesssim |z|^{p-1}$, $p(0) \leq p < p(1/2)$ for $1/2 \leq s \leq 3/2$.

(b) $f \in C^\kappa$, $|\partial^j f(z)| \lesssim |z|^{p_1 - j} + |z|^{p_2 - j}$, $0 \leq j \leq {}^\exists\kappa (> [s])$, for $p(1/2) \leq p_1 \leq p_2 \leq p(s)$ and $1/2 \leq s < n/2$.

The above results show the SDGS for the nonlinearities in (0.1) with $p(0)$ replaced by $p(1/2)$. However, they deal only the single power for p with $p(0) \leq p < p(1/2)$. We put the following assumptions for nonlinearities to describe the main theorem.

$$\begin{array}{rcl} N(s,p) & : & f \in C^{[s]}(\mathbb{C};\mathbb{C}), \ \partial^j f(z) = O(|z|^{p-j}) \ \text{ for } \ 0 \leq j \leq \min\{[s], p\}. \\ & & \text{Moreover } \partial^j f(z) = 0 \ \text{ for } \ p < j \leq [s] \ \text{ if } \ p < [s]. \\ & & |\partial^{[s]} f(z) - \partial^{[s]} f(w)| \leq \begin{cases} (|z| \vee |w|)^{p-[s]-1}|z-w| & \text{if } \ [s]+1 \leq p, \\ |z-w|^{p-[s]} & \text{if } \ s < p < [s]+1, \\ 0 & \text{if } \ p \leq s. \end{cases} \end{array} \tag{0.2}$$

$$N(s,p_1,p_2) \ : \ f \text{ can be rewritten by finite sum of } \sum_j f_j \text{ with } f_j \text{ satisfying } N(s,p_j)$$
$$\text{for some } \ p_j \ \text{ with } \ p_1 \leq p_j \leq p_2.$$

Main theorem *Let $n \geq 2$, $1/2 \leq s < n/2$, $p(0) \leq p_1 \leq p_2 \leq p(s)$. Let f satisfy $N(s-1/2, p_1, p_2)$. Then SDGS holds.*

Remark In the theorem, the smallness of the data is assumed for the norm in $\mathcal{H}^{n/2-2/(p_2-1)}$. The corresponding scattering theory is also obtained. The following nonlinearities are also in the scope of the theorem : $f(u) = c|u|^p \log|u|$ with $\max\{[s+1/2], p_1\} < p < p_2$.

The theorem is obtained refining the Strichartz estimates of the equations as

$$\max_{q_0\le q\le q_1} \| \int_0^t \frac{U(t-\tau)}{\omega} h(\tau)d\tau; X^s_q \equiv L^\infty H^s \cap L^q B^{s-1/2}_q \| \lesssim \min_{q_0\le q\le q_1} \|h; L^{q'} B^{s-1/2}_{q'}\|, \qquad (0.3)$$

where $q_0 = 2(n+2)/n,\ q_1 = 2(n+1)/(n-1),\ U(t) \equiv \exp(it\omega),\ \omega \equiv \sqrt{1-\Delta},\ B^s_r \equiv B^s_{r,2}(\mathbb{R}^n)$. The well-known Strichartz estimates are given by the form :

$$\| \int_0^t \frac{U(t-\tau)}{\omega} h(\tau)d\tau; X^s_q\| \lesssim \|h; L^{q'} B^{s-1/2}_{q'}\| \ \text{ for } \ q_0 \le \forall q \le q_1. \qquad (0.4)$$

With the use of (0.4), SDGS is shown for $f(u) = c|u|^{p(0)}$ with $q = q_0$, and for $f(u) = c|u|^{p(s)}$ with $q = q_1$, however q which deals $p(0)$ and $p(s)$ at the same time does not exist. On the other hand, with the use of estimate (0.3), both norms of LHS in (0.4) for $q = q_0$ and $q = q_1$ are dominated by the minimum of the norms of RHS in (0.4). Therefore we can treat $c|u|^{p(0)}$ and $c|u|^{p(s)}$ at the same time. The estimate (0.3) is based on the following lemma.

Lemma 1 [2, 3, 6] *Let* $(n+1)/2 \le \lambda \le (n+2)/2,\ 2n-2\lambda \le \sigma \le 2\lambda - 2,\ \sigma > 0$. *Then* $U(t)$ *satisfies* $\|U(t)\phi; B^{-\lambda/2}_\infty\| \lesssim |t|^{-\sigma/2}\|\phi; B^{\lambda/2}_1\|,\ \ \|U(t)\phi; L^q B^{-\lambda\alpha(r)}_r\| \lesssim \|\phi; B^0_2\|$, *where* $\alpha(r) = 1/2 - 1/r,\ 1/r + 2/\sigma q = 1/2,\ 2 < q \le \infty$.

Let $\lambda_j,\ \sigma_j, j = 1,2$, be the indices in Lemma 1. Let $r_*, q_*, r_j, q_j, j = 1,2$, be the numbers which satisfy $1/r_j + 2/\sigma_j q_j = 1/2,\ 2 < q_j \le \infty,\ r_* = 2r_1/r_2,\ 1/r_* + 2/\sigma_1 q_* = 1/2$. By Lemma 1, we have

$$\begin{aligned} &\||t|^{(\sigma_1+\sigma_2)/4} U(t)\phi; L^\infty(dw, B^{-\lambda_1/2}_\infty)\| \lesssim \|\phi; B^{\lambda_2/2}_1\|, \\ &\||t|^{(\sigma_1+\sigma_2)/4} U(t)\phi; L^{q_*}(dw, B^{-\lambda_1\alpha(r_*)}_{r_*})\| \lesssim \|\phi; B^0_2\|, \end{aligned}$$

where $dw(t) = |t|^{-(\sigma_1+\sigma_2)q_*/4}dt$, $\|v; L^q(dw, X)\| = \{\int \|v(t); X\|^q dw(t)\}^{1/q}$. Interpolating these inequalities, and applying the bilinear estimates, we have

$$\| \int U(t-\tau)h(\tau)d\tau; L^{q_1} B^{-\lambda_1\alpha(r_1)}_{r_1}\| \lesssim \|h; L^{q_2'} B^{-\lambda_2\alpha(r_2')}_{r_2'}\|.$$

The estimate (0.3) follows by the duality argument. The key point is that λ_1 and λ_2 can be taken independently under the assumption in Lemma 1. (0.4) is derived with the restriction $\lambda_1 = \lambda_2$.

Remark The estimate (0.3) is also shown by Machihara-Nakanishi-Ozawa in [5] with the method by Keel–Tao in [4].

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